GENERALIZED STOCHASTIC BURGERS' EQUATION WITH NON-LIPSCHITZ DIFFUSION COEFFICIENT

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ABSTRACT. In this article, we study the existence of weak solutions to the one-dimensional generalized stochastic Burgers' equation with polynomial nonlinearity perturbed by space-time white noise with Dirichlet boundary conditions and α-Hölder continuous coefficient in noise term, where $\alpha \in [1/2, 1)$. The existence of weak solutions is shown by solving an equivalent martingale problem.

1. Introduction

The generalized stochastic Burgers' equation is given as

$$\frac{\partial f(t,x)}{\partial t} = \frac{\partial^2 f(t,x)}{\partial x^2} + h(t,x,f(t,x)) + \frac{\partial g(t,x,f(t,x))}{\partial x} + \sigma(t,x,f(t,x))\frac{\partial^2 W(t,x)}{\partial t \partial x} \tag{1.1}$$

with following boundary conditions

$$f(t,0) = f(t,1) = 0, \qquad t \geq 0, \tag{1.2}$$

and initial data

$$f(0,x) = f_0(x), \qquad 0 \leq x \leq 1, \tag{1.3}$$

where $\frac{\partial^2 W(t,x)}{\partial t \partial x}$ is a white noise with respect to space and time both as in [33] and $h = h(t,x,r), g = g(t,x,r)$, and $\sigma = \sigma(t,x,r)$ are Borel-measurable functions on $\mathbb{R}^+ \times [0,1] \times \mathbb{R}$. For $g = 0$, (1.1) becomes stochastic reaction-diffusion equation e.g. [4, 16, 28, 33]. When $h = \sigma = 0$ and $g(t,x,r) = \frac{r^2}{2}$, equation (1.1) gives the classical Burgers' equation, which basically shows the Newton's second law and describes the relationship between the changing momentum and force on fluids elements and it is used as a simple version of the Navier-Stokes equation which represents the hydrodynamical turbulent model, see [7, 8, 14, 20] and references there in. Next, for $h = 0, \sigma \neq 0$, $g(t,x,r) = \frac{r^2}{2}$, equation (1.1) is known as stochastic Burger's equation, which is preferred as a better model over the deterministic Burgers' equation because it also describes the chaotic phenomena in the fluid, see, e.g., [9, 11, 21, 23]. One dimensional form of the classical stochastic Burgers'

equation has been intensively studied in the literature [1, 2, 6, 12, 13, 17, 26], where existence and uniqueness of solutions were discussed under the various assumption on the diffusion coefficients σ. Here [2, 12, 13, 17] deal with σ having Lipschitz continuity while [5, 6, 26] have used non-Lipschitz diffusion coefficient σ. But if we include the polynomial nonlinearity in the equation, then this becomes more complicated issue to study the existence and uniqueness of solutions. To the best of our knowledge, there are very few results [18, 19, 25], which discuss the stochastic Burgers' equation with polynomial nonlinearity. In 1999, Gyöngy [18] has shown the existence and uniqueness of solutions for the class of a quasi linear stochastic partial differential equation with polynomial nonlinearity having white noise with respect to time only. Further, in 2006, Kim [25] has discussed about the Cauchy problem for the stochastic Burger equation by considering the nonlinear term with the polynomial growth on whole real line having white noise with respect to time. In these two works, it is assumed that the diffusion coefficient σ is either a constant or a Lipschitz function with linear growth conditions. Later in 2013, the existence and uniqueness of the global solution for Stochastic Burgers' equation with polynomial nonlinearity driven by additive Lévy process (a stochastic process with jumps) is obtained by Hausenblas and Giri [19]. However, in the present work we also deal with stochastic Burgers' equation with polynomial type nonlinearity but with the non-Lipschitz σ.

The equations (1.1)-(1.3), with quadratic nonlinearity in g under the condition that σ is $\frac{1}{2}$-Hölder continuous (non-Lipschitz) function, was studied by Kolkovska [26] in 2002, where she has established the existence of weak solutions by showing the tightness for a sequence of polygonal approximation for the equation and then solving an equvivalent martingale problem. Further, in 2014, Boulanba and Mellouk [6], extended the work of Kolkovska [26] in d-dimension ($d \geq 2$) with more general noise.

The novelty of the present work is that it generalizes the work of [26] by extending the quadratic nonlinearity to a class of polynomial nonlinearity and using more general diffusion coefficient σ. This work also differs from the work of [18, 25], where they have shown existence of weak solutions of stochastic Burgers' equation with polynomial nonlinearity having white noise with respect to time and Lipschitz continuity in the diffusion coefficient σ, whereas the present work considers the α-Hölder continuity in the diffusion coefficient σ, where $\alpha \in [1/2, 1)$ along with a space-time white noise.

In this article, we study the one dimensional stochastic Burger equation with polynomial nonlinearity perturbed by a space-time white noise i.e.

$$\frac{\partial f}{\partial t}(t, y) = \frac{\partial^2 f}{\partial y^2}(t, y) + \lambda \frac{\partial f^p}{\partial y}(t, y) + \sigma(f(t, y)) \frac{\partial^2 W}{\partial t \partial y}(t, y) \tag{1.4}$$

$$f(t, 0) = f(t, 1) = 0, \qquad t \in [0, T], \tag{1.5}$$

$$f(0, y) = f_0(y), \qquad y \in (0, 1) \tag{1.6}$$

where $p \geq 2$ is a fixed integer and $T > 0$. Here, the existence of weak solutions to equations (1.4)–(1.6) is established. The proof is mainly motivated by the technique used in Funaki [16].

