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A Nonpreemptive Priority Multiserver Queueing System with Accessible and Non-Accessible Bulk Service and Heterogeneous Arrivals

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Abstract: In this paper a multiserver non-preemptive priority queueing system consisting of two types of customer units with Poisson arrival and exponential service time distribution is analysed. The lower priority units are served by a bulk service rule. This rule admits each batch served to have not less than 'a' and not more than 'b' customers such that the arriving customers can enter service station without affecting the service time if the size of the batch being served is less than 'd' ($a \le d \le b$). The higher priority units have non-preemptive priority over lower priority units. The higher priority units are served singly. The steady state probability vectors of the number of customers in the queue are obtained using the modified matrix geometric method. The stability condition and mean queue length of customers are obtained. Numerical results are also presented.

Keywords: Bulk Service Queues, Accessible and Non-accessible Batch Service, Matrix Geometric Method, Steady State Solution.

AMS (2000) Subject Classification number: 60k25 and 65k30

1. Introduction

The most common queue discipline that can be observed in everyday life is first in first out. Some of the other disciplines in common usage are last in first out; selection for service in random order independent of the time of arrival to the queue and a variety of priority schemes. There are two general situations in priority disciplines. In the first which is called preemptive priority, the customer with the highest priority is allowed to enter service immediately suspending even the service of a customer, if he is of lower priority, who may be already receiving service. In the second which is called nonpreemptive priority, the highest priority customer goes to the head of the queue but gets into service only after completion of the service to the customer who is presently in service even if this customer has a lower priority.

Heathcote (1959) has analysed a preemptive priority queuing model with two priorities. Sivasamy (1986) has discussed a model with single server facility in which two independent Poisson classes of customers type I and type II arrive and form separate queues. Type I customers have preemptive priority over type II customers. The type I customers are served by following bulk service rule and the type II customers are served one at a time. As soon as a type I customer arrives, the server breaks his service to a type II customer if service is going on and remains idle and waits for the queue size of type I customers to became 'a', no matter how many type II customers are present. Audsin Mohana Dhas *et al.*, 1992 have modified the above model in which the server stops servicing a type II customer only if the size of queue of type I customer reaches the minimum required for bulk service.

The queueing model analysed in this paper consists of a group of 'c'servers operating in parallel and two independent Poisson classes of customers type I (low priority) and type II (high priority) arrive and form separate queues. Type I customers can have a queue of infinite length and type II customers can wait in a queue of finite capacity N. The higher priority customers (type II) have nonpreemptive priority over lower priority customers (type I). The type I customers are served by a bulk service rule. This rule admits each batch served to have not less than 'a' and not more than 'b' customers such that the arriving customers can enter service station without affecting the service time, if the size of batch being served is less than 'd' ($a \leq d \leq b$).The higher priority customers are served singly.

2. Description of the Model

This model has multiserver facility in which two independent Poisson classes of customers with low priority and high priority arrive with different rates λ_1 and λ_2 and form types I and II customer queues respectively. The waiting room capacity of the type I customer is infinite and customers in type I unit are served by a bulk service rule. This bulk service rule is assumed to operate as follows: the server starts service only when minimum number of customers 'a' is present in the queue and maximum service capacity is 'b' such a rule for bulk service, first introduced by Neuts (1967), may be called a General Bulk Service Rule (GBSR).Here the GBSR is further assumed to allow the late entries to join a batch in course of ongoing service as long as the number of customers in that batch is less than 'd' (called maximum accessible limit). If the server is free and the queue length is 'a' or more, but less than or equal to 'd', the entire queue is taken for service and admits the subsequent arrivals in the batch while the service is going on till the service of current batch is over or till the accessible limit 'd' ($a \leq d \leq b$) is reached, whichever occurs first, such a batch is called an accessible batch. The waiting room capacity of the type II customer is finite with maximum capacity N and they are served singly. Customer in type II unit have nonpreemptive priority over customers in type I unit. That is every server after completion of each service looks for customer in type II units and serves if such customer is available. If no customer in type II unit is available, then the server serves the customers in type I unit according the above bulk service rule. Each server has an independently and identically distributed exponential service time distribution with parameter μ.

