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DECOMPOSITION OF θ-CONTINUOUS FUNCTIONS IN TOPOLOGICAL SPACES

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Abstract

In the present paper, we introduce and study the $\theta\omega\alpha$ *-continuous and* $\theta\omega\alpha$ *-irresolute functions. The class of* $\theta\omega\alpha$ *-continuous functions properly placed between the classes of* θ *-continuous and* θg *-continuous functions in topological spaces. Moreover, we have introduced and obtained the characterizations of the* $\theta\omega\alpha$ *-closed,* $\theta\omega\alpha$ *-open functions and* $\theta\omega\alpha$ *-homeomorhisms. Also we observed the relation of these functions with the existing functions between topological spaces.*

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Keywords and Phrases: $\theta\omega\alpha$ *-open sets,* θ *-continuous functions,* θg *-continuous functions,* $\theta\omega\alpha$ *-continuous functions,* $\theta\omega\alpha$ *-irresolute functions,* $\theta\omega\alpha$ *-closed functions,* $\theta\omega\alpha$ *-open functions and* $\theta\omega\alpha$ *-homeomorphisms.*

1. INTRODUCTION

The concept of continuity is very important in general topology. Fomin [7] introduced the notions of θ-continuous functions in 1943. These functions acts as natural tools for studying the almost compact spaces of Alexandroff and Urysohn. Later, these functions were investigated by Iliadis [8], Iliadis and Fomin [9] and Arockiarani et.al. [1].

The main aim of this paper is to introduce and study the properties of new spaces such as $T_{\theta\omega\alpha}$ -spaces, ${}_{\theta\omega\alpha}T$ -spaces, ${}_{\theta g}\mathrm{T}_{\theta\omega\alpha}$ -spaces and ${}_{g}\mathrm{T}_{\theta\omega\alpha}$ -spaces, the

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generalisation of θ-continuous functions called $\theta\omega\alpha$-continuous functions in topological spaces. Further, the concepts of $\theta\omega\alpha$-irresolute, $\theta\omega\alpha$-closed, $\theta\omega\alpha$-open functions and $\theta\omega\alpha$-homeomorphisms are studied and also obtained some of their characterizations.

2. PRELIMINARY

Throughout this paper the spaces (X, τ), (Y, η) and (Z, γ) (or simply X, Y and Z) always represents the topological space on which no separation axioms are assumed unless otherwise mentioned. A set A$\subseteq$X is said to be $\theta\omega\alpha$-closed [12] if $Cl_\theta(A) \subseteq$ U whenever A$\subseteq$U and U is $\omega\alpha$-open in X.

Definition 2.1: A function f: X$\rightarrow$Y is called a (i) θg-continuous [6] (respectively θ-continuous [1], g-continuous [2], α-continuous [11], $\omega\alpha$-continuous [4]) if f^{-1}(V) is θg-closed (respectively θ-closed, g-closed, α-closed, $\omega\alpha$-closed) in X for every closed set V in Y.

Remark 2.2: 1. [3] Every open set is $\omega\alpha$-open.

2. [6] Every θ-closed set is closed.

3. $\theta\omega\alpha$-SPACES IN TOPOLOGICAL SPACES

Definition 3.1: A topological space X is said to be a $T_{\theta\omega\alpha}$-space (respectively ${}_{\theta\omega\alpha}T$-space) if every $\theta\omega\alpha$-closed set is closed (respectively $\omega\alpha$-closed)in X.

Example 3.2: Let X $=\{a, b, c\}$ with $\tau =\{\phi, \{a\}, \{a, b\}, X\}$. Then the space (X, τ) is $T_{\theta\omega\alpha}$-space and ${}_{\theta\omega\alpha}T$-space.

Definition 3.3: If every θg-closed set (respectively g-closed) is $\theta\omega\alpha$-closed (respectively $\omega\alpha$-closed) in a topological space X, then X is called ${}_{\theta g}T_{\theta\omega\alpha}$-space (respectively ${}_{g}T_{\theta\omega\alpha}$-space).

Example 3.4: Let X $=\{a, b, c\}$ and $\tau =\{\phi, \{a, b\}, X\}$. Then the space (X, τ) is ${}_{\theta g}T_{\theta\omega\alpha}$-space and ${}_{g}T_{\theta\omega\alpha}$-space.

Remark 3.5: From Definitions 3.1 and 3.3, we observed that

$$T_{\frac{1}{2}}\text{-space} \Rightarrow T_{\theta\omega\alpha}\text{-space} \Rightarrow {}_{\theta\omega\alpha}T\text{-space.}$$

Reverse implications are need not be true which can be seen in the following examples.

Example 3.6: A topological space X = $\{a, b, c\}$ with $\tau = \{\phi, \{a\}, X\}$ is $T_{\theta\omega\alpha}$-space but not $T_{\frac{1}{2}}$ -space.

Example 3.7: Let X =$\{a, b, c\}$ with $\tau = \{\phi, \{a, b\}, X\}$ is $_{\theta\omega\alpha}T$ -space but not $T_{\theta\omega\alpha}$ -space.

Example 3.8: A topological space X =$\{a, b, c\}$ with $\tau = \{\phi, \{a\}, \{a, b\}, \{a, c\}, X\}$ is $_{\theta g}T_{\theta\omega\alpha}$ -space but not $_{g}T_{\theta\omega\alpha}$ -space.

Remark 3.9: $T_{\theta\omega\alpha}$ -spaces are independent from $_{g}T_{\theta\omega\alpha}$ -Spaces and $_{\theta g}T_{\theta\omega\alpha}$ -Spaces. We can see in the following examples.