The structure of the paper is the following: In the coming section, we give the rigorous formulation of the problem. In Section 3, the discretized form of (1.4)–(1.6) is obtained, which gives a system of stochastic differential equations in finite dimension. Further, the existence of unique strong solution to this system of stochastic differential equation is established by showing the existence and pathwise uniqueness of weak solution for the system, which is motivated by [16]. Next, in Section 4, we have shown the tightness property of the family of approximating solutions by satisfying the multi-dimensional Totoki Kolmogorov criterion. At last, in Section 5, the existence of weak solutions of the original problem (1.4)–(1.6) is shown by solving an equivalent martingale problem. With no loss of generality, we suppose that $\lambda = 1$.

2. Formulation of the Problem

Definition 2.1. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$ be a stochastic basis with filtration $(\mathcal{F}_t) = \{\mathcal{F}_t, t \in [0, T]\}$. Then the *Brownian sheet* $W(t, x) = \{W(t, x) : t \in [0, T], x \in \mathbb{R}\}$ is defined as a continuous, $(\mathcal{F}_t)$ adapted and centered Gaussian random field with covariance

$$\mathbb{E}\left(W(s, x)W(t, y)\right) = (s \wedge t)(x \wedge y)$$

in the sense of Walsh [33].

Remark 2.2. By the properties of W, it can be proved that white noise with respect to the filtration $(\mathcal{F}_t)$ is a martingale measure over $([0, T] \times \mathcal{B}[0, 1])$, where $\mathcal{B}[0, 1]$ is bounded Borel subset of $[0, 1]$, [24, 33]. Now, the equation (1.4) can be interpreted in the weak sense by the following equation (2.1).

Definition 2.3. A continuous stochastic process $\{f(t, x); t \in [0, T], x \in [0, 1]\}$, which is $(\mathcal{F}_t)$-adapted, is said to be *solution* of equation (1.4) in a weak sense, if for every $\phi \in \mathcal{C}^2[0, 1]$, such that $\phi(0) = \phi(1) = 0$, and a.s. for each $t \in [0, T]$, and $x \in [0, 1]$, we have

$$\int_0^1 f(t, y)\phi(y)dy = \int_0^1 f(0, y)\phi(y)dy + \int_0^t \int_0^1 f(s, y)\phi''(y)dyds$$
$$- \lambda \int_0^t \int_0^1 f^p(s, y)\phi'(y)dyds + \int_0^t \int_0^1 \sigma(f(s, y))\phi(y)W(ds, dx). \tag{2.1}$$

Further, we assume following conditions on σ. First condition is that the diffusion coefficient σ satisfies Hölder's continuity of order $\alpha \in [1/2, 1)$ on the interval $[0, 1]$ i.e. there exist a constant $c \geq 0$ such that

$$|\sigma(r_1) - \sigma(r_2)| \leq c|r_1 - r_2|^\alpha \qquad \forall\, r_1, r_2 \in [0, 1], \tag{2.2}$$

and second one is

$$\sigma(0) = \sigma(1) = 0. \tag{2.3}$$

Example 2.4. Let $\sigma : \mathbb{R} \to \mathbb{R}$ such that

$$\sigma(r) := \begin{cases} \sqrt{r(1-r)} & \text{if } 0 \leq r \leq 1. \\ 0 & \text{otherwise.} \end{cases} \tag{2.4}$$

Then, σ satisfies conditions (2.2) and (2.3).

Note: Other examples of such functions can be seen in [6].

3. The Discretization Processes

Let M ≥ 1 be a fixed integer and define the set $\left\{\frac{i}{M}; i = 0, 1, \cdots, M\right\}$. On this set, consider the descretized form of (1.4)–(1.6) as

$$dY(t, x_i) = \left(\Delta_M Y(t, x_i) + \nabla_M(Y^p(t, x_i))\right)dt + \sqrt{M}\sigma(Y(t, x_i))dB_i(t), \qquad (3.1)$$

$$Y(t, x_0) = Y(t, x_M) = 0, \qquad (3.2)$$

$$Y(0, x_i) = f_0(x_i), \qquad (3.3)$$

for every $i = 1, 2, \cdots, M - 1$ and $t \geq 0$. Here $x_i := \left\{\frac{i}{M}\right\}$ and $\{B_i(t) : i = 1, 2, \cdots M - 1\}$ is the system of Brownian motions, derived from the Brownian sheet $W(x, t)$ and it is defined as

$$\mathbb{E}(B_i(t)) := 0 \; \forall \; i = 1, 2, \cdots M,$$

$$\text{and } \mathbb{E}(B_i(t)B_j(s)) := \begin{cases} 0 & \text{if } i \neq j, \\ \min\{t, s\} & \text{if } j = i, \end{cases}$$

while ∇_M and Δ_M denote the approximation of the first and second order derivative respectively with respect to the variable x in the discrete sense and defined as

$$\Delta_M Y(t, x_i) := \frac{Y(t, x_i + \frac{1}{M}) - 2Y(t, x_i) + Y(t, x_i - \frac{1}{M})}{\frac{1}{M^2}}$$

and

$$\nabla_M(Q(t, x_i)) := \frac{Q(t, x_i + \frac{1}{M}) - Q(t, x_i)}{\frac{1}{M}},$$

for all $i = 1, 2, \cdots, M - 1$.