3. Steady State Probability Vector

The steady state process under consideration can be formulated as a continuous time Markov chain with state space

$$
\begin{aligned}
S = \; & \{(i, j, m);\ 0 \le i \le a-1,\ j = 0,\ 0 \le m \le c-1\} \\
& U\{(i, j, n, m);\ i = 0,\ j = 0,\ a \le n \le d-1,\ 0 \le m \le c-1\} \\
& U\{(i, j, m);\ i = 0,\ 1 \le j \le N,\ a \le n \le d-1,\ m = c-1\} \\
& U\{(i, j, m);\ i \ge 0,\ 0 \le j \le N,\ m = c\}
\end{aligned}
$$

where i denotes number of type I customers waiting in the queue and j denotes the number of type II customers waiting in the queue, n denotes number of customers in the accessible service batch (the number of customers in the service batch is less than 'd') and m denotes the number of busy servers with number in each service batch is greater than or equal to maximum accessible limit 'd'.

The infinitesimal generator **Q** of the continuous time Markov chain has the following block partitioned structure.

$$
Q = \begin{array}{c}
 \\ 00 \\ 0 \\ 1 \\ 2 \\ \cdot \\ \cdot \\ \cdot \\ a-1 \\ a \\ \cdot \\ \cdot \\ \cdot \\ b \\ b+1 \\ \cdot \\ \cdot \\ \cdot \\ a+b+1 \\ \cdot \\ \cdot \\ \cdot
\end{array}
\begin{array}{c}
\begin{array}{ccccccccccc}
00 & 0 & 1 & 2 & \cdots & a-1 & a & a+1 & \cdots & b & \cdots
\end{array} \\
\left[\begin{array}{ccccccccccc}
D & D_0 & D_1 & D_2 & & D_{a-1} & & & & & \\
C_0 & A_1 & A_0 & & & & & & & & \\
C_1 & & A_1 & A_0 & & & & & & & \\
C_2 & & & \cdots & \cdots & & & & & & \\
 & & & & & & & & & & \\
 & & & & & & & & & & \\
 & & & & & A_0 & & & & & \\
C_{a-1} & & & & & A_1 & A_0 & & & & \\
 & A_2 & & & & & A_1 & A_0 & & & \\
 & & & & & & & A_1 & & & \\
 & & & & & & & & & & \\
 & & & & & & & & & A_0 & \\
 & A_2 & & & & & & & & A_1 & \\
 & & A_2 & & & & & & & & \\
 & & & A_2 & \cdots & & & & & & \\
 & & & & & & & & & & \\
 & & & & & & & & & & \\
 & & & & & A_2 & & & & & \\
 & & & & & & A_2 & \cdots & & & \\
 & & & & & & & & & & \\
 & & & & & & & & & &
\end{array}\right]
\end{array}
$$

where $D, D_0, D_1, D_2, \ldots, D_{a-1}, C_0, C_1, C_2, \ldots, C_{a-1}, A_0, A_1$, and A_2 are matrices.