Example 3.10: A topological space X =$\{a, b, c\}$ with $\tau = \{\phi, \{a\}, X\}$ is $T_{\theta\omega\alpha}$ -space but not $_{g}T_{\theta\omega\alpha}$ -space and $_{\theta g}T_{\theta\omega\alpha}$ -space.

Example 3.11: A topological space X =$\{a, b, c\}$ with $\tau = \{\phi, \{a, b\}, X\}$ is $_{g}T_{\theta\omega\alpha}$ -space and $_{\theta g}T_{\theta\omega\alpha}$ -spaces but not $T_{\theta\omega\alpha}$ -space.

Remark 3.12 $_{\theta\omega\alpha}T$ -spaces are independent from $_{g}T_{\theta\omega\alpha}$ -Spaces and $_{\theta g}T_{\theta\omega\alpha}$ -Spaces. We can see in the following examples.

Example 3.13: A topological space X =$\{a, b, c\}$ with $\tau = \{\phi, \{a\}, \{a, b\}, X\}$ is $T_{\theta\omega\alpha}$ -space but not $_{g}T_{\theta\omega\alpha}$ -space and $_{\theta g}T_{\theta\omega\alpha}$ -space.

Example 3.14: A topological space X =$\{a, b, c\}$ with $\tau = \{\phi, \{a\}, \{b, c\}, X\}$ is $_{g}T_{\theta\omega\alpha}$ -space and $_{\theta g}T_{\theta\omega\alpha}$ -spaces but not $_{\theta\omega\alpha}T$ -space .

Theorem 3.15: Let (X, τ) be a topological space then the following properties are equivalent:

(i) X is a $T_{\theta\omega\alpha}$ -space

(ii) Every singleton set of X is either open or $\omega\alpha$ -closed [3].

Proof : (i)$\Rightarrow$(ii) : Let $y \in$ X. Suppose $\{y\}$ is not a $\omega\alpha$ -closed then X-$\{y\}$ is not an $\omega\alpha$ -open in X. But X is only an $\omega\alpha$ -open set containing X-$\{y\}$. Therefore Cl_θ(X-$\{y\}$)$\subseteq$X. Thus X -$\{y\}$ is $\theta\omega\alpha$ -closed. By hypothesis X is $T_{\theta\omega\alpha}$ -space then X-$\{y\}$ is closed it implies that $\{y\}$ is open in X.

(ii) $\Rightarrow$ (i) : Let A be a $\theta\omega\alpha$ -closed set in X and $x \in Cl_\theta$(A) then we have following two cases:

Case (a) : Let $\{x\}$ be an open set in X. Since x$\in Cl_\theta$(A) then $\{x\}\cap$A$\neq\phi$ which implies that $x\in$ A.

Case (b) : Let $\{x\}$ be $\omega\alpha$-closed. Suppose $x\notin$A then we have $x\in Cl_\theta$(A)-A which is not possible according to Theorem 3.20 [12].Therefore x$\in$A. Therefore in both cases we have Cl_θ(A)$\subseteq$A. But A$\subseteq Cl_\theta$(A) is always true. Therefore A = Cl_θ(A) which implies that A is closed. Hence X is $T_{\theta\omega\alpha}$-space.

4. $\theta\omega\alpha$ -CONTINUOUS FUNCTIONS IN TOPOLOGICAL SPACES

Definition 4.1: A function $f:X\to Y$ is called $\theta\omega\alpha$-continuous function if for each closed set V of Y, f^{-1}(V) is $\theta\omega\alpha$-closed in X.

Theorem 4.2: Every θ-continuous function is $\theta\omega\alpha$-continuous but not conversely.

Proof : Let $f:X\to Y$ be a θ-continuous function and A be a closed set in Y then $f^{-1}(A)$ is θ-closed in X. From Theorem 2.2 [12], $f^{-1}(A)$ is $\theta\omega\alpha$-closed. Hence f is a $\theta\omega\alpha$-continuous function.

Example 4.3: Let X = Y =$\{a, b, c\}$ with $\tau=\{\phi, \{a\}, \{b\}, \{a, b\}, X\}$ and $\eta=\{\phi, \{a, b\}, Y\}$.Then the identity function f: (X, τ)$\to$(Y, η) is $\theta\omega\alpha$-continuous but not θ-continuous, since for a closed set $\{c\}$ in Y, $f^{-1}(\{c\})$ = $\{c\}\notin\theta$C(X, τ).

Theorem 4.4: Every $\theta\omega\alpha$-continuous is θg-continuous function and hence g-continuous but not conversly.

Proof: Let A be closed in Y and $f:X\to Y$ be a $\theta\omega\alpha$-continuous then $f^{-1}(A)$ is $\theta\omega\alpha$-closed in X. From Theorem 2.4 and 2.6 [12] implies that $f^{-1}(A)$ is θg-closed and g-closed. Therefore f is θg-continuous and g-continuous.

Example 4.5: Let X = Y =$\{a, b, c\}$ with $\tau=\{\phi, \{a\}, \{a, b\}, X\}$ and $\eta=\{\phi,\{a\}, Y\}$. Let f: (X, τ) $\to$ (Y, η) be a function defined by $f(a)=b$, $f(b)=a$ and $f(c)=c$. Clearly, $f^{-1}(\{b, c\})=\{a, c\}\notin\theta\omega\alpha C$ (X, τ). Hence f is both θg-continuous and g-continuous but not $\theta\omega\alpha$-continuous.

Remark 4.6: Converse of the Theorem 4.4 holds if (X, τ) is ${}_{\theta g}T_{\theta\omega\alpha}$-space.

