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ON THE BEHAVIOR OF MEMBERS AND THEIR STOPPING TIMES IN COLLATZ SEQUENCES GENERATED BY VARIATIONS IN POWERS OF TWO

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Abstract: This paper proposes some results that are related to the Collatz sequences generated by some variations in consecutive powers of two. We then examine these series from the perspective of strings, and match patterns and generate formulas that further prove the former claims. This paper further attempts to prove that the Collatz conjecture holds true for values greater than those which it has been computationally tested.

Keywords: Thabit numbers, Collatz, Collatz Sequences, Patterns.

1. INTRODUCTION

The Collatz conjecture is named after Lothar Collatz, who first proposed it in 1937. The conjecture is also known as the $3n + 1$ conjecture, the Ulam conjecture (after Stanislaw Ulam), Kakutani's problem (after Shizuo Kakutani), the Thwaites conjecture (after Sir Bryan Thwaites), Hasse's algorithm (after Helmut Hasse), or the Syracuse problem.

Take any natural number n. If n is even, divide it by 2 to get $n/2$. If n is odd, multiply it by 3 and add 1 to obtain $3n + 1$. The Conjecture is that irrespective of the initial number, upon the application of this function, the result will always end up as 1. This property is known as oneness[1-3].

Consider a function:

$$\acute{\varepsilon}(n) = n/2 \quad \text{if} \quad n \equiv 0 \pmod 2,$$

$$\acute{\varepsilon}(n) = 3n + 1 \quad \text{if} \quad n \equiv 1 \pmod 2.$$

Define the iterate of $\acute{\varepsilon}$:

$$\acute{\varepsilon}^{(0)}(n) = n$$

$$\acute{\varepsilon}^{(i+1)}(n) = (\acute{\varepsilon}^{i}(n)).$$

The Problem is to prove that for all integral n, there exists a finite number k, which is defined as the stopping time of n, such that:

$$\acute{\varepsilon}^{(k)}(n) = 1.$$

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A thabit number is any number of the form $3.2^n - 1$.

Consider the function:

$$f(n) = 3.2^n - 1.$$

However this paper is only concerned with numbers of the form

$$3.2^n$$

i.e., numbers of the form $f(n) + 1$.

Collatz sequences are defined only for positive numbers. There is no closed-form formula to predict either the sequence or the stopping time of a number. There are also very few patterns that have been observed when it comes to Collatz sequences. Purely because the stopping times of numbers are not generated in a uniform manner. This paper does not attempt to prove the Collatz conjecture, but rather seeks to provide some insight into some interesting patterns related to thabit numbers and collatz sequences that have, to the best of the author's knowledge, not been observed before. It has also been written in the hope that it provides some hint for solving the problem.

In the following sections, we show some patterns in Collatz sequences generated by numbers of the form $(a.2^n)$-(3.2^m) and $(a.2^{(n+1)})$-(3.2^m) such that the value of $(a.2^n)$-(3.2^m) is at least the sixth smallest positive element of the set of values that this expression can take.

2. CODE AND LOGIC

2.1 Generating Collatz Sequences

Here we present the simple C++ code to return the stopping time of any integer n, along with its entire cycle.

```
k:
     a=0;
     c=0;
     cin >> a;

     step:
  if(n%2==0)
  {
            n = n/2;
            c++;
            if(n!=1){
            cout << n << " ";
```

```
            goto step;}
            else {cout << "1" << endl << c << endl;
            goto k;
            }
            }
    else {
        n = (3*n) + 1;
        c++;
        if(n!=1){
            cout << n << " ";
             goto step;}
             else { cout << " " << c << endl;
               goto k;
               }
      }
```

This gives us an idea of how the Collatz sequences are generated. Most results in this paper were derived using this program.

2.2 Generating Numbers of the Form (2^n.3)

```
for(k=1; k<k_n; k++)
t=((pow(2,k))*3 + 1)
```

The above code will return the variation in thabit numbers that we require (the addition of 1) to show the results mentioned earlier.

3. RESULTS AND OBSERVATIONS

Theorem 1: For any p and q of opposite parities, such that

$$(2^p) - (3.2^q)$$

is the 6th smallest positive element of the set of values that it can take

$$\acute{\varepsilon}((2^{p+1})\text{-}(3.2^q)) - \acute{\varepsilon}((2^p)\text{-}(3.2^q)) = 1.$$

Theorem 2: After the first $2p - q - 1$ terms of the Collatz sequence formed by $((2^p)\text{-}(3.2^q))$ and the first $2p - q$ terms of the Collatz sequence formed by $((2^{p+1})\text{-}(3.2^q))$, the cycles will meet and the terms that follow the sequence will be common.