The square matrix D of order $(ac + (d - a)(c + N))$ is given by

$$
D = \begin{array}{c} \\ \bar{0} \\ \bar{1} \\ \bar{2} \\ \vdots \\ \overline{a-2} \\ \overline{a-1} \\ \bar{\bar{0}} \\ \bar{\bar{1}} \\ \bar{\bar{2}} \\ \vdots \\ \overline{\overline{c-2}} \\ \overline{\overline{c-1}} \\ \underline{1} \\ \underline{2} \\ \vdots \\ \underline{N} \end{array}
\begin{array}{c}
\begin{array}{cccccccccccccc} \bar{0} & \bar{1} & \bar{2} & \cdots & \overline{a-1} & \bar{\bar{0}} & \bar{\bar{1}} & \bar{\bar{2}} & \cdots & \overline{\overline{c-1}} & \underline{1} & \underline{2} & \cdots & \underline{N} \end{array} \\
\left[\begin{array}{cccccccccccccc}
d & L & & & & & & & & & & & & \\
 & d & L & & & & & & & & & & & \\
 & & d & L & & & & & & & & & & \\
 & & & \ddots & \ddots & & & & & & & & & \\
 & & & & \cdot & L & & & & & & & & \\
 & & & & d & L_1 & L_2 & L_3 & \cdots & L_c & & & & \\
B_0 & & & & & B_{00} & C & & & & & & & \\
B_1 & & & & & B_{10} & B_{11} & C & & & & & & \\
B_2 & & & & & & B_{21} & B_{22} & C & & & & & \\
\vdots & & & & & & & \ddots & \ddots & \ddots & & & & \\
B_{c-2} & & & & & & & & & & & & & \\
B_{c-1} & & & & & & & & B_{c-1c-2} & B_{c-1c-1} & C & & & \\
 & & & & & & & & & & B & & & \\
 & & & & & & & & & & b & B & & \\
 & & & & & & & & & & & \ddots & \ddots & \\
 & & & & & & & & & & & & \cdot & B \\
 & & & & & & & & & & & & b & B_{NN}
\end{array} \right]
\end{array}
$$

The square matrix d is of order c and is given by

$$
d = \begin{array}{c} \\ i00 \\ i01 \\ i02 \\ \vdots \\ i0c-1 \end{array}
\begin{array}{c}
\begin{array}{ccccccc} i00 & i01 & i02 & i03 & \cdots & i0c-2 & i0c-1 \end{array} \\
\left[\begin{array}{ccccccc}
-\lambda & \lambda_2 & & & & & \\
\mu & -\lambda-\mu & \lambda_2 & & & & \\
 & 2\mu & -\lambda-2\mu & \lambda_2 & & & \\
 & & & \ddots & & & \\
 & & & & & (c-1)\mu & -\lambda-(c-1)\mu
\end{array} \right]
\end{array}
$$

where $\lambda = \lambda_1 + \lambda_2$. In all matrices the unmarked entries are treated as zero.

The matrix L is given by $L = \lambda_1 \mathbf{I}$, where $\mathbf{I}$ is unit matrix of order $(ac + (d - a)(c + N))$

The matrix L_i $(0 \le i \le c)$, is matrix of order $(c \times (d - a))$ with λ_1 in $(i, 1)^{\text{th}}$ place and zero in the remaining places.

The matrix $B_i (0 \le i \le c - 2)$, is matrix of order $((d - a) \times c)$ with μ in the $(i + 1)^{\text{th}}$ column, λ_1 in the $(d - a, i + 2)^{\text{th}}$ place and zero in the remaining places.

The matrix B_{c-1}, is matrix of order $((d - a) \times c)$ with μ in the last column and zero in the remaining places.

The square matrix $B_{ii} (0 \le i \le c)$, is of order $(d - a)$ whose diagonal elements are $-(\lambda + (i + 1)\mu)$, the upper diagonal elements are λ_1 and zero in the remaining places.

The square matrix B_{ii-1} $(0 \le i \le c - 1)$, is of order $(d - a)$ whose diagonal elements are $i\mu$ and zero in the remaining places.

Let the matrix $B = B_{c-1\,c-1}$ and $b = B_{c-1\,c-2}$

The square matrix B_{NN} is of order $(d - a)$ whose diagonal elements are $-(\lambda_1 + c\mu)$, the upper diagonal elements are λ_1 and zero in the remaining places.