Theorem 4.7: Every $\theta\omega\alpha$ -continuous function is $g\omega\alpha$ -continuous function but not conversly.

Proof : The proof follows from the Definition 4.1 and the Remark 2.9 [12].

Example 4.8: Let X = Y = $\{a, b, c\}$ with $\tau = \{\phi, \{a\}, \{a, b\}, \{a, c\}, X\}$, $\eta = \{\phi, \{a, b\}, Y\}$. Let f: (X, τ) → (Y, η) be a function defined by $f(a) = a$, $f(b) = c$ and $f(c) = b$. Clearly, f is $g\omega\alpha$ -continuous but $f^{-1}(\{c\}) = \{b\}$ $\notin \theta\omega\alpha$ C(X). Hence f is not $\theta\omega\alpha$ -continuous.

Theorem 4.9: A function $f : X \to Y$ is $\theta\omega\alpha$ -continuous if and only if for every open set U of Y, $f^{-1}(U)$ is $\theta\omega\alpha$ -open in X.

Proof : The proof is obvious.

Theorem 4.10: $f : X \to Y$ and $g : Y \to Z$ are θωα -continuous functions then their composition need not be a $\theta\omega\alpha$ -continuous as seen from the following example.

Example 4.11: Let X = Y = Z = $\{a, b, c\}$ with $\tau = \{\phi, \{a\}, \{b\}, \{a, b\}, \{a, c\}, X\}$, $\eta = \{\phi, \{a, b\}, Y\}$ and $\gamma = \{\phi, \{a, b\}, Z\}$. Define a function f : (X, τ) → (Y, η) by $f(a) = b$, $f(b) = c$ and $f(c) = a$ and g : (Y, η) → (Z, γ) is an identity function. But (gof): (X, τ) → (Z, γ) is not $\theta\omega\alpha$ -continuous, since for the closed set $\{b, c\}$ in Z, $(gof)^{-1}(\{b, c\})$ = $f^{-1}(g^{-1}(\{b, c\}))$ = $f^{-1}(\{b, c\}) = \{a, b\}$ is not $\theta\omega\alpha$ -closed in X.

Remark 4.12: Let $f : X \to Y$ and $g : Y \to Z$ be $\theta\omega\alpha$ -continuous functions then gof : X → Z is $\theta\omega\alpha$ -continuous if Y is $T_{\theta\omega\alpha}$ -space.

Theorem 4.13: Composition of $\theta\omega\alpha$ -continuous and θ -continuous function is a $\theta\omega\alpha$ -continuous function.

Proof : Let $f : X \to Y$ be $\theta\omega\alpha$ -continuous and $g : Y \to Z$ be θ -continuous. Let A ⊆ Z be closed, then g^{-1}(A) is θ -closed in Y as g is θ -continuous function. But from Remark (ii) in 2.2, g^{-1}(A) is closed in Y. Since f is $\theta\omega\alpha$ -continuous, $f^{-1}(g^{-1}(A)) = (gof)^{-1}(A)$ is $\theta\omega\alpha$ -closed in X. Then gof is $\theta\omega\alpha$ -continuous.

Theorem 4.14: Let f : X→Y be a function then the following statements are equivalent:

(i) f is $\theta\omega\alpha$ -continuous

(ii) for each point $x \in$ X and each open set V in Y containing $f(x)$, there exists an $\theta\omega\alpha$ - open set U in X such that $x \in$ U, $f(U) \subseteq$ V.

Proof : (i) $\Rightarrow$ (ii) : Let V be an open set in Y containing $f(x)$. Since f is $\theta\omega\alpha$ -continuous, $x \in f^{-1}$(V) is $\theta\omega\alpha$ -open in X. Put U = f^{-1}(V) then $x \in$ U $f(U) \subseteq$ V.

(ii) $\Rightarrow$ (i) : Let V be an open set in Y and $x \in f^{-1}(V)$ then $f(x) \in$ V, there exists an $\theta\omega\alpha$ -open set U_x in X such that $f(U_x) \subseteq$ V. Then $x \in U_x \subseteq f^{-1}$(V) and f^{-1}(V) $= \cup U_x$. Therefore f^{-1}(V)is $\theta\omega\alpha$ -open set in X. Which implies that f is $\theta\omega\alpha$ -continuous.

Theorem 4.15: Let X be a $T_{\theta\omega\alpha}$ -space and f : X$\rightarrow$Y be a function then following statements are equivalent:

(i) f is $\theta\omega\alpha$ -continuous

(ii) $f(\theta\omega\alpha - Cl(A)) \subset Cl(f(A))$, for every subset A of X

(iii) $\theta\omega\alpha$ -$Cl(f^{-1}$(B))$\subset f^{-1}(Cl(B))$, for every subset B of Y

(iv) f^{-1}(int(B))$\subset \theta\omega\alpha$ -int(f^{-1}(B)).

Proof : (i) $\Rightarrow$ (ii) : Let A be any subset of X. Since A$\subset f^{-1}(f(A)) \subset f^{-1}(Cl(f(A)))$. Now $Cl(f(A))$ is closed in Y and f is $\theta\omega\alpha$ -continuous which implies that $f^{-1}(Cl(f(A)))$ is a $\theta\omega\alpha$ -closed in X containing A. Consequently $\theta\omega\alpha - Cl(A) \subset f^{-1}(Cl(f(A)))$. Therefore $f(\theta\omega\alpha - Cl(A)) \subset f(f^{-1}(Cl(f(A)))) \subset Cl(f(A))$. Hence $f(\theta\omega\alpha - Cl(A)) \subset Cl(f(A))$.