By setting $Y(t, x_i \pm \frac{1}{M}) := y_{i\pm1}(t)$, we can write (3.1) as

$$dy_i(t) = \left(M^2[y_{i+1}(t) - 2y_i(t) + y_{i-1}(t)] + M[y_{i+1}(t)^p - y_i(t)^p]\right)dt$$
$$+ \sqrt{M}\sigma(y_i(t))dB_i(t) \qquad (3.4)$$

Re-writing (3.4) in more compact form as

$$dy_i(t) = \left(\sum_{j=1}^{M-1} \alpha_{ij}y_j(t) + \beta_{ij}y_j(t)^p\right)dt + \sqrt{M}\sigma(y_i(t))dB_i(t) \qquad (3.5)$$

with

$$y_0(t) = y_M(t) = 0 \qquad\qquad t \in [0, T], \qquad (3.6)$$

and

$$y_i(0) = f_0(i/M), \qquad\qquad 1 \le i, j \le M - 1, \qquad\qquad (3.7)$$

where

$$\alpha_{ij} := \begin{cases} M^2 & \text{if } j = i+1, i-1 \\ -2M^2 & \text{if } j = i \\ 0 & \text{otherwise} \end{cases} \qquad\qquad (3.8)$$

and

$$\beta_{ij} := \begin{cases} M & \text{if } j = i+1 \\ -M & \text{if } j = i \\ 0 & \text{otherwise.} \end{cases} \qquad\qquad (3.9)$$

It is noticed that the drift and diffusion coefficients in the (3.5) do not satisfy the Lipschitz continuity, which restrict us to apply the classical results on the existence and uniqueness for the solution of (3.5). The following theorem is the main result of this section.

Theorem 3.1. *Let* $Y(0) = (y_0(0), y_1(0), \cdots, y_M(0)) \in [0,1]^{M+1}$, *be some given initial random condition and (2.2) and (2.3) hold. Then for each* $T > 0$ *and any integer* $M \ge 1$, *the system*

$$dy_i(t) = \left(\sum_{j=1}^{M-1} \alpha_{ij} y_j(t) + \beta_{ij} y_j(t)^p \right) dt + \sqrt{M} \sigma(y_i(t)) dB_i(t) \qquad (3.10)$$

$$y_0(t) = y_M(t) = 0 \qquad \forall\ t \in [0,T], \qquad\qquad (3.11)$$

$$y_i(0) = y_i, \qquad\qquad (3.12)$$

where $i = 1, 2, \cdots M - 1$, *admits a unique strong solution*

$$Y(t) = (y_0(t), y_1(t), \cdots, y_M(t)) \in \mathcal{C}([0,T], [0,1]^{M+1}). \qquad (3.13)$$

Proof. We consider the following modified form of stochastic differential equations (3.10)–(3.12)as

$$dy_i(t) = \left(\sum_{j=1}^{M-1} \alpha_{ij} y_j(t) + \beta_{ij} g(y_j(t)) \right) dt + \sqrt{M} K(y_i(t)) dB_i(t) \qquad (3.14)$$

$$y_0(t) = y_M(t) = 0 \qquad \forall\ t \in [0,T], \qquad\qquad (3.15)$$

$$y_i(0) = y_i, \qquad\qquad (3.16)$$

where $i = 1, 2, \cdots, M - 1$, $g : \mathbb{R} \to \mathbb{R}$ is defined as $g(x) = x^p \mathbf{1}_{\{-1 \le x \le 1\}}$ and $K : \mathbb{R} \to \mathbb{R}$ is defined as $K(x) = \sigma(x) \mathbf{1}_{\{0 \le x \le 1\}}$. Since, the coefficients of (3.14)–(3.16) are continuous and satisfies the linear growth conditions, by the [15, Theorem 3.10, Chapter 5], there exists a weak solution $Y(t)$ to (3.14)–(3.16).

Next, it is shown that for every weak solution $Y(t) = (y_0(t), y_1(t), \cdots, y_M(t))$ of the (3.14)–(3.16), $y_i(t) \in [0,1]$ for every $i = 0, \cdots, M$ and $t \in [0,T]$. In order to prove this the following lemma [29] is required.

