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# FUZZY QUASI-UNIFORMITIES BY ENTOURAGE

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**Abstract:** In this paper we present a theory of quasi-uniformities on topological spaces to fuzzy topological space in the style of weil that is called fuzzy weil quasi uniform space. In particular, we show every fuzzy topological space is fuzzy weil quasiuniformizable.

**Keywords:** Weil entourages, fuzzy weil quasi uniformity, fuzzy weil quasi uniformizable.

## INTRODUCTION

Fuzzy uniformity was studied by four authors: B. Huton, R. Lowen, U. Höhle and A. K. Katsara, while U. Höhle [3, 4] and R. Lowen [10] starting point took a certain counterpart of the filter approach to based on uniform covers [Tukey [16]]. B. Hutton [5] and A. K. Katsara [8] presented an equivalent notion of fuzzy uniformity in terms of certain families of maps from the $L^X$ into itself. Recently, other authors continued this concept as an approach to fuzzy $L$-uniform space.

The contents of this paper are summarized as follows:

In the first section we recall some preliminary ideas. In Section 2, we present a theory of fuzzy quasi uniform space via entourage in the style of weil that is called fuzzy weil uniform space and we show that every fuzzy topological space is weil quasi-uniformizable.

## 1. PRELIMINARIES

**Definition 1.1 [7]:** Let $L$ be a frame. Recall [7] that the coproduct of the frame $L$ by itself

$$L \xrightarrow{u_1^L} L \oplus L \xrightarrow{u_2^L} L$$

can be constructed as follows:

Take the Cartesian product $L \times L$ with the usual order $A$ down Set $A$ of $L \times L$ is a $C$-ideal if $(\{x\} \times S \subseteq A \Rightarrow (x, \vee S) \in A)$ and $(S \times \{y\} \subseteq A \Rightarrow (\vee S, y) \in A)$. Put $L \oplus L$ as the frame of all $C$-ideals of $L \times L$. Observe that the case $S = \phi$ implies that every $C$-ideal contains the set $IO := \downarrow \{(1, 0)\} \cup \downarrow \{(0, 1)\}$. Obviously, each $\downarrow \{(x, y)\} \cup IO$ is a $C$-ideal. It is denoted by $x \oplus y$. Finally put $u_1^L(x) = x \oplus 1$ and $u_2^L(y) = 1 \oplus y$.

The following clear facts are useful.

- For every $A \in L \oplus L$, $A = \vee\{x \oplus y/(x, y) \in A\}$ and so every element of $L \oplus L$ is join-generated by some family of elements $x \oplus y$.
- $IO \neq x \oplus y \subseteq z \oplus w$ implies $x \leq z$ and $y \leq w$.

For any frame homomorphism $f: L \to M$, we write $f \oplus f: L \oplus L \to M \oplus M$ for the frame homomorphism given by $(f \oplus f) \cdot u_i^L = u_i^M \cdot f (i = 1, 2)$. Obviously, $(f \oplus f)(\vee_\gamma(x_\gamma \oplus y_\gamma) = \vee_\gamma( f(x_\gamma) \oplus f(y_\gamma))$.

Given $A$, $B$ in the lattice $D(L \times L)$ of all down-sets $L \times L$ we denoted by $k(A) = \cap \{B \in L \oplus L/A \subseteq B\}$ the $C$-ideal generated by $A$ and by $AoB$ the $C$-ideal generated by $A \cdot B = \{(x, y) \in L \times L/\exists z \in L\backslash\{0\} : (x, z) \in A, (z, y) \in B\}$, that is, $\vee\{x \oplus y/\exists z \in L\backslash\{0\} : (x, z) \in A, (z, y) \in B\}$.

**Lemma 1.2 [11]:** For any $A, B \in D(L \times L) \cdot k(A)\, ok\,(B) = AoB$.

**Definitions 1.3 [11]:** For $A \subset L$ and $x, y \in L$ we define $A$ is called a cover of $L$ if $\vee A = 1$

$$st\,(x, A) = \vee\{y \in L/(y, y) \in A, y \wedge x \neq 0\}.$$

**Definition 1.4 [11]:** For $e: L \to L$, $\mu \in L$, $\mu$ is $e$-small if $\mu < e(\lambda)$ whenever $\mu \wedge \lambda \neq 0$.