(ii) $\Rightarrow$ (iii) : Let B be any subset of Y then f^{-1}(B) is subset of X. From (ii) f(θωα -Cl(f^{-1}(B))) $\subset Cl(f(f^{-1}(B))) \subset Cl(B)$. Therefore $\theta\omega\alpha$ -$Cl(f^{-1}(B)) \subset f^{-1}(Cl(B))$.

(iii) $\Rightarrow$ (iv) : Let B $\subset$ Y then $(Y - B) \subset$Y. Since X is $T_{\theta\omega\alpha}$ -space and by hypothesis, $\theta\omega\alpha$ (f^{-1}(Y - B))$\subset f^{-1}(Cl(Y - B))$. This implies X-($\theta\omega\alpha$ -int (f^{-1}(B)))$\subset$X-(f^{-1}(int(B))). Therefore f^{-1}(int(B))$\subset \theta\omega\alpha$ -int(f^{-1}(B)).

(iv) $\Rightarrow$ (i) : Let F be a closed set in Y then (Y - F) is an open set in Y. Therefore f^{-1}(Y – F) = f^{-1}(int(Y – F))$\subset \theta\omega\alpha$ -int(f^{-1}(Y – F)) = X - $\theta\omega\alpha$ -

$Cl(f^{-1}(F))$. This implies $f^{-1}(F)$ is $\theta\omega\alpha$-closed set. Therefore f is $\theta\omega\alpha$-continuous function.

Theorem 4.16: The following examples shows that the class of $\theta\omega\alpha$-continuous functions are independent with the class of continuous and α-continuous functions.

Example 4.17: Let X = Y = $\{a, b, c\}$ with $\tau = \{\phi, \{a, b\}, X\}$, $\eta = \{\phi, \{a\}, \{a, b\}, Y\}$. Let $f:(X, \tau) \to (Y, \eta)$ be an identity function, then f is $\theta\omega\alpha$-continuous but not continuous and α-continuous function, since for the closed set A = $\{b, c\}$ in Y, $f^{-1}(\{b, c\}) = (\{b, c\})$ is closed and α-closed in X.

Example 4.18: Let X = Y = $\{a, b, c\}$ with $\tau = \{\phi, \{a\}, \{a, b\}, \{a, c\}, X\}$, $\eta = \{\phi, \{a, b\}, Y\}$. Let $f:(X, \tau) \to (Y, \eta)$ be the identity function, then f is continuous and α-continuous but not $\theta\omega\alpha$-continuous function, since for the closed set A = $\{c\}$ in Y, $f^{-1}(\{c\}) = (\{c\})$ is not $\theta\omega\alpha$-closed in X.

Theorem 4.19: Following examples shows the concept of $\theta\omega\alpha$-continuous function is independent with $\omega\alpha$-continuous.

Example 4.20: Let X = Y = $\{a, b, c\}$ with $\tau = \{\phi, \{a\}, \{a, b\}, \{a, c\}, X\}$, $\eta = \{\phi, \{a, b\}, Y\}$. A function $f:(X, \tau) \to (Y, \eta)$ defined by $f(a) = a$, $f(b) = c$ and $f(c) = b$. For the closed set A = $\{c\}$, $f^{-1}(\{c\}) = \{b\}$ is not a $\theta\omega\alpha$-closed in X. Therefore f is not $\theta\omega\alpha$-continuous but it is a $\omega\alpha$-continuous function.

Example 4.21: Let X = Y = $\{a, b, c\}$ with $\tau = \{\phi, \{a\}, \{b, c\}, X\}$, $\eta = \{\phi, \{a\}, Y\}$. Let $f:(X, \tau) \to (Y, \eta)$ be the function defined by $f(a) = b$, $f(b) = a$ and $f(c) = c$. Then f is $\theta\omega\alpha$-continuous but not $\omega\alpha$-continuous, since for the closed set A = $\{b, c\}$ in Y, $f^{-1}(\{b, c\}) = (\{a, c\})$ is not $\omega\alpha$-closed in X.

Remark 4.22: Converse of the Theorems 4.16 and 4.19 holds if X is $T_{\theta\omega\alpha}$-Space.

Remark 4.23: From the above results, it follows that the $\theta\omega\alpha$-continuous functions are properly placed between θ-continuous and θg-continuous functions.we can see in the below Figure 1.

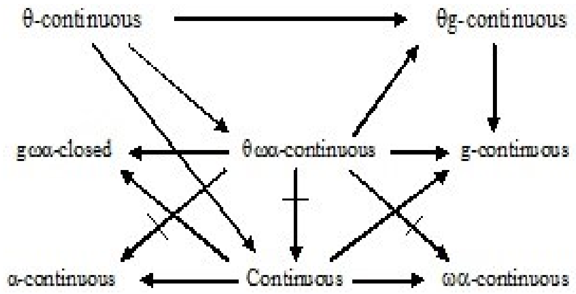

Figure 1

5. $\theta\omega\alpha$ -IRRESOLUTE FUNCTIONS IN TOPOLOGICAL SPACES

Definition 5.1: A function f : X$\to$Y is called $\theta\omega\alpha$ -irresolute if f^{-1}(V) is $\theta\omega\alpha$ -closed in X, for every $\theta\omega\alpha$ -closed in Y.

Theorem 5.2: A function f : X$\to$Y is $\theta\omega\alpha$ -irresolute if and only if for every $\theta\omega\alpha$ -open set A in Y, f^{-1}(A) is $\theta\omega\alpha$ -open in X.