The matrix D_i, $(0 \le i \le a - 1)$ of order $((ac + (d - a)(c + N)) \times (N + 1))$

$$D_0 = \begin{array}{c} \bar{0} \\ \bar{1} \\ \cdot \\ \cdot \\ \cdot \\ \overline{a-1} \\ \bar{\bar{0}} \\ \bar{\bar{1}} \\ \cdot \\ \cdot \\ \overline{\overline{c-2}} \\ \overline{\overline{c-1}} \\ \cdot \\ \cdot \\ \underline{N} \end{array} \begin{bmatrix} H_0 \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ H_{c-1,0} \\ H_1 \\ \cdot \\ \cdot \\ H_N \end{bmatrix}$$

where H_0, is a matrix of order $(c \times (N + 1))$ with λ_2 in the $(c, 1)^{\text{th}}$ place and zero in the remaining places, $H_{c-1,0}$ is a matrix of order $(c \times (N + 1))$ with λ_1 in the $(c, 1)^{\text{th}}$

place and zero in the remaining places, and $H_i\,(1 \le i \le N)$, is a matrix of order $((d-a) \times (N+1))$ with μ in the i^{th} column, λ_1 in the $(d-a, i+1)$th place and zero in the remaining places.

$$D_i = \begin{array}{c} \overline{0} \\ \overline{1} \\ \cdot \\ \cdot \\ \cdot \\ \overline{i} \\ \cdot \\ \cdot \\ \cdot \\ \overline{a-1} \\ \overline{\overline{0}} \\ \overline{\overline{1}} \\ \cdot \\ \cdot \\ \cdot \\ \overline{\overline{c-1}} \\ \underline{1} \\ \underline{2} \\ \cdot \\ \cdot \\ \cdot \\ \underline{N} \end{array} \begin{bmatrix} \\ \\ \\ \\ \\ P_i \\ \\ \\ \\ \\ 0 \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \end{bmatrix}, \quad 1 \le i \le a-1$$

where P_i is a matrix of order $(c \times (N+1))$

$$P_i = \begin{array}{c} \\ i00 \\ i01 \\ i02 \\ \cdot \\ \cdot \\ \cdot \\ i0c-1 \end{array} \begin{array}{c} \begin{array}{ccccc} 0c & 1c & 2c & \ldots & Nc \end{array} \\ \begin{bmatrix} & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ \lambda_2 & & & & \end{bmatrix} \end{array}$$

The matrix $C_i\,(0 \le i \le a-1)$, of order $((N+1) \times (d-a)\,(c+N))$ is given by

$$
\begin{array}{cccccccccccccccccc}
 & \bar{0} & \bar{1} & . & . & . & \bar{i} & . & . & . & \overline{a-1} & \bar{\bar{0}} & . & . & . & \overline{\overline{c-1}} & \underline{1} & . & . & . & \underline{N}
\end{array}
$$

$$
C_i = \begin{bmatrix} & & & & Q_i & & & & \\ & & & & & & & & \\ & & & & & & & & \end{bmatrix}
$$

where Q_i is a matrix of order $(N+1) \times c$ given by

$$
Q_i = \begin{array}{c} \\ i00 \\ i01 \\ i02 \\ . \\ . \\ . \\ i0c-1 \end{array}
\begin{array}{c}
\begin{array}{ccccc} i00 & i01 & i02 & \cdots & i0c-1 \end{array} \\
\begin{bmatrix}
 & & & & c\mu \\
 & & & & \\
 & & & & \\
 & & & & \\
 & & & & \\
 & & & & \\
 & & & &
\end{bmatrix}
\end{array}
$$

The square matrix A_0 of order $(N+1)$ is given by $A_0 = \lambda_1 I$ where I is unit matrix. The square matrix A_1 is of order $(N+1)$ given by

$$
A_1 = \begin{bmatrix}
-\lambda - c\mu & \lambda_2 & & & & & \\
c\mu & -\lambda - c\mu & \lambda_2 & & & & \\
 & c\mu & . & . & & & \\
 & & . & . & . & & \\
 & & & . & . & . & \\
 & & & & . & -\lambda - c\mu & \lambda_2 \\
\lambda_2 & & & & & c\mu & -\lambda_1 - c\mu
\end{bmatrix}
$$

The square matrix $\mathbf{A_2}$ is of order $(N+1)$ with $c\mu$ in the $(1, 1)^{\text{th}}$ place and zero in the remaining places.