**Definition 1.5 [9]:** Let $X$ be a nonempty ordinary set, $L$ be a frame with an order-reversing involution', $\delta \subset L^X$, $\delta$ is called a $L$-fuzzy topology on $X$, and $(L^X, \delta)$ is called an $L$-fuzzy topological space, if $\delta$ satisfies the following three conditions:

(LFT1) $\bar{0}, \bar{1} \in \delta$

(LFT2) $\forall \mathcal{A} \subset \delta, \vee \mathcal{A} \in \delta$

(LFT3) $\forall \cup, \vee \in \delta, \cup \wedge \vee \in \delta$

Every element in $\delta$ is called an open set in $L^X$.

**Definition 1.6 [9]:** Let $X$ be a nonempty ordinary set, $L$ be a frame, $i$, $c: L^X \to L^X$ mappings on $L^X$, $i$ is called an interior operator on $L^X$, if it fulfills the following conditions:

(IO1) $i(\bar{1}) = \bar{1}$

(IO2) $\forall A \in L^X, i(A) \leq A$

(IO3) $\forall A, B \in L^X, i(A \wedge B) = i(A) \wedge i(B)$

(IO4) $\forall A \in L^X, i(i(A)) = i(A)$.

For an interior operator $i$ on $L^X$, define the $L$-fuzzy topology generated by $i$ as $\delta = \{A \in L^X : i(A) = A\}$ and $i(A) = \text{Int}\,(A)$.

$c$ is called a closure operator on $L^X$, if it fulfills the following condition,

(C01) $c(\bar{0}) = \bar{0}$

(C02) $\forall A \in L^X, A \leq c(A)$

(C03) $\forall A, B \in L^X, c(A \vee B) = c(A) \vee c(B)$

(C04) $\forall A \in L^X, c(c(A)) = c(A)$.

For a closure operator $c$ on $L^X$, define the $L$-fuzzy topology generated by $c$ as $\delta = \{A \in L^X; c(A') = A'\}$ and $c(A) = cl(A)$.

## 2. FUZZY WEIL QUASI UNIFORM SPACE

Since $L^X$ is a frame [5] $\bar{0}$ and $\bar{1}$ denote the least and the greatest element in $L^X$ we can define weil entourage on $L^X$.

**Definition 2.1:** $E \in L^X \oplus L^X$ is weil entourage of $L^X$ if and only if $\{\mu \in L^X/(\mu, \mu) \in E\}$ is a cover of $L^X$. That is $\vee\{\mu \in L^X/(\mu, \mu) \in E\} = \bar{1}$.

The collection $wE_n(L^X)$ of all weil entourage of $L^X$ may be partially ordered by inclusion.

**Definition 2.2:** We define the composition of fuzzy weil entourage as follows:

$$EoF = \vee\{f \oplus g/\exists h \in L^X \setminus \bar{0}, (f, h) \in E, (h, g) \in F\}$$

the inverse of a fuzzy weil entourage $E$ has the natural definition $E^{-1} = \{(g, f)/(f, g) \in E\}$.

We also consider a new partial order in $L^X$, induced by a family $\mathcal{E}$ of fuzzy weil entourages:

$g \overset{\mathcal{E}}{\Delta} f$ ($g$ is $\mathcal{E}$-strongly below $f$) if there is $E \in \mathcal{E}$ such that $Eo(g \oplus g) \subseteq f \oplus f$.

When $\mathcal{E}$ is symmetric ($E \in \mathcal{E}$ implies $E^{-1} \in \mathcal{E}$) this is equivalent to saying there is $E \in \mathcal{E}$ such that $(f \oplus f)oE \subseteq g \oplus g$.

**Proposition 2.3:** Let $E$ be a fuzzy weil entourage. Then

(a) for any $f \in L^X, f \leq st(f, E)$

(b) $E^n \subseteq E^{n+1}$ for every natural $n$.

(c) For any down set $A$ of $L^X \times L^X$, $A \subseteq (EoA) \cap (AoE)$.

(d) for any $f \in L^X$, $st(st(f, F), F) \leq st(f, F^2)$.

**Proof:** (a) Consider $f \in L^X$, we have

$$f = f \wedge \vee\{g \in L^X \mid (g, g) \in E\} = \vee\{f \wedge g \mid (g, g) \in E, f \wedge g \neq 0\} \leq st(f, E)$$

(b) It suces to prove that $E \subseteq E^2$ consider $(f, g) \in E$. By (a) $g \leq st(g, E)$. It is trivial $f \oplus st(g, E) \subseteq (f \oplus g)oE \subseteq E^2$. Consequently, $(f, g) \in E^2$.