Proof : The proof is obvious.

Theorem 5.3: If f : X$\to$Y is $\theta\omega\alpha$ -continuous and Y is $T_{\theta\omega\alpha}$-space, then f is $\theta\omega\alpha$ -irresolute.

Proof: Let Y be $T_{\theta\omega\alpha}$ -space and F be $\theta\omega\alpha$ -closed in Y then F is closed in Y. Since *f* is $\theta\omega\alpha$ -continuous, f^{-1}(F) is $\theta\omega\alpha$ -closed in X. Therefore *f* is $\theta\omega\alpha$ -irresolute.

Theorem 5.4: Let a function f : X$\to$Y be $\theta\omega\alpha$ -irresolute, closed and onto. If X is $T_{\theta\omega\alpha}$ -space then Y is $T_{\theta\omega\alpha}$ -space.

Proof : Let F be a $\theta\omega\alpha$ -closed set in Y. Since *f* is $\theta\omega\alpha$ -irresolute then f^{-1}(F) is $\theta\omega\alpha$ -closed in X. But X is $T_{\theta\omega\alpha}$ -space, f^{-1}(F) is closed in X. By hypothesis, f closed and onto then F = $f(f^{-1}$(F)) is closed in Y. Therefore Y is $T_{\theta\omega\alpha}$ -space.

Remark 5.5: From below examples we have observed that the concept of $\theta\omega\alpha$ -continuous function is independent of $\theta\omega\alpha$ -irresolute.

Example 5.6: Let X = Y $=\{a, b, c\}$, $\tau=\{\phi, \{a\}, \{b\}, \{a, b\}, \{a, c\}, X\}$ with $\eta=\{\phi, \{a\}, \{b, c\}, Y\}$. Define a function f : (X, τ)$\to$(Y, η) by $f(a)=b$, $f(b)=a$ and $f(c)=c$. Clearly, f is $\theta\omega\alpha$ -continuous but not $\theta\omega\alpha$ -irresolute, since for the subset $\{b\}$ in Y, $f^{-1}(\{b\}) = (\{a\}) \notin \theta\omega\alpha$ C(X, τ).

Example 5.7: Let X = Y = $\{a, b, c\}$, $\tau = \{\phi, \{a, b\}, X\}$ with $\eta = \{\phi, \{a\}, \{a, b\}, \{a, c\}, Y\}$. Let f : (X, τ)→(Y, η) be an identity function then f is $\theta\omega\alpha$-irresolute but not a $\theta\omega\alpha$-continuous, since for the subset $\{b\}$ in Y, $f^{-1}(\{b\}) = (\{b\}) \notin \theta\omega\alpha$ C(X, τ).

Theorem 5.8: Let X, Y and Z be topological spaces. Let f : X→Y and g : Y→ Z be two functions then the following results are holds:

(i) If f is $\theta\omega\alpha$-irresolute and g is $\theta\omega\alpha$-continuous then (gof): X→Z is $\theta\omega\alpha$-continuous.

(ii) If f and g are $\theta\omega\alpha$-irresolute functions then (gof) is $\theta\omega\alpha$-irresolute.

(iii) (gof) is $\theta\omega\alpha$-irresolute, if f is $\theta\omega\alpha$-continuous, g is $\theta\omega\alpha$-irresolute and Y is $T_{\theta\omega\alpha}$-space.

Proof: (i) Let U be an open set in Z then g^{-1}(U) is $\theta\omega\alpha$-open in Y as g is $\theta\omega\alpha$-continuous. Since f is $\theta\omega\alpha$-irresolute then, $f^{-1}(g^{-1}$(U) is $\theta\omega\alpha$-open in X. But $f^{-1}(g^{-1}$(U)) = $(gof)^{-1}$(U). Therefore $(gof)^{-1}$(U) is $\theta\omega\alpha$-open in X. Hence (gof) is $\theta\omega\alpha$-continuous.

(ii) Let V ⊂ Z be an $\theta\omega\alpha$-open then g^{-1}(V) and $f^{-1}(g^{-1})$(V) are $\theta\omega\alpha$-open, since g and f are $\theta\omega\alpha$-irresolute functions. But $f^{-1}(g^{-1})$(V) = $(gof)^{-1}$ (V) is $\theta\omega\alpha$-open in X. Therefore (gof) is $\theta\omega\alpha$-irresolute.

(iii) Let F be a $\theta\omega\alpha$-closed in Z. Since g is $\theta\omega\alpha$-irresolute and Y is $T_{\theta\omega\alpha}$-space, (g^{-1}(F)) is closed in Y. But f is $\theta\omega\alpha$-continuous then $f^{-1}(g^{-1})$(F) is $\theta\omega\alpha$-closed in X. We have $f^{-1}(g^{-1})$(F) = $(gof)^{-1}$(F). Therefore $(gof)^{-1}$ (F) is $\theta\omega\alpha$-closed in X. Hence (gof) is $\theta\omega\alpha$-irresolute.

6. $\theta\omega\alpha$-CLOSED AND $\theta\omega\alpha$-OPEN FUNCTIONS IN TOPOLOGICAL SPACES

Definition 6.1: A function $f : X \to Y$ is called $\theta\omega\alpha$-closed if for each closed set A in X, $f(A)$ is $\theta\omega\alpha$-closed in Y.

Theorem 6.2: A function $f : X \to Y$ is $\theta\omega\alpha$-closed if and only if for each subset A of Y and for each open set U containing $f^{-1}(A)$ then there exists an $\theta\omega\alpha$-open set V of Y such that A ⊆ V and $f^{-1}(V) \subseteq$ U.