Lemma 3.2. *Let* $Z=\{Z(t),\ t\geq 0\}$ *be a real valued semi-martingale. Suppose that there exist a function* $\rho:[0,\infty)\to[0,\infty)$ *such that* $\int_0^\epsilon \frac{1}{\rho(u)}\,du=+\infty$ *for all* $\epsilon\geq 0$, *and* $\int_0^t \frac{1_{\{Z_s>0\}}}{\rho(Z_s)}d\langle Z\rangle_s<\infty$ *for all* $t\geq 0$ *a.s. Then the local time of* Z *at zero, i.e.* $L_t^0(Z)$, *is identically zero for all* t *a.s..*

Let us apply the above Lemma 3.2 for the semi-martingale y_i and take $\rho(y_i)=y_i$, then $\int_0^\epsilon \frac{1}{\rho(u)}du=+\infty$ for all $\epsilon\geq 0$, and

$$\int_0^t \frac{\mathbf{1}_{\{y_i(s)>0\}}}{\rho(y_i(s))}d\langle y_i\rangle_s=\int_0^t \frac{\mathbf{1}_{\{y_i(s)>0\}}}{(y_i(s))}M\sigma^2(y_i(s))ds \quad <\infty.$$

Therefore local time $L_t^0(y_i)$ is zero. Again we use Lemma 3.2 and Tanaka's formula [30, Theorem 1.2 (Chapter IV)] for $(y_i(t))^-:=\max[0,-y_i(t)]$ and summing over indices $i=1\cdots,M-1$, we get

$$\sum_{i=1}^{M-1}(y_i(t))^-=-\int_0^t\sum_{i=1}^{M-1}\mathbf{1}_{\{y_i(s)\leq 0\}}\sum_{j=1}^{M-1}(\alpha_{ij}y_j(s)+\beta_{ij}g(y_j(s)))ds$$

$$=-\int_0^t\left[\sum_{i=1}^{M-1}\mathbf{1}_{\{y_i(s)\leq 0\}}\sum_{j=1}^{M-1}\alpha_{ij}y_j(s)\right]ds$$

$$-\int_0^t\left[\sum_{i=1}^{M-1}\mathbf{1}_{\{y_i(s)\leq 0\}}\sum_{j=1}^{M-1}\beta_{ij}g(y_j(s))\right]ds$$

$$\leq\int_0^t\left[\sum_{i=1}^{M-1}\mathbf{1}_{\{y_i(s)\leq 0\}}\sum_{j=1}^{M-1}\alpha_{ij}(y_j(s))^-\right]ds$$

$$+\int_0^t\left[\sum_{i=1}^{M-1}\mathbf{1}_{\{-1\leq y_i(s)\leq 0\}}\sum_{j=1}^{M-1}(-\beta_{ij})(y_j(s))\right]ds$$

$$\leq\int_0^t\left[\sum_{i=1}^{M-1}\mathbf{1}_{\{y_i(s)\leq 0\}}\sum_{j=1}^{M-1}\alpha_{ij}(y_j(s))^-\right]ds$$

$$+M\int_0^t\left[\sum_{i=1}^{M-1}\mathbf{1}_{\{-1\leq y_i(s)\leq 0\}}\sum_{j=1}^{M-1}(y_j(s))\right]ds$$

$$\leq M\int_0^t\left[\sum_{i=1}^{M-1}(y_i(s))^-\right]ds. \tag{3.17}$$

Finally, Gronwall's inequality gives

$$\sum_{i=1}^{M-1}(y_i(t))^-=0,$$

i.e. $(y_i(t))_{\{i\in\{1,2,\cdots M-1\}\}}$ is always non-negative for each $t\in[0,T]$. Again, solving the equation (3.17) for $(1-y_i(t))^-$, we can obtain $y_i(t)\leq 1$ for every $t\in[0,T]$ and $1\leq i\leq M-1$.

Since, for $y_i(t) \in [0,1]$, the system (3.14) coincide with system (3.10), therefore, the system (3.10) has a weak solution with trajectories lies in $\mathcal{C}\left([0,T], [0,1]^{M+1}\right)$.

3.1. Pathwise uniqueness of the solution for the descretized equations.

In order to show the pathwise uniqueness of solutions to (3.10), suppose that $Y^{(1)} = (y_0^{(1)}, \cdots, y_M^{(1)})$ and $Y^{(2)} = (y_0^{(2)}, \cdots, y_M^{(2)})$ are two different weak solutions of (3.10)-(3.12), with the same Brownian motion and same initial data.

Set $v_i := y_i^{(1)} - y_i^{(2)}$, $i = 1, 2, \cdots, M-1$, and $t \in [0,T]$. Then, we have

$$v_i(t) = \sum_{j=1}^{M-1} \alpha_{ij} \int_0^t v_j(s)ds + \sum_{j=1}^{M-1} \beta_{ij} \int_0^t (y_i^{(1)}(s)^p - y_i^{(2)}(s)^p)ds$$

$$+ \int_0^t \sqrt{M}\left(\sigma(y_i^{(1)}(s)) - \sigma(y_i^{(2)}(s))\right)dB_i(s) \qquad i = 1, 2, \cdots, M-1.$$

$$(3.18)$$

The quadratic variation $\langle V \rangle_t$ of $v_i(t)$,

$$\langle V \rangle_t = \int_0^t \left[\sqrt{M}\sigma(y_i^{(1)}(s)) - \sqrt{M}\sigma(y_i^{(2)}(s))\right]^2 ds$$

satisfies

$$\int_0^t \frac{\left[\sqrt{M}\sigma(y_i^{(1)}(s)) - \sqrt{M}\sigma(y_i^{(2)}(s))\right]^2}{y_i^{(1)}(s) - y_i^{(2)}(s)} \mathbf{1}_{\{y_i^{(1)}(s) - y_i^{(2)}(s) > 0\}} ds$$

$$= M \int_0^t \frac{\left(\sigma(y_i^{(1)}(s)) - \sigma(y_i^{(2)}(s))\right)^2}{y_i^{(1)}(s) - y_i^{(2)}(s)} \mathbf{1}_{\{y_i^{(1)}(s) - y_i^{(2)}(s) > 0\}} ds.$$