(c) Let $(f, g) \in A$. The case $f = 0$ or $g = 0$ are trivial. If f, $g \neq 0$, since $f = \vee\{f \wedge e/(e, e) \in E, f \wedge e \neq 0\}$ and for any $(e, e) \in E$ with $f \wedge e \neq 0$, $(e, g) \in EoA$, we have, by definition of $G$-ideal, that $(f, g) \in EoA$. Similarly $A \subseteq AoE$.

(d) we observe $st(st(\mu, F), F) \leq st(\mu, F^2)$

$$st(st(\mu, F), F) = \vee\{\lambda \in L^X / ((\lambda, \lambda) \in F, \lambda \wedge st(\mu, F) \neq \bar{0}\}$$

Consider $\lambda \in L^X$ with $(\lambda, \lambda) \in F$ and $\lambda \wedge st(\mu, F) \neq \bar{0}$. Then there is $\gamma \in L^X$ such that $(\gamma, \gamma) \in F$, $(\gamma \wedge \mu) \neq \bar{0}$ and $(\gamma \wedge \lambda) \neq \bar{0}$, therefore $(\lambda, \lambda \wedge \gamma) \in F$ and $(\lambda \wedge \gamma, \gamma) \in F$ thus $(\lambda, \gamma) \in F^2$, similarly $(\gamma, \lambda) \in F^2$. Also $(\lambda, \lambda)$, $(\gamma, \gamma) \in F^2$. But $F^2$ is a $C$-ideal so $(\lambda \vee \gamma, \lambda \vee \gamma) \in F^2$. In conclusion $(\lambda \vee \gamma, \lambda \wedge \gamma) \in F^2$ and $(\lambda \vee \gamma) \wedge \mu \geq \mu \neq \bar{0}$ hence $\lambda \leq st(\mu, F^2)$ and $st(st(\mu, F), F) \leq st(\mu, F^2)$.

**Definition 2.4:** Let $X$ be a nonempty set and $\mathcal{E} \subset wEnt(L^X)$ we say $(X, \mathcal{E})$ is a fuzzy weil quasi uniformity on $X$ if it satisfies the following conditions:

(FWQE$_1$) $\mathcal{E}$ is a filter of $(w\, Ent(L^X), \subseteq)$

(F w QE$_2$) For each $E \in \mathcal{E}$ there is $F \in \mathcal{E}$ such that $FoF \subseteq E$.

The pair $(X, \mathcal{E})$ is said to be a fuzzy weil quasi-uniform space.

A fuzzy weil quasi uniform space $(X, \mathcal{E})$ is called a fuzzy weil uniform space if it satisfies

(FWE$_3$) for any $E \in \mathcal{E}$, $E^{-1}$ is also in $\mathcal{E}$.

It is useful to note that the symmetric fuzzy weil entourages $E$ of $\mathcal{E}$ form a basis for $\mathcal{E}$. Infact, if $E \in \mathcal{E}$ then $E^{-1} \in \mathcal{E}$ so $E \cap E^{-1}$ is a symmetric fuzzy weil entourage of $\mathcal{E}$ contained in $E$.

**Definition 2.5:** Let $(X, \mathcal{E})$, $(X', \mathcal{E}')$ be two fuzzy weil uniform space. A mapping $f: X \to X'$ is said to be uniformly homomorphic such that $(\vec{f} \oplus \vec{f})(E) \in \mathcal{E}'$ whenever $E \in \mathcal{E}, \vec{f} : L^X \to L^{X'}, \vec{f}(\mu)(y) = \vee_{f(x) = y}\, \mu(x)$ we will denote by $F$ weil-UNIF the category whose objects are fuzzy weil uniform spaces and morphisms are uniformly homomorphism mappings.

Now we define the Fuzzy topology generated by a Fuzzy weil quasi uniformity.

**Theorem 2.6:** Let $(X, \mathcal{E})$ be a fuzzy weil quasi uniform space, mapping $i: L^X \to L^X$ be defined as follows:

$$\forall A \in L^X\ i(A) = \vee\{C \in L^X | \exists E \in \mathcal{E}, st(C, E) \leq A\}$$

then $i$ is an interior operator on $L^X$.