Proof : Let A $\subseteq$ Y and U $\subseteq$ X is an open set such that $f^{-1}(A) \subseteq$ U. Then V = Y - $f\,(U^c)$ is an $\theta\omega\alpha$ -open in Y. So $f^{-1}(V)$ = X – (U^c) = $((U^c)^c)$ = U such that $f^{-1}(V) \subseteq$ U.

Conversely, let S be a closed set in X then $f^{-1}\,((f(S))^c) \subseteq S^c$ where S^c is open in X. By hyp, there exists $\theta\omega\alpha$ -open set V of Y such that $(f(S))^c \subseteq$ V and $f^{-1}(V) \subseteq S^c$. So S $\subseteq$ (f^{-1}(V)) ^{c}. Therefore $V^c \subseteq f(S) \subseteq f^{-1}\,((f(S))^c) \subseteq V^c$ then $f(S) = V^c$. Since V^c is $\theta\omega\alpha$ -closed, $f(S)$ is $\theta\omega\alpha$ -closed. Hence f is $\theta\omega\alpha$ -closed.

Theorem 6.3: If f : X $\rightarrow$ Y is continuous, surjective function and composition of any function g : Y $\rightarrow$ Z with f is $\theta\omega\alpha$ -closed then g is $\theta\omega\alpha$ -closed.

Proof : (i) Let A $\subseteq$ Y be closed set then f^{-1}(A) is closed in X. Since (gof) is $\theta\omega\alpha$ -closed function, $(gof)(f^{-1}(A))$ is $\theta\omega\alpha$ -closed in Z. Now $(gof)(f^{-1}(A))$ = $g(f(f^{-1}(A)))$ = $g(A)$ is $\theta\omega\alpha$ -closed in Z. Therefore g is $\theta\omega\alpha$ -closed function.

Theorem 6.4: If the composition of two functions f : X $\rightarrow$ Y and g : Y $\rightarrow$ Z is a $\theta\omega\alpha$ -closed and g is $\theta\omega\alpha$ -irresolute and injective then f is $\theta\omega\alpha$ -closed.

Proof : Let A be a closed set in X. Since (gof) is $\theta\omega\alpha$ -closed, $(gof)(A)$ is $\theta\omega\alpha$ -closed in Z. $g^{-1}(gof)(A)$ is $\theta\omega\alpha$ -closed in Y as g is $\theta\omega\alpha$ -irresolute. Hence $f(A)$ is $\theta\omega\alpha$ -closed in Y. Therefore f is $\theta\omega\alpha$ -closed function.

Theorem 6.5: Composition of two $\theta\omega\alpha$ -closed functions need not be $\theta\omega\alpha$ - closed which can be observed from the below example.

Example 6.6: Let X = Y = Z = $\{a, b, c\}$, $\tau = \{\phi, \{a\}, X\}$ and $\eta = \{\phi, \{b\}, Y\}$ and $\gamma = \{\phi, \{a\}, \{a, b\}, \{a, c\}, Z\}$. Define f : (X, τ) $\rightarrow$ (Y, η) as $f(a) = a$, $f(b) = c$ and $f(c) = b$ and g : (Y, η) $\rightarrow$ (Z, γ) as $g(a) = b$, $g(b) = a$ and $g(c) = c$. Clearly, f and g are $\theta\omega\alpha$ -closed functions but their composition (gof) : (X, τ) $\rightarrow$ (Z, γ) is not a $\theta\omega\alpha$ -closed, since for a closed set $\{b, c\} \subset$ X, $(gof)(\{b, c\}) = \{a, c\} \notin \theta\omega\alpha$ C(Z, γ).

Remark 6.7: Composition of continuous function f : X $\rightarrow$ Y and $\theta\omega\alpha$ -closed function g : Y $\rightarrow$ Z is a $\theta\omega\alpha$ -closed.

Definition 6.8: A function f : X$\rightarrow$Y is called a $\theta\omega\alpha$ -open if the image $f(A)$ is $\theta\omega\alpha$ -open in Y for each open set A in X.

Theorem 6.9: Let f : X$\rightarrow$Y be any bijective function then the following statements are equivallent:

(i) f^{-1}: Y$\rightarrow$X is $\theta\omega\alpha$ -continuous

(ii) f is a $\theta\omega\alpha$ -open function

(iii) f is a $\theta\omega\alpha$ -closed function.

Proof : (i) $\Rightarrow$ (ii) : Let U$\subseteq$X be an open set. By hypothesis, (f^{-1})$^{-1}$(U) = $f(U)$ is $\theta\omega\alpha$ -open in Y. Therefore f is $\theta\omega\alpha$ -open function.

(ii) $\Rightarrow$ (iii) : Let K$\subseteq$X be a closed set then K^c is open in X. By hypothesis, $f(K^c)$ is $\theta\omega\alpha$ -open in Y. We have $f(K^c) = (f(K))^c$ is $\theta\omega\alpha$ -open in Y. So $f(K)$ is $\theta\omega\alpha$ -closed in Y. Hence f is $\theta\omega\alpha$ -closed function.

(iii) $\Rightarrow$ (i) : Let K be a closed in X then $f(K)$ is $\theta\omega\alpha$ -closed in Y. But $f(K)$ = (f^{-1})$^{-1}$(K), which implies that (f^{-1})$^{-1}$(K) is $\theta\omega\alpha$ -closed in Y. Therefore f^{-1} is $\theta\omega\alpha$ -continuous function.