Since $y_i^{(1)}, y_i^{(2)} \in [0,1]$, implies that $(y_i^{(1)} - y_i^{(2)}) \in [-1,1]$. Further, using condition (2.2) and then simplifying, we have

$$\int_0^t \frac{\left[\sqrt{M}\sigma(y_i^{(1)}(s)) - \sqrt{M}\sigma(y_i^{(2)}(s))\right]^2}{y_i^{(1)}(s) - y_i^{(2)}(s)} \mathbf{1}_{\{y_i^{(1)}(s) - y_i^{(2)}(s) > 0\}} ds$$

$$\leq M \int_0^t \left(y_i^{(1)}(s) - y_i^{(2)}(s)\right)^{2\alpha - 1} \mathbf{1}_{\{y_i^{(1)}(s) - y_i^{(2)}(s) > 0\}} ds < 2MT < \infty. \quad (3.19)$$

Therefore, applying Lemma 3.2 to $v_i(t) = y_i^{(1)} - y_i^{(2)}$ with $\rho(v_i) = v_i$, we obtain that local time $L_t^0(y_i^{(1)} - y_i^{(2)}) = 0$ for all $i = 1, 2, \cdots, M-1$. Using Tanaka's formula

for the continuous semi-martingale $v_i(t)$, we get

$$
\mid v_i(t) \mid = \sum_{j=1}^{M-1} \alpha_{ij} \int_0^t \mathrm{sgn}(v_i(s)) v_i(s) ds
$$

$$
+ \sum_{j=1}^{M-1} \beta_{ij} \int_0^t \mathrm{sgn}(v_i(s))(y_j^{(1)}(s)^p - y_j^{(2)}(s)^p) ds
$$

$$
+ \sqrt{M} \int_0^t \mathrm{sgn}(v_i(s)) \Big[\sigma(y_i^{(1)}(s)) - \sigma(y_i^{(2)}(s)) \Big] dB_i(s)
$$

where $i = 1, 2, \cdots, M - 1$. Using the fact that α_{ij} and β_{ij} are bounded, summing over all $i = 1, 2, \cdots, M - 1$, and taking the expectation, we obtain

$$
\mathbb{E}\left(\sum_{i=1}^{M-1} \mid v_i(t) \mid \right) = \mathbb{E}\left(\sum_{i,j=1}^{M-1} \alpha_{ij} \int_0^t v_i(s) ds \right)
$$

$$
+ \mathbb{E}\left(\sum_{i,j=1}^{M-1} \beta_{ij} \int_0^t \mathrm{sgn}(v_i(s))(y_i^{(1)}(s)^p - y_i^{(2)}(s)^p) ds \right)
$$

$$
:= I_1 + I_2. \tag{3.20}
$$

For I_1, we have

$$
I_1 \leq \mathbb{E} \int_0^t \sum_{i=1}^{M-1} \mid v_i(s) \mid \left(\sum_{j=1}^{M-1} \mid \alpha_{ij} \mid \right)
$$

$$
\leq 4M^2 \mathbb{E} \int_0^t \sum_{i=1}^{M-1} \mid v_i(s) \mid ds. \tag{3.21}
$$

Since the values of solutions lie in interval $[0, 1]$, therefore, for I_2, we estimate

$$
I_2 \leq \mathbb{E} \int_0^t \sum_{i=1}^{M-1} \mid v_i(s) \mid \left(\sum_{j=1}^{M-1} \mid \beta_{ij} \mid \right)
$$

$$
\leq 2M \, \mathbb{E} \int_0^t \sum_{i=1}^{M-1} \mid v_i(s) \mid ds. \tag{3.22}
$$

Inserting (3.21) and (3.22) in to (3.20) and applying Gronwall's inequality, we have

$$
\mathbb{E}\left(\sum_{i=1}^{M-1} \mid v_i(t) \mid \right) = 0,
$$

i.e., the weak solutions are pathwise unique. Finally, by a standard theorem of Yamada and Watanabe [34](or see [10, pages 8-9]), the existence of a unique strong solution is obtained. □

4. Tightness of the Approximating Processes

In this section, we demonstrate the tightness of the family of the strong solutions for system of stochastic differential equations (3.5)–(3.7). Let us denote the polygon approximation of $y_i(t)$ by $f_M(t, y)$ which is defined as

$$f_M(t, y) := Y\left(t, \frac{[My] + 1}{M}\right)(My - [My]) + Y\left(t, \frac{[My]}{M}\right)([My] + 1 - My), \quad (4.1)$$

where $t \in [0, T]$, $y \in [0, 1]$ and $[y] = \frac{k}{M}$ for $\frac{i}{M} \le y < \frac{i+1}{M}$, so that we have $Y\left(t, \frac{i}{M}\right) = y_i(t) = f_M\left(t, \frac{i}{M}\right)$, for every $t \in [0, T]$ and $0 \le i \le M$.