The steady state probability vector $\underline{\mathbf{X}}$, can be found by solving the system

$$
\underline{\mathbf{X}}\,\mathbf{Q} = \mathbf{0} \quad \text{and} \quad \underline{\mathbf{X}}\,\underline{\mathbf{e}} = \mathbf{1}
$$

where $\underline{\mathbf{e}}$ is a column vector of appropriate dimension with all the elements equal to *one*. The system of linear equations $\underline{\mathbf{X}}\,\mathbf{Q} = 0$ is solved using the matrix geometric method Neuts (1978), since the rate matrix $\mathbf{Q}$ has a special block tridiagonal structure. It is noted that $\mathbf{Q}$ is irreducible.

Let us partition $\underline{\mathbf{X}}$ as $(\underline{\mathbf{X}}_{00}, \underline{\mathbf{X}}_0, \underline{\mathbf{X}}_1, \underline{\mathbf{X}}_2, \ldots)$ where

$$\underline{\mathbf{X}}_{00} = [X_{000}, X_{001}, X_{002}, \ldots, X_{00c-1,} X_{100}, X_{101}, X_{102}, \ldots, X_{10C-1}, \ldots, X_{a-101}, X_{a-102}, \ldots, X_{a-10c-1}, \ldots, X_{00a0}, X_{00a+10}, \ldots, X_{00d-10}, \ldots, X_{00a1}, X_{00a+11}, \ldots, X_{00d-11}, \ldots, X_{00ac-1}, \ldots, X_{00a+1c-1}, \ldots, X_{00d-1c-1}, X_{01ac-1}, X_{01a+1c-1}, \ldots, X_{01d-1c-1}, X_{02ac-1}, \ldots, X_{02d-1c-1}, \ldots, X_{0Nac-1}, X_{0Na+1c-1}, \ldots, X_{0Nd-1c-1}]$$

and the vector $\underline{\mathbf{X}}_{00}$ is associated with states corresponding to

$$\{(i, j, m);\ 0 \le i \le a-1, j = 0, 0 \le m \le c-1\}$$

$$U\{(i, j, n, m);\ i = 0, j = 0, a \le n \le d-1, 0 \le m \le c-1\}$$

$$U\{(i, j, m);\ i = 0, 1 \le j \le N, a \le n \le d-1, m = c-1\}$$

and $$\underline{\mathbf{X}}_{\mathbf{i}} = [X_{i0c}, X_{i1c}, X_{i2c}, \ldots, X_{iNc}] \qquad i \ge 0.$$

Following Neuts (1978), we examine the existence of a solution of the form

$$\underline{\mathbf{X}}_{\mathrm{i}} = \underline{\mathbf{X}}_{\mathbf{a-1}} R^{i-\mathrm{a}+1} \quad \text{for} \quad i \ge a. \tag{1}$$

The steady state probability vector $\underline{\mathbf{X}}$ is called the matrix geometric probability vector. If a matrix geometric solution exists, the system $\underline{\mathbf{X}}\mathbf{Q} = \underline{0}$ gives

$$A_0 + RA_1 + R^{b+1} A_2 = \underline{0}. \tag{2}$$

The square matrix R is of order $(N+1)$ and is the unique minimal non-negative solution to the matrix non linear equation (2), with $R \ge 0$ Wallace (1969). It is an irreducible non-negative matrix of spectral radius less than *one*.

The matrix R is computed by successive substitutions in the recurrence relation

$$R(0) = 0$$

$$R(n+1) = -A_0 A_1^{-1} - R^{b+1}(n) A_2 A_1^{-1} \quad \text{for} \quad n \ge 0 \tag{3}$$

It may noted that $A_0 + A_1 + A_2$ is irreducible. For Markov process with such generators $\mathbf{Q}$, Neuts (1978) has obtained the following stability condition

$$\underline{\Pi}\, A_0\, \underline{\mathbf{e}} < (k-1)\, \underline{\Pi}\, A_k\, \underline{\mathbf{e}}$$

The corresponding stability condition in our case becomes

$$\underline{\Pi}\, \mathbf{A_0}\, \underline{\mathbf{e}} < \mathbf{b}\, \underline{\Pi}\, \mathbf{A_2}\, \underline{\mathbf{e}} \tag{4}$$

where the row vector $\underline{\Pi}$ is defined by $\underline{\Pi}\ \mathbf{A} = \underline{\mathbf{0}}$ and $\underline{\Pi}\ \underline{\mathbf{e}} = 1$.