Theorem 6.10: A function f : X$\rightarrow$Y is $\theta\omega\alpha$ -open if and only if for any subset A of Y and any closed set K containing f^{-1}(A) then ther exists an $\theta\omega\alpha$ -closed set B$\subseteq$Y containing A such that f^{-1}(B)$\subseteq$K.

Proof : The proof is obvious.

Theorem 6.11: A function f : X$\rightarrow$Y is $\theta\omega\alpha$ -open if and only if f^{-1}($\theta\omega\alpha$ -Cl(A))$\subset$Cl(f^{-1}(A)) for every subset A of Y.

Proof : Let A$\subset$Y then f^{-1}(A)$\subset$Cl(f^{-1} (A)). By Theorem 6.10, there exists a $\theta\omega\alpha$ -closed set B $\subset$ Y such that A$\subset$B and f^{-1}(B)$\subset$Cl(f^{-1} (A)).

Conversely, let A$\subset$Y and K be any closed set containing f^{-1}(A) in X. Now put M = $\theta\omega\alpha$ -Cl(A). Then M is $\theta\omega\alpha$ -closed and A$\subset$M. By hyp, f^{-1}(M) = f^{-1}($\theta\omega\alpha$ -Cl(A))$\subset$Cl(f^{-1}(A)$\subset$K. By Theorem 6.10, f is $\theta\omega\alpha$ -open function.

Definition 6.12: A function f : X$\rightarrow$Y is said to be $\theta\omega\alpha^*$-closed if for every $\theta\omega\alpha$ -closed set K in X, $f(K)$ is $\theta\omega\alpha$ -closed in Y.

Example 6.13: Let X = Y = $\{a, b, c\}$, $\tau = \{\phi, \{a\}, \{a, b\}, X\}$ and $\eta = \{\phi, \{a\}, \{b\}, \{a, b\}, Y\}$. Define a function $f : (X, \tau) \rightarrow (Y, \eta)$ by $f(a) = b$, $f(b) = a$ and $f(c) = c$. Clearly, f is $\theta\omega\alpha^*$-closed function.

Remark 6.14: A $\theta\omega\alpha^*$-closed function $f : X \rightarrow Y$ is a $\theta\omega\alpha$-closed function if X is a $T_{\theta\omega\alpha}$-space.

Theorem 6.15: The following statements are equivalent for any bijective function $f : X \rightarrow Y$,

(i) $f^{-1} : Y \rightarrow X$ is $\theta\omega\alpha$-irresolute

(ii) f is a $\theta\omega\alpha^*$-open function

(iii) f is a $\theta\omega\alpha^*$-closed function.

Proof: (i) $\Rightarrow$ (ii) : Let V $\subseteq$ X is $\theta\omega\alpha$-open. Since f^{-1} is $\theta\omega\alpha$-irresolute, $(f^{-1})^{-1}$ (V) = $f(V)$ is $\theta\omega\alpha$-open in Y. Therefore f is $\theta\omega\alpha^*$-open function.

(ii) $\Rightarrow$ (iii) : Let F be a $\theta\omega\alpha$-closed in X. Since f is $\theta\omega\alpha^*$-open, $f(F^c)$ is $\theta\omega\alpha$-open in Y. But $f(F^c) = (f(F))^c$, $(f(F))^c$ is $\theta\omega\alpha$-open in Y. Which implies that $f(F)$ is $\theta\omega\alpha$-closed in Y. Hence f is $\theta\omega\alpha^*$-closed.

(iii) $\Rightarrow$ (i) : Let F$\subseteq$X be a $\theta\omega\alpha$-closed set. $f(F)$ is $\theta\omega\alpha$-closed in Y as f is $\theta\omega\alpha^*$-closed map. But $f(F) = ((f^{-1})^{-1})$(F) is $\theta\omega\alpha$-closed in Y. Therefore f^{-1} is $\theta\omega\alpha$-irresolute function.

Theorem 6.16: If $f : X \rightarrow Y$ is $\theta\omega\alpha$-closed and $\omega\alpha$-irresolute then $f(A)$ is $\theta\omega\alpha$-closed in Y for every $\theta\omega\alpha$-closed set A of X.

Proof: Let A any $\theta\omega\alpha$-closed in X and U$\subseteq$Y be $\omega\alpha$-open set containing $f(A)$, then A$\subset f^{-1}$(U). Since f is $\omega\alpha$-irresolute, f^{-1}(U) is $\omega\alpha$-open in X. We have, A is $\theta\omega\alpha$-closed, $\mathrm{Cl}_\theta(A) \subset f^{-1}(U)$. Now $f(\mathrm{Cl}_\theta(A)) \subset f(f^{-1}(U))$ = U. Since f is $\theta\omega\alpha$-closed, $\mathrm{Cl}_\theta(f(\mathrm{Cl}_\theta(A))) \subset$U which implies $cl_\theta f(A) \subset$U. Therefore $f(A)$ is $\theta\omega\alpha$-closed in Y.

7. $\theta\omega\alpha$ -HOMEOMORPHISM IN TOPOLOGICAL SPACES

Definition 7.1: A function f : X $\rightarrow$Y is called $\theta\omega\alpha$ -homeomorphism if both f and f^{-1} are $\theta\omega\alpha$ -continuous and f is bijective.

Theorem 7.2: The below statements are equivalent for any bijective function f : X $\rightarrow$ Y :

(i) f is $\theta\omega\alpha$ -homeomophism

(ii) f is $\theta\omega\alpha$ -continuous and $\theta\omega\alpha$ -open

(iii) f is $\theta\omega\alpha$ -continuous and $\theta\omega\alpha$ -closed.

Proof : The proof follows from Definition 7.1 and Theorem 6.8.