Suppose $q_M\left(t, \frac{i}{M}, \frac{j}{M}\right)$, $t \in [0, T]$, $0 \le i, j \le M$ is the fundamental solution of the discrete heat equation such that

$$\frac{\partial}{\partial t} q_M\left(t, \frac{i}{M}, \frac{j}{M}\right) = \Delta_M q_M\left(t, \frac{i}{M}, \frac{j}{M}\right) \qquad t \ge 0, 1 \le i, j \le M - 1. \quad (4.2)$$

$$q_M\left(0, \frac{i}{M}, \frac{j}{M}\right) = M\delta_{ij}, \quad (4.3)$$

with boundary conditions

$$q_M\left(t, 0, \frac{j}{M}\right) = q_M\left(t, 1, \frac{j}{M}\right) = 0 \quad (4.4)$$

for all $t \in [0, T]$, $1 \le j \le M - 1$.

Then (3.5)–(3.7) can be re-written as

$$y_i(t) = \sum_{j=1}^{M-1} \frac{1}{M} q_M\left(t, \frac{i}{M}, \frac{j}{M}\right) y_i(0)$$
$$+ \int_0^t \sum_{j=1}^{M-1} \frac{1}{M} q_M\left(t - s, \frac{i}{M}, \frac{j}{M}\right) \beta_{ij}(y_i(s))^p$$
$$+ \int_0^t \sum_{j=1}^{M-1} \sqrt{M} q_M\left(t - s, \frac{i}{M}, \frac{j}{M}\right) \sigma(y_i(s)) dB_i(s), \quad 1 \le i \le M - 1,$$

where the last integral on the right hand side represents the sum of Itô stochastic integrals. Let us define the re-scaled formulation of the Green function G_M to the heat kernel q_M in $[0, 1]$ by

$$G_M\left(t, y, \frac{j}{M}\right) = q_M\left(t, \frac{[My] + 1}{M}, \frac{j}{M}\right)(My - [My])$$
$$+ q_M\left(t, \frac{[My]}{M}, \frac{j}{M}\right)([My] + 1 - My). \quad (4.5)$$

Therefore the linear interpolation of the $f_M(t, y)$, for $y \in \left[\frac{i}{M}, \frac{i+1}{M}\right)$ is

$$
\begin{aligned}
f_M(t, y) = & \sum_{j=1}^{M-1} \frac{1}{M} G_M\left(t, \frac{i}{M}, \frac{j}{M}\right) y_i(0) \\
& + \int_0^1 \sum_{j=1}^{M-1} \left[\frac{1}{M} q_M\left(t-s, \frac{i+1}{M}, \frac{j}{M}\right) \beta_{(i+1)j}(y_i(s))^p (My - [My])\right. \\
& \qquad\qquad \left. \frac{1}{M} q_M\left(t-s, \frac{i}{M}, \frac{j}{M}\right) \beta_{ij}(y_i(s))^p ([My] + 1 - My)\right] ds \\
& + \int_0^1 \sum_{j=1}^{M-1} \sqrt{M} G_M\left(t-s, \frac{i}{M}, \frac{j}{M}\right) \sigma(y_i(s)) dB_i(s), \quad 1 \le i \le M-1, \\
& := f_M^1(t, y) + f_M^2(t, y) + f_M^3(t, y), \hspace{3cm} (4.6)
\end{aligned}
$$

where $\{f_M^l\}$, for $l = 1, 2, 3$, denote the first, second and third summation on the right hand side respectively.

Proposition 4.1. *For every $M \ge 1$, the sequence $\{f_M(t, y) : t \in [0, T]\}$ is tight in the space $\mathcal{C}\big([0, T], A\big)$, where $A = \mathcal{C}\big([0, 1], [0, 1]\big)$.*

Proof. By the hypothesis (2.2), we have

$$
\sigma(f_M(t, y)) \le c \min((f_M(t, y))^\alpha, (1 - f_M(t, y))^\alpha), \hspace{2cm} \forall\, \alpha \in [1/2, 1). \tag{4.7}
$$

It can easily be seen from (4.1) that $f_M \in [0, 1]$ and consequently it implies through condition (4.7) that σ is also bounded by some positive constant. Further, by using the same technique as used in the proof of Lemma 2.2 and Proposition 2.1 in [16], for every $0 < T < \infty$ and $\mu \in \mathbb{N}$, we obtain that there exists $K := K(\mu, T)$ such that

$$
\mathbb{E}\,|\, f_M^3(t_1, x) - f_M^3(t_2, y)\,|^{2\mu} \le K\left(\,|\, t_1 - t_2\,|^{\mu/2} + |\, x - y\,|^{\mu/2}\,\right) \tag{4.8}
$$

for every $x, y \in [0, 1]$ and $t_1, t_2 \in [0, T]$, and $\mu \in \mathbb{N}$ and

$$
\lim_{M \to \infty} \sup_{(t,y) \in [0,T] \times [0,1]} |\, f_M^1(t, y) - f(t, y)\,| = 0. \tag{4.9}
$$

Here, f represents the fundamental solution of

$$
\frac{\partial f}{\partial y} = \frac{\partial^2 f}{\partial y^2}. \tag{4.10}
$$

Also, $f_M^2(t, 0) = f_M^2(t, 1) = 0$ and

$$
f_M^2\left(t, \frac{k}{M}\right) = \int_0^t \left[q_M\left(t-s, \frac{k}{M}, \frac{k+1}{M}\right) y_{k+1}(s)^p - q_M\left(t-s, \frac{k}{M}, \frac{k}{M}\right) y_k(s)^p\right] ds.
$$