**Proof:** (I01) Since $st(C, E) \leq 1$ for every $E \in \mathcal{E}$ so $i(\bar{1}) = \bar{1}$.

(I02) $i(A) = \vee\{C \in L^X | \exists E \in \mathcal{E}, st(C, E) \leq A\}$ since $C \leq st(C, E)$ for all $C \in L^X$ then $i(A) \leq A$.

(I03) We need to prove $i(A) \wedge i(B) \le i(A \wedge B)$ for arbitrary $A, B \in L^X$. Infact, since for arbitrary $E, F \in \mathcal{E}$ and arbitrary $A, B, C, D \in L^X$ such that $st(C, E) \le A$ and $st(D, F) \le B$, we have $st(C \wedge D, E \cap F) \le st(C, E) \wedge st(D, F) \le A \wedge B$ so $i(A) \wedge i(B) = \vee\{C \wedge D \mid C, D \in L^X, \exists E, F \in \mathcal{E}, st(C, E) \le A, st(D, E) \le B\} \le \vee\{C \wedge D \mid C, D \in L^X, \exists E, F \in \mathcal{E}\, st(C \wedge D, E \cap F) \le A \wedge B\} = i(A \wedge B)$.

(I04) By (I02) we have $i(i(A)) \le i(A)$. We want to show that $i(A) \le i(i(A))C \in L^X$, $E \in \mathcal{E}, st(C, E) \le A$ by ($FWQE_2$) we have $\exists F \in \mathcal{E}, FoF \subset E$ then $st(C, FoF) < st(C, E)$ by Proposition 2.3(d) $st(st(C, F), F) < st(C, FoF) \le A$ then $st(C, F) \le i(A)$ then $C \le i(i(A))$ so

$$i(A) = \vee\{C \in L^X \mid \exists E \in \mathcal{E} \quad st(C, E) \le A\} \le i(i(A)).$$

**Definition 2.7:** Let $\mathcal{E}$ be an $F$ weil quasi uniformity on $X$ the interior operator defined in [2.5] is called the interior operator on $L^X$ generated by $F$ weil quasi uniformity $\mathcal{E}$. The $L$-fuzzy topology generated by the fuzzy weil quasi-uniformity $\mathcal{E}$, denoted by $\delta(\mathcal{E})$, $(L^X, \delta(\mathcal{E}))$ is called the $L$-fts corresponding to $(L^X, \mathcal{E})$.

**Theorem 2.8:** Let $(L^X, \mathcal{E})$ be an fuzzy weil quasi-uniform spaces, mapping $c: L^X \to L^X$ be defined as

$$\forall A \in L^X, c(A) = \wedge\{st(A, E) \mid E \in \mathcal{E}\}$$

then $c$ is a closure operator on $L^X$.

**Proof:** (C01) Since for every $E \in \mathcal{E}\ \ st(\bar{0}, E) = \bar{0}$ so $c(\bar{0}) = \bar{0}$.

(C02) Since $A \le st(A, E)$ for every $E \in \mathcal{E}$ then $A \le c(A)$.

(C03) We need only to prove $c(A \vee B) \le c(A) \vee c(B)$ for arbitrary $A, B \in L^X$. It is trivial.

$st(A \vee B, E_1 \cap E_2) \le st(A, E_1) \vee st(B, E_2)$ suppose $e \in L^X$ such that $e \not\le c(A) \vee c(B)$ then $e \not\le c(A), e \not\le c(B)$ then there exist $E, F \in \mathcal{E}$ such that $e \not\le st(A, E), e \not\le st(B, F)$ then $e \not\le st(A \vee B, E \cap F), e \not\le c(A \vee B)$ so $c(A \vee B) \le c(A) \vee c(B)$.

(C04) For every $A \in L^X$ we have $st(st(C, F), F)) < st(C, FoF)$. For every $A \in L^X$, $E \in \mathcal{E}$, $c(A) = \wedge\{st(A, E) \mid E \in \mathcal{E}\}$. By ($FWQE_2$) there exist $F \in \mathcal{E}$ such that $FoF \subset\ E\ c(c(A)) \le \wedge\{st(st(A, F), F) \mid F \in \mathcal{E}\} \le \wedge\{st(A, FoF) \mid F \in \mathcal{E}\} \le \wedge\{st(A, E), E \in \mathcal{E}\} = c(A)$ and by (C02) $c(A) \le c(c(A))$ so $c(c(A)) = c(A)$.