Combined Proofs:

Let us begin with the first term in the cycles of these two cases:

We split the cycles into sub-cycles to assess and determine the relation between these sequences.

Sub-cycle 1:

For $((2^{p+1})-(3.2^q))$ and $((2^p)-(3.2^q))$ it is obvious that these integers are even. So, we divide by 2 to get:

$$((2^p)-(3.2^q)) \rightarrow ((2^{p-1})-(3.2^{q-1})) \rightarrow q \text{ iterations...} \rightarrow (2^{p-q}-3)$$

and

$$((2^{p+1})-(3.2^q)) \rightarrow ((2^p)-(3.2^{q-1})) \rightarrow q \text{ iterations...} \rightarrow (2^{p-q+1}-3).$$

Observations:

1. And so, the values of q and p will continue to decrease by 1 until q reaches the value of zero, as before that value is reached, the value of both expressions will be even.
2. At this point, we can clearly see that after q iterations, both the terms acquire values that are odd.

Sub-cycle 2:

$$(2^{p-q}-3) \rightarrow (3.2^{p-q}-2^3) \rightarrow (3.2^{p-q-1}-2^2) \rightarrow (3.2^{p-q-2}-2) \rightarrow (3.2^{p-q-3}-1)$$

and

$$(2^{p-q+1}-3) \rightarrow (3.2^{p-q+1}-2^3) \rightarrow (3.2^{p-q}-2^2) \rightarrow (3.2^{p-q-1}-2) \rightarrow (3.2^{p-q-2}-1).$$

Observations:

1. At this point, seeing as the cycle length is already at a minimum of 6 (taking $q = 0$), we observe that the theorem we present will **hold true only for values of $(2^p)-(3.2^q)$ that are greater than or equal to the 6th positive element in the set of values that this expression can take.**
2. The difference between the n^{th} term for the sequence $(2^{p-q}-3)$ and $n+1^{th}$ $(2^{p-q+1}-3)$ decreases from 2^2 for $n = 2$, to 2^1 for $n = 3$, to 2^0 for $n = 4$.

Sub-cycle 3:

$$(3.2^{p-q-3}-1) \rightarrow (3^2.2^{p-q-3}-2) \rightarrow (3^2.2^{p-q-4}-1) \rightarrow (3^3.2^{p-q-4}-2) \rightarrow (3^3.2^{p-q-5}-1)$$
$$\rightarrow 2.(p-q-3) \text{ iterations...} \rightarrow 3^{p-q-2}-1$$

and

$$(3.2^{p-q-2}-1) \rightarrow (3^2.2^{p-q-2}-2) \rightarrow (3^2.2^{p-q-3}-1) \rightarrow (3^3.2^{p-q-3}-2) \rightarrow (3^3.2^{p-q-4}-1)$$
$$\rightarrow 2.(p-q-2) \text{ iterations...} \rightarrow 3^{p-q-1}-1$$

Observations:

1. As we can observe, in this series the functions $\acute{\varepsilon}(n) = n/2$ and $\acute{\varepsilon}(n) = 3n + 1$ are applied alternatively, as the parity changes continually when the first term is of the form $(k.2^{\mathrm{l}}\text{-}1)$ for any k, $i \in N$. As a result, the exponent of 3 increments continually, that is, after every alternate term; and the exponent of 2 decrements every alternate term as well, eventually reaching the value of 0.
2. The difference between the $(2n + 1)^{\text{th}}$ term of the $(3.2^{p-q-3}\text{-}1)$ sequence and the $(2n + 2)^{\text{th}}$ term of the $(3.2^{p-q-2}\text{-}1)$ sequence is consistently 1. We observe that in these cases, the exponents of 3 and 2 are equal, and the only difference is in the constant term, which alternated between 1 and 2, leading to the unit difference.

Sub-cycle 4:

Let us define $(p - q - 2)$ as g.

So, as we know, p and q are of opposite parities. So $p - q - 2$ will always be odd.

Therefore, $g + 1$ will be even.