Definition 7.3: A function f : X$\rightarrow$Y is said to be $\theta\omega\alpha^{*}$-homeomorphism if it satisfies following two conditions,

(i) f is bijective and

(ii) f and f^{-1} both are $\theta\omega\alpha$ -irresolute.

Theorem 7.4: The $\theta\omega\alpha$-homeomorphism is independent of $\theta\omega\alpha^{*}$-homeomorphism.

Proof : It clears from Remark 5.5.

Theorem 7.5: The composition (gof) : X$\rightarrow$Z is $\theta\omega\alpha^{*}$-homeomorphism if both functions f : X $\rightarrow$Y and g : Y$\rightarrow$Z are $\theta\omega\alpha^{*}$-homeomorphism.

Proof : Let V be a $\theta\omega\alpha$ -open in Z. Now $(gof)^{-1}$(V) = $f^{-1}(g^{-1}$(V)) = f^{-1}(U) where g^{-1}(V) = U. By hyp, U is $\theta\omega\alpha$ -open in Y and f^{-1}(U) $\theta\omega\alpha$ -open in X. Therefore (gof) is $\theta\omega\alpha$ -irresolute. Also for a $\theta\omega\alpha$ -open set W in X, We have $(gof)(W)=g(f(W))=g(A)$ where $A=f(W)$. Again by hyp, $f(W)$ and $g(W)$ are $\theta\omega\alpha$ -open in Y and Z respectively. Therefore $(gof)^{-1}$ is $\theta\omega\alpha$ -irresolute. Hence (gof) $\theta\omega\alpha^{*}$-homeomorphism.

Theorem 7.6: If a function f : X $\rightarrow$Y is $\theta\omega\alpha$ -homeomorphism where X and Y are $T_{\theta\omega\alpha}$-spaces then f is $\theta\omega\alpha^{*}$-homeomorphism.

Proof : Let V be a $\theta\omega\alpha$ -closed in Y then V is closed. By hyp and from Theorem 7.2 implies that f is bijectve, $\theta\omega\alpha$ -continuous and $\theta\omega\alpha$ -open. Therefore

f^{-1}(V) is $\theta\omega\alpha$ -closed in X. Hence f is a $\theta\omega\alpha$ -irresolute function. Let V$\subseteq$X be a $\theta\omega\alpha$ -open then V is open in X and f (V)$\subseteq$Y is $\theta\omega\alpha$ -open. By hyp, (f^{-1})$^{-1}$ (V) is $\theta\omega\alpha$ -open in Y. Therefore f^{-1} is $\theta\omega\alpha$ -irresolute function.

Theorem 7.7: If f : X$\rightarrow$Y is $\theta\omega\alpha^*$-homeomorphism then the following properties are true:

(i) for every subset A of Y, $\theta\omega\alpha$ – Cl(f^{-1}(A)) = f^{-1}($\theta\omega\alpha$ – Cl(A))

(ii) for every subset B of X , $\theta\omega\alpha \mathrm{Cl}(f(B))$ = $f(\theta\omega\alpha - \mathrm{Cl}(B))$.

Proof : Let A $\subseteq$ Y. By Theorem 3.4 (i) [12], $\theta\omega\alpha - \mathrm{Cl}(f(A))$ is $\theta\omega\alpha$ -closed in Y. Since f is $\theta\omega\alpha^*$-homeomorphism, f is $\theta\omega\alpha$ -irresolute and $f^{-1}(\theta\omega\alpha - \mathrm{Cl}(f(A)))$ is $\theta\omega\alpha$ -closed in X. We have f^{-1}(A) $\subseteq$ f^{-1}($\theta\omega\alpha$ -cl(A)). So $\theta\omega\alpha$ -Cl(f^{-1}(A))$\subseteq$ f^{-1}($\theta\omega\alpha$ -cl(A)) from Theorem 3.4 [(v) and (viii)] [12].

Again by hypothesis, f^{-1} is $\theta\omega\alpha$ -irresolute. Since $\theta\omega\alpha$ -Cl(f^{-1}) is $\theta\omega\alpha$ -closed in X, (f^{-1})$^{-1}$($\theta\omega\alpha$ -Cl(f^{-1}(A))) = $f(\theta\omega\alpha$ -Cl(f^{-1}(A))) is $\theta\omega\alpha$ -closed in Y. Now A$\subseteq$(f^{-1})$^{-1}$(f^{-1}(A))$\subseteq$(f^{-1})$^{-1}$($\theta\omega\alpha$ -Cl(f^{-1}(A))) = $f(\theta\omega\alpha$ -Cl (f^{-1})). Therefore $\theta\omega\alpha$ -Cl(A) $\subseteq$ $f(\theta\omega\alpha$ -Cl(f^{-1}(A))). Hence f^{-1}($\theta\omega\alpha$ -Cl(A))$\subseteq$ f^{-1}($f(\theta\omega\alpha$ -Cl(f^{-1}(A))))$\subseteq$ $\theta\omega\alpha$ -Cl(f^{-1}(A)). Thus $\theta\omega\alpha$ -Cl (f^{-1}(A)) = f^{-1}($\theta\omega\alpha$ -Cl(A)).

(ii) Let B be a subset of X. Since f is $\theta\omega\alpha^*$-homeomorphism, f^{-1} is also $\theta\omega\alpha^*$- homeomorphism. From the above result, $\theta\omega\alpha$ -Cl(f^{-1}(B)) = (f^{-1})$^{-1}$ ($\theta\omega\alpha$ -Cl(B)), for every subset B of X. Therefore $\theta\omega