In this case

$$\underline{\Pi} = [\Pi_{i0c}, \Pi_{i1c}, \Pi_{i2c}, \ldots, \Pi_{iNc}].$$

Since $\underline{\Pi}\, A_0 \underline{\mathrm{e}} = \lambda_1$ and

$$\underline{\Pi}\, A_2 \underline{\mathrm{e}} = c\mu\,(1-\rho)/(1-\rho^{N+1}) \quad \text{where} \quad \rho = \lambda_2 / c\mu$$

the stability condition in equation (4) takes the form

$$\lambda_1 < c\mu\,(1-\rho)/(1-\rho^{N+1})$$

To determine the vectors $\underline{\mathbf{X}}_{00}$, $\underline{\mathbf{X}}_{0}$, $\underline{\mathbf{X}}_{1}$, ..., $\underline{\mathbf{X}}_{a-1}$, we proceed as below:

Let $\mathbf{Q}^*$ be

$$Q^* = \begin{bmatrix} D & D_0 & D_1 & D_2 & . \; . \; . & D_{a-1} \\ C_0 & & A_0 & & & \\ C_1 & & A_1 & A_0 & & \\ \cdots & \cdots & \cdots & \cdots & & \cdots \\ C_{a-1} & (R+R^2+...+R^{b-a+1})A_2 & R^{b-a+2}A_2 & \cdots & & A_1+R^bA_2 \end{bmatrix}$$

To prove that $\mathbf{Q}^*\underline{\mathbf{e}} = 0$ it is enough to consider the last row of $\mathbf{Q}^*$ since the other rows are identical to that of $\mathbf{Q}$

(Last row of $\mathbf{Q}^*$)$\underline{\mathbf{e}}$

$$= C_{a-1}\,\underline{e} + (R + R^2 + \ldots + R^b)A_2\,\underline{e} + A_1\,\underline{e}$$

$$= A_2\,\underline{e} + (R+R^2+R^3+\ldots+R^b)A_2\,\underline{e} + A_1\,\underline{e} + \sum_{i=0}^{\infty} R^i(A_0 + RA_1 + R^{b+1}A_2)\,\underline{e}$$

(since $C_{a-1}\,\underline{e} = A_2\,\underline{e}$)

$$= (I-R)^{-1}\,(A_0 + A_1 + A_2)\,\underline{e}$$

$$= \underline{0}.$$

Therefore, $\mathbf{Q}^*$ is an infinitesimal generator and it is also irreducible. Let $\underline{\mathbf{X}}^* = (\underline{\mathbf{X}}_{00}, \underline{\mathbf{X}}_{0}, \underline{\mathbf{X}}_{1}, \underline{\mathbf{X}}_{2}, ..., \underline{\mathbf{X}}_{a-1})$ be a solution of $\underline{\mathbf{X}}^*\,\mathbf{Q}^* = \underline{\mathbf{0}}$. Then we get

$$\underline{\mathbf{X}}_{00}D + \underline{\mathbf{X}}_{0}C_0 + \underline{\mathbf{X}}_{1}\,C_1 + \ldots + \underline{\mathbf{X}}_{a-1}C_{a-1} = 0$$

$$\underline{\mathbf{X}}_{00}D_0 + \underline{\mathbf{X}}_{0}A_1 + \underline{\mathbf{X}}_{a-1}(R + R^2 + \ldots + R^{b-a+1}) = 0$$

$$\underline{\mathbf{X}}_{00}D_1 + \underline{\mathbf{X}}_{0}A_0 + \underline{\mathbf{X}}_{1}A_1 + \underline{\mathbf{X}}_{a-1}R^{b-a+2}A_2 = 0$$

$$\underline{\mathbf{X}}_{00}D + \underline{\mathbf{X}}_{a-1}A_0 + \underline{\mathbf{X}}_{a-1}C_{a-1}(A_1 + R^bA_2) = 0$$