Since (4.2)–(4.4) imply that q_M is the fundamental solution of heat kernel associated to Δ_M, we have

$$
\frac{\partial}{\partial t} f_M^2\left(t, \frac{k}{M}\right) = \Delta_M f_M^2\left(t, \frac{k}{M}\right) + f_M\left(t, \frac{k+1}{M}\right)^p - f_M\left(t, \frac{k}{M}\right)^p.
$$

From Theorem 4.2 in [22], we obtain

$$\max_{1 \le k \le M} \left| f_M^2 \left(t, \frac{k}{M} \right) \right| \le e^{pt} \int_0^t \max_{1 \le k \le M} \left| f_M^1 \left(s, \frac{k}{M} \right) + f_M^3 \left(s, \frac{k}{M} \right) \right| ds. \quad (4.11)$$

Hence, from (4.8)-(4.9), and the polygonal form of f_M^2, we conclude that for any finite $T > 0$ and $\mu \in \mathbb{N}$, there exists $K = K(T, \mu)$ in such a way that

$$\mathbb{E} |f_M^2(t_1, x) - f_M^2(t_2, y)|^{2\mu} \le K \left(|t_1 - t_2|^{\mu/2} + |x - y|^{\mu/2} \right) \quad (4.12)$$

for every $t_1, t_2 \in [0, T]$, and $0 \le x, y \le 1$, and $M \in \mathbb{N}$. Substituting estimates (4.8), (4.9) and (4.12) into (4.6) and using the multidimensional Totoki-Kolmogorov criterion [31, 32] on tightness we conclude that for every $T > 0$, $f_M(t, x) \in \mathcal{C}([0, T], A)^1$ and the sequence $\{f_M(t, x), M \in \mathbb{N}\}$ is tight.   □

5. The Weak Solution

In Section 3, it is shown that the sequence $f_M = \{f_M(t, y), M \ge 1\}$ is tight in $\mathcal{C}([0, T], A)$ and hence by Prokhorov's Theorem [3, page 59 (Chapter 1)], f_M is relatively compact in $\mathcal{C}([0, T], A)$, As a consequence there exist a convergent subsequence $f_M^k = \{f_M^k(t, y), M \ge 1, k \ge 1\}$ of f_M in $\mathcal{C}([0, T], A)$, which converges weakly to a stochastic process $\widetilde{f}$ in $\mathcal{C}([0, T], A)$. Applying the well known Skorohod's representation Theorem, we get another probability space $(\bar{\Omega}, \bar{\mathcal{F}}, (\bar{\mathcal{F}}_t), \bar{P})$ and a sequence of processes $\widetilde{f}_M(t, y)$ and $f(t, y)$ adapted to the filtration $\{\bar{\mathcal{F}}_t\}_{\{t \in [0, T]\}}$, in such a way that $f_M \overset{\mathrm{D}}{\equiv} \widetilde{f}_M(t, y)$, $\widetilde{f} \overset{\mathrm{D}}{\equiv} f$ and $\{\widetilde{f}_M(t, y)\}$ converges to f almost surely on compact subset of $\mathcal{C}([0, T], A)$ for any $T > 0$ as $M \to \infty$. Also f satisfies the given boundary conditions in (1.5). Next, by solving an equivalent martingale problem to (1.4)–(1.6), we show that $f(t, y)$ is the required weak solution to (1.4)–(1.6).

Proposition 5.1. *For every $\phi \in \mathcal{C}^2([0, 1])$ such that $\phi(1) = \phi(0) = 0$, we have*

$$\mathcal{M}_\phi(t) = \int_0^1 f(t, y)\phi(y)dy - \int_0^1 f(0, y)\phi(y)dy$$
$$- \int_0^t \int_0^1 f(s, y)\phi''(y)dyds + \int_0^t \int_0^1 f^p(s, y)\phi'(y)dyds \quad (5.1)$$

is a martingale with the quadratic variation

$$\langle \mathcal{M}_\phi \rangle_t = \int_0^t \int_0^1 \sigma^2(f(t, y))\phi^2(y)dyds. \quad (5.2)$$

Proof. Using the Skorohod representation theorem after multiplying both the side by $\frac{1}{M}\phi\left(\frac{k}{M}\right)$ in (3.1) and summing over all $k = 1, 2, \cdots, M - 1$, we get, for fixed

---

$^1\mathcal{C}([0, T] \times [0, 1], [0, 1])$ and $\mathcal{C}([0, T], A)$ both have equal topologies (see [16, page 145])