The value of $(3^g - 1)/2$ is always odd for an odd g.

$$3^g - 1 \rightarrow (3^g - 1)/2 \rightarrow (3^{g+1}\text{-}1)/2$$

and

$$3^{g+1}\text{-}1 \rightarrow (3^{g+1}\text{-}1)/2.$$

Conclusion:

After this point, the remaining terms in the series formed by both these expressions will be same, and the number of terms will ergo be equal. In the final sub-cycle, the $\acute{\varepsilon}(n)$ function has to be applied one more time for $3^g - 1$ so as to equate the values.

At this time, the sub-cycle for each of the terms has reached the same number. Let us define the stopping time of this number, $((3^{g+1}\text{-}1)/2)$ as k. Now, the stopping times of the full cycles of these expressions will be equal to the sum of the stopping times of each sub-cycle and k.

So we can clearly see that the stopping times for the expressions up to the $((3^{g+1}\text{-}1)/2)^{\text{th}}$ terms are (We subtract 1 to prevent double counting in sub-cycle 4):

$$((2^p)\text{-}(3.2^q)):\ q + 4 + 2\,(p - q - 3) + 2 - 1 = 2\text{p} - q - 1$$

$$((2^{p+1})\text{-}(3.2^q)):\ q + 4 + 2\,(p - q - 2) + 1 - 1 = 2p - q.$$

Theorem 2 is Hence Proved.

Let us assess the stopping times of these sequences sub-cycle wise:

$$\acute{\varepsilon}((2^{p})\text{-}(3.2^{q})) = q + 4 + 2(p - q - 3) + 2 - 1 + k = 2p - q + k - 1$$

$$\acute{\varepsilon}((2^{p+1})\text{-}(3.2^{q})) = q + 4 + 2(p - q - 2) + 1 - 1 + k = 2p - q + k.$$

Therefore:

$$\acute{\varepsilon}((2^{p+1})\text{-}(3.2^{q})) - ((2^{p})\text{-}(3.2^{q})) = 1.$$

Theorem 1 is hence proved.

Extension of Theorem 1and 2: For all odd a, and any p and q of opposite parities, such that

$$(a)(2^{p}) - (3.2^{q})$$

is the 6th smallest positive element of the set of values that it can take

$$\acute{\varepsilon}((a)(2^{p+1})\text{-}(3.2^{q})) - \acute{\varepsilon}((a)(2^{p})\text{-}(3.2^{q})) = 1.$$

After the first $2p - q - 1$ terms of the Collatz sequence formed by $(a(2^{p})\text{-}(3.2^{q}))$ and the first $2p - q$ terms of the Collatz sequence formed by $(a(2^{p+1})\text{-}(3.2^{q}))$, the cycles will meet and the terms that follow the sequence will be common.

Proof: The observations from Theorems 1 and 2 still hold for this proof. All one needs to do is repeat the procedures for Theorem 1 but with a as the coefficient of the term 2^{p} and 2^{p+1}.

Sub-cycle 1:

For $(a(2^{p+1})\text{-}(3.2^{q}))$ and $(a.(2^{p})\text{-}(3.2^{q}))$ it is obvious that these integers are even. So, we divide by 2 to get:

$$(a(2^{p})\text{-}(3.2^{q})) \to (\mathrm{a}(2^{p-1})\text{-}(3.2^{q-1})) \to q \text{ iterations...} \to (a.2^{p-q}\text{-}3)$$

and

$$(a(2^{p+1})\text{-}(3.2^{q})) \to (a(2^{p})\text{-}(3.2^{q-1})) \to q \text{ iterations...} \to (a.2^{p-q+1}\text{-}3)$$

Sub-cycle 2:

$$(a.2^{p-q}\text{-}3) \to (a.3.2^{p-q}\text{-}2^{3}) \to (a.3.2^{p-q-1}\text{-}2^{2}) \to (a.3.2^{p-q-2}\text{-}2) \to (a.3.2^{p-q-3}\text{-}1)$$

and

$$(a.2^{p-q+1}\text{-}3) \to (a.3.2^{p-q+1}\text{-}2^{3}) \to (a.3.2^{p-q}\text{-}2^{2}) \to (a.3.2^{p-q-1}\text{-}2) \to (a.3.2^{p-q-2}\text{-}1)$$