**Definition 2.9:** Let $\mathcal{E}$ be an fuzzy weil quasi-uniformity on $X$. The closure operator defined in (2.8) is called the closure operator on $L^X$ generated by the fuzzy weil uniformity $\mathcal{E}$.

Theorems (2.6), (2.8) shows that every $L$-fuzzy quasi uniformity can generates an $L$-fuzzy topology, but the unexpected result is that its reverse is also true.

**Theorem 2.9:** Let $(L^X, \delta)$ be an $L$-fts, then there exists an fuzzy weil quasi uniformity $\mathcal{E}$ on $X$ such that $\delta = \delta(\mathcal{E})$.

**Proof:** For every $U \in \delta$, define a self mapping $f_U$ on $L^X$ as follows:

$$\forall A \in L^X, \qquad f_U(A) = \begin{cases} \overline{1} & A \not\leq U \\ U & \overline{0} \neq A \leq U \\ \overline{0} & A = \overline{0} \end{cases}$$

It is easy to find that $f_U$ is value increasing,

$$f_U(\vee A) = \vee_{A \in \mathcal{A}} f_U(A), \qquad f_U\, o f_U = f_U.$$

Let $D = \{f | \exists \mathcal{A} \in [\delta]^{<w}, f \geq \wedge_{U \in \mathcal{A}} f_U\}$ then for all $g, f \in \mathcal{D}$ there exist $h \in \mathcal{D}$ such that $h \leq g \wedge f (1)$ take $\mathcal{A} \in [\delta]^{<w}$ such that $f \geq \wedge_{U \in \mathcal{A}} f_U = \wedge_{U \in \mathcal{A}}(f_U\, o f_U)$ since for every $V \in \mathcal{A}$ $f_V\, o f_V \geq (\wedge_{U \in \mathcal{A}} f_U)\, 0\, (\wedge_{U \in \mathcal{A}} f_U)$ so take $g = \wedge_{U \in \mathcal{A}} f_U$ we ahve $g \in \mathcal{D}$ and $gog \leq f (2)$

Now we define $E = \cup\{\alpha \oplus \alpha / \alpha \in U_f\}$ such that $U_f$ be the cover of all $f$-small elements of $L^X$ and denote $\mathcal{E}_\delta = \{E \in L^X \oplus L^X / E = \cup_{\alpha \in U_f} \alpha \oplus \alpha\}$.

($FQW_1$) It is obviously satisfied by (1).

($FQW_2$) Let $E_e \in \mathcal{E}_\delta$ we can take $f \in \mathcal{D}$ such that $f^3 \leq e$. By Lemma 1.4, we have $E_f\, o E_f = (\cup_{\alpha \in U_f} \alpha \oplus \alpha)\, o\, (\cup_{\alpha \in U_f} \alpha \oplus \alpha)$. Let $(a, c) \in E_f\, o E_f$ then $(a, b) \leq (\alpha, \alpha)$ and $(b, c) \leq (\beta, \beta)$ where $\alpha, \beta \in U_f$ then $a < \alpha < st(\alpha, E_f)$, $c < B < st(\alpha, E_f)$ we prove $st(\alpha, E_f)$ is $e$-small.

Let $\lambda \wedge st(\alpha, E_f) \neq \overline{0}$, $(\gamma, \gamma) \in E_f$ with $\gamma \wedge \alpha \neq \overline{0}$ and $\gamma \wedge \lambda \neq \overline{0}$ then $\alpha, \gamma$ is $f$-smallness then $\gamma < f(\lambda)$, $\alpha < f(\gamma)$ then $\alpha < f^2(\lambda)$. Therefore, for every $(\gamma', \gamma') \in E_f$ such that $\gamma' \wedge \alpha \neq \overline{0}$ we have $\gamma' \leq f(\alpha) < f^3(\lambda) \leq e(\lambda)$. Then $st(\alpha, E_f)$ is $e$-small then $st(\alpha, E_f) \oplus st(\alpha, E_f) \in E_e$ so $(a, c) \in E_e$.

This theorem can be restated as "every $L$-fts is fuzzy weil quasi-uniformizable".

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