$M \geq 1$,

$$\mathcal{M}_\phi^M(t) = \sum_{k=1}^{M-1} f_M\left(t, \frac{k}{M}\right)\phi\left(\frac{k}{M}\right)\frac{1}{M} - \sum_{k=1}^{M} f_M\left(0, \frac{k}{M}\right)\phi\left(\frac{k}{M}\right)\frac{1}{M}$$

$$- \int_0^t \sum_{k=1}^{M-1} \Delta_M f_M\left(t, \frac{k}{M}\right)\phi\left(\frac{k}{M}\right)\frac{1}{M} - \int_0^t \sum_{k=1}^{M-1} \nabla_M f_M^p\left(t, \frac{k}{M}\right)\phi\left(\frac{k}{M}\right)\frac{1}{M}$$

$$\xrightarrow{D} \sum_{k=1}^{M-1} \widetilde{f}_M\left(t, \frac{k}{M}\right)\phi\left(\frac{k}{M}\right)\frac{1}{M} - \sum_{k=1}^{M-1} \widetilde{f}_M\left(0, \frac{k}{M}\right)\phi\left(\frac{k}{M}\right)\frac{1}{M}$$

$$- \int_0^t \sum_{k=1}^{M-1} \widetilde{f}_M\left(s, \frac{k}{M}\right)\Delta_M\phi\left(\frac{k}{M}\right)\frac{1}{M} - \int_0^t \sum_{k=1}^{M-1} \widetilde{f}_M^p\left(s, \frac{k}{M}\right)\nabla_M\phi\left(\frac{k}{M}\right)\frac{1}{M}$$

$$= \sum_{k=1}^{M-1} \phi\left(\frac{k}{M}\right)\frac{1}{M}\int_0^t \sqrt{M}\sigma\left(\widetilde{f}_M\left(s, \frac{k}{M}\right)\right)dB_k(s). \tag{5.3}$$

Since the right-hand side on (5.3), each integral in the summation, is an Itô integral and hence these are martingale also. Therefore, $\mathcal{M}_\phi^M(t)$ is also a martingale.

Moreover, ϕ^2 is also an integrable function, therefore

$$(\mathcal{M}_\phi^M(t))^2 = \left(\sum_{k=1}^{M-1} \phi\left(\frac{k}{M}\right)\frac{1}{M}\int_0^t \sqrt{M}\sigma\left(\widetilde{f}_M\left(s, \frac{k}{M}\right)\right)dB_k(s)\right)^2$$

$$\mathbb{E}(\mathcal{M}_\phi^M(t))^2 = \left(\sum_{k=1}^{M-1} \phi^2\left(\frac{k}{M}\right)\frac{1}{M}\int_0^t \sigma^2\left(\widetilde{f}_M\left(s, \frac{k}{M}\right)\right)\right)ds$$

$$\leq T\sum_{k-1} \frac{1}{M-1}\phi^2\left(\frac{k}{M}\right)$$

$$< c(\phi, T), \tag{5.4}$$

where $c(\phi, t)$ is a finite constant free from M and depends only on ϕ and T. Therefore, $\mathcal{M}_\phi^M(t) \to \mathcal{M}_\phi(t)$ as $M \to \infty$, where $\mathcal{M}_\phi(t)$ is given by (5.1). Now, since the quadratic variation of $\mathcal{M}_\phi^M(t)$ is given by

$$\langle \mathcal{M}_\phi^M(t)\rangle = \left\langle \sum_{k}^{M-1} \phi\left(\frac{k}{M}\right)\frac{1}{M}\int_0^t \sqrt{M}\sigma\left(\widetilde{f}_M\left(s, \frac{k}{M}\right)\right)dB_k(s)\right\rangle_t$$

$$= \int_0^t \left(\sum_{k}^{M-1} \frac{1}{M}\sigma^2\left(\widetilde{f}_M\left(s, \frac{k}{M}\right)\right)\phi^2\left(\frac{k}{M}\right)\right)ds. \tag{5.5}$$

therefore, we have

$$\lim_{M\to\infty}\langle \mathcal{M}_\phi^M(t)\rangle = \int_0^t \int_0^1 \sigma^2(f(s,y))\phi^2(y)dyds = \langle \mathcal{M}_\phi(t)\rangle. \tag{5.6}$$

$\square$

Now, the main result of the persent work, is as follows:

Theorem 5.2. *Let $f_0 : [0, 1] \to [0, 1]$ be a continuous function and σ satisfies the conditions (2.2)–(2.3). Then f(t,x) is a weak solution of (1.4)–(1.6).*

Proof. From Chapter 2 in Walsh [33], for the quadratic variation $\langle \mathcal{M}_\phi(t) \rangle$, we can find a martingale measure $\mathcal{M}(ds, dx)$ with quadratic variation

$$\nu(dx, dt) = \sigma(f(t, x)) dt ds.$$

Now, as in Kono and Siga [27], we can establish a space-time white noise $\widetilde{W}_t$ independent of $\mathcal{M}(dx, ds)$ such that

$$W_t(\phi) = \int_0^1 \int_0^t \frac{1}{\sigma(f(s, x))} 1_{\{f(s,x) \neq \{0,1\}\}} \phi(x) \mathcal{M}(ds, dx)$$

$$+ \int_0^1 \int_0^t 1_{\{f(s,x) = \{0,1\}\}} \phi(x) \widetilde{W}(ds, dx) \qquad (5.7)$$

where W_t corresponds to the space-time white noise $W(ds, dx)$ such that

$$\mathcal{M}_t(\phi) = \int_0^1 \int_0^t \sigma(f(s, x)) \phi(x) W(ds, dx) \qquad (5.8)$$

Therefore, from Proposition 5.1 and Definition 2.3, it is proved that f is the weak solution to (1.4)–(1.6). This complete the proof of Theorem 5.2. $\square$

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