Sub-cycle 3:

$$(a.3.2^{p-q-3}\text{-}1) \to (a.3^{2}.2^{p-q-3}\text{-}2) \to (a.3^{2}.2^{p-q-4}\text{-}1) \to (a.3^{3}.2^{p-q-4}\text{-}2)$$
$$\to (a.3^{3}.2^{p-q-5}\text{-}1) \to 2.(p - q - 3) \text{ iterations...} \to a.3^{p-q-2}\text{-}1$$

and

$$(a.3.2^{p-q-2}\text{-}1) \rightarrow (a.3^2.2^{p-q-2}\text{-}2) \rightarrow (a.3^2.2^{p-q-3}\text{-}1) \rightarrow (a.3^3.2^{p-q-3}\text{-}2)$$
$$\rightarrow (a.3^3.2^{p-q-4}\text{-}1) \rightarrow 2.(p-q-2) \text{ iterations...} \rightarrow a.3^{p-q-1}\text{-}1.$$

Sub-cycle 4:

Now, at this point, we know that a id odd. It is clear that both values are even.

$$a.3^{p-q-2}\text{-}1 \rightarrow (a.3^{p-q-2}\text{-}1)2 \rightarrow (a.3^{p-q-1}\text{-}1)/2$$

and

$$a.3^{p-q-1}\text{-}1 \rightarrow (a.3^{p-q-1}\text{-}1)/2.$$

After this point, the remaining terms in the series formed by both these expressions will be same, and the number of terms will ergo be equal. In the final sub-cycle, the $\acute{\varepsilon}(n)$ function has to be applied one more time for $3^{p-q-2} - 1$ so as to equate the values.

At this time, the sub-cycle for each of the terms has reached the same number. Let us define the stopping time of this number, $((3^{p-q-1}\text{-}1)/2)$ as k. Now, the stopping times of the full cycles of these expressions will be equal to the sum of the stopping times of each sub-cycle and k. (We subtract 1 to prevent double counting in sub-cycle 4):

$$(a\,(2^p)\text{-}(3.2^q)): q + 4 + 2\,(p - q - 3) + 2 - 1 = 2p - \text{q} - 1$$
$$(a\,(2^{p+1})\text{-}(3.2^q)): q + 4 + 2\,(p - q - 2) + 1 - 1 = 2p - q.$$

Extension of Theorem 2 is Hence Proved.

Let us assess the stopping times of these sequences sub-cycle wise:

$$(a\,(2^p)\text{-}(3.2^q)) = q + 4 + 2\,(p - q - 3) + 2 - 1k = 2p - q + k - 1$$
$$(a\,(2^{p+1})\text{-}(3.2^q)) = q + 4 + 2\,(p - q - 2) + 1 - 1 + k = 2p - q + k.$$

Therefore:

$$(a\,(2^{p+1})\text{-}(3.2^q)) - (a\,(2^p)\text{-}(3.2^q)) = 1.$$

Extension of Theorem 1 is hence proved.

4. EXAMPLES AND ILLUSTRATIONS

Example 1: $p = 7$; $q = 2$

$$((2^p)\text{-}(3.2^q)) = 2^7\text{-}3.2^2$$
$$((2^p)\text{-}(3.2^q)) = 2^8\text{-}3.2$$

Expression	Sequence	Stopping time	Terms upto common sequence
116	**58 29 88 44 22 11 34 17 52 26 13** 40 20 10 5 16 8 4 2 1	20	11
244	**122 61 184 92 46 23 70 35 106 53 160 80** 40 20 10 5 16 8 4 2 1	21	12

Example 2: $a = 5$; $p = 6$; $q = 1$

$$((5)(2^p)\text{-}(3.2^q)) = 5.2^6\text{-}3.2^1$$

$$((5)(2^p)\text{-}(3.2^q)) = 5.2^7\text{-}3.2^1$$

Expression	Sequence	Stopping time	Terms before common sequence
314	**157 472 236 118 59 178 89 268 134 67** 202 101 304 152 76 38 19 58 29 88 44 22 11 34 17 52 26 13 40 20 10 5 16 8 4 2 1	37	10
634	**317 952 476 238 119 358 179 538 269 808 404** 202 101 304 152 76 38 19 58 29 88 44 22 11 34 17 52 26 13 40 20 10 5 16 8 4 2 1	38	11

5. CONCLUSION

Generally, patterns in the Collatz series are very rare. In this paper we have not only found some interesting results on the sequences of a large set of numbers as per theorems 1 and 2 and their extensions, but we have also found patterns within the sequences generated. With the expansion of Theorem 1 in section 3, we have also shown that the Collatz Conjecture holds true for values greater than it has been computationally tested for. Analyzing the stopping time and the properties of these sequences will have provided some insight into this beautiful conjecture.

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