The vectors $\underline{\mathbf{X}}_{i}$ $(0 \le i \le a-2)$ and $\underline{\mathbf{X}}_{00}$ can be calculated in terms of $\underline{\mathbf{X}}_{a-1}$ using the above set of equations and $\underline{\mathbf{X}}_{a-1}$ be normalised by

$$\sum_{i=0}^{a-2} \underline{X}_i\,\underline{e} + \underline{X}_{a-1}\,(I-R)^{-1}\,\underline{e} + \underline{X}_{00}\,\underline{e} = 1.$$

Thus, the vector $\underline{\mathbf{X}}_{00}$, $\underline{\mathbf{X}}_{0}$, $\underline{\mathbf{X}}_{1}$, $\underline{\mathbf{X}}_{2}$, ..., $\underline{\mathbf{X}}_{a-1}$ are uniquely determined.

4. Mean Queue Length

When the number of busy servers is less than c, the expected number of type I customers is

$$\sum_{i=0}^{a-1} i \left\{ \sum_{m=0}^{c-1} X_{iom} \right\}.$$

When all servers are busy, the expected number of type I customers is

$$\sum_{i=0}^{\infty} i \left[\sum_{j=0}^{N} X_{ijc} \right] = \sum_{i=0}^{a-1} i \left[\sum_{j=0}^{N} X_{ijc} \right] + \sum_{i=a}^{\infty} i \left[\sum_{j=0}^{N} X_{ijc} \right]$$

$$= \sum_{i=0}^{a-1} i\varphi_i + \sum_{i=a}^{\infty} i\varphi_i$$

where

$$\Phi_i = \underline{\mathbf{X}}_{\mathbf{i}}\underline{\mathbf{e}}, \qquad 0 \le i \le a-1$$

$$= \underline{\mathbf{X}}_{\mathbf{a-1}} R^{i-a+1}\underline{e}, \qquad i \ge a.$$

Mean queue length of type I customers is

$$L = \sum_{i=0}^{a-1} i \left[\sum_{m=0}^{c-1} X_{iom} \right] + \sum_{i=0}^{a-1} i\phi_i \, [a X_{a-1} + X_{a-1} R (I-R)^{-1}] R (I-R)^{-1} \underline{e}.$$

When all servers are busy the expected number of type II customers is

$$\sum_{j=0}^{N} j \left(\sum_{i=0}^{\infty} X_{ijc} + \sum_{n=a}^{d-1} X_{0\,jnc-1} \right).$$

5. Numerical Results

By Theorem 1 of Latouche and Neuts (1980) R is the limit of the sequence $\{R(n)\}$, $n \ge 0$ of matrices defined by the relation in equation (3).

For Numerical illustration, let us take $\lambda_1 = 6$, $\lambda_2 = 4$, $\mu = 2$, $c = 4$, $a = 3$, $d = 5$, $b = 6$, $N = 7$. Then $A_0 = 6I$, where I is unit matrix of order (8×8). The matrix R is given by $R = [C_1, C_2, ..., C_8]$ where

C_1 = [0.3833798563, 0.0958450173, 0.0239613609, 0.0059905535, 0.0014980649, 0.0003753693, 0.0001554856, 0.0000272994$]^T$

C_2 = [0.2075319155, 0,4268832164, 0,1067212793, 0.0266812696, 0.0066722172, 0.0016718539, 0.0004255624, 0.0001215893 $]^T$

$$C_3 = [\,0.1267097863, 0.2191784035, 0.4297965135, 0.1074529546, 0.0268708896, 0.0067330244, 0.0017138608, 0.0004896745\,]^T$$

$$C_4 = [\,0.0920596349, 0.1167687288, 0.2166998223, 0.4291902358, 0.1073281193, 0.0268931506, 0.0068455292, 0.0019558655\,]^T$$

$$C_5 = [\,0.0788538653, 0.0666037323, 0.1104314652, 0.2151689304, 0.4289143608, 0.1074728467, 0.0273567246, 0.0078162070\,]^T$$

$$C_6 = [\,0.0748855577, 0.0422199331, 0.0575520706, 0.1083821925, 0.2150838978, 0.4297476737, 0.1093903169, 0.0312543762\,]^T$$

$$C_7 = [0.0743815317, 0.0305582829, 0.0315653707, 0.0557429427, 0.1096389358, 0.2188161343, 0.4375168342, 0.1250048098\,]^T$$

$$C_8 = [0.0747323707, 0.0255190400, 0.0200516547, 0.0323567030, 0.0627767545, 0.1250693462, 0.2500176518, 0.0000504345\,]^T$$

$$\underline{X}_{00} = [0.0122783533, 0.0330692981, 0.0461131019, 0.0534458096, 0.0182320556, 0.0543252178, 0.0599258965, 0.0574971779, 0.0219785342, 0.0551965045, 0.0621231527, 0.0482304568, 0.0178889563, 0.0104335124, 0.0413981350, 0.0089342059, 0.0442097593, 0.0253033113, 0.0281707822, 0.0173096001, 0.0068087169, 0.0068891439, 0.0016456293, 0.0023189815, 0.0003977435, 0.0007185526, 0.0009614431, 0.0002119268, 0.0002327059, 0.0006060107, 0.0005715585, 0.0001724806, 0.0000163302, 0.0000562788]$$

$$\underline{X}_0 = [0.0534874892, 0.0170424788, 0.0054438220, 0.0017090500, 0.0005075192, 0.0001564856, 0.0004857036, 0.0001628919]$$

$$\underline{X}_1 = [0.0449470505, 0.0018836820, 0.0071274628, 0.0025355149, 0.0008593882, 0.0002852269, 0.0009470228, 0.0003403887]$$

$$\underline{X}_2 = [0.0222764538, 0.0146949368, 0.0077977652, 0.0048519061, 0.0051162703, 0.0084411144, 0.0162204520, 0.0322044331]$$

The remaining vectors $\underline{\mathbf{X}}_{\mathbf{i}}$, $i \geq 3$ are evaluated using the relation in equation (1). For the chosen parameters $\underline{\mathbf{X}}_{75} \to 0$ and the sum of the steady state probabilities is found to be 1.00000000 (correct to eight decimals).

References

[1] Audsin Mohana Dhas D., R. Nadarajan, and Ho Woo Lee, A Preemptive Priority Queue with General Bulk Service and Heterogeneous Arrivals, *Computers Indust,* **18** (1992), 327-333.

[2] Heathcote C. R., The Time Dependent Problem for a Queue with Preemptive Priorities, *Operations Research*, **7** (1959), 670-680.

[3] Latouche G., and M. F. Neuts, Efficient Algorithmic Solution to Exponential Tandem Queues with Blocking, *SIAM J. Algebr. Discr .Math.*, **1** (1980), 93-106.

[4] Neuts M. F., A General Class of Bulk Queues with Poisson Input, *Ann. Math.Statist.*, **38** (1967), 759-770.

[5] Neuts M. F., Markov Chains with Applications in Queueing Theory which have Matrix-Geometric Invariant Probability Vector, *Adv. Appl. Prob.*, **10** (1978), 185-212.

[6] Sivasamy R., A Preemptive Priority Queue with General Bulk Service Rule, *Bull. Austral. Math. Soc.*, **33** (1986), 237-243.

[7] Wallace V., The Solution of Quasi Birth and Death Process Arising from Multiple Access Computer System, Ph.D. Thesis, *System Engineering Laboratory*, University of Micihigan (1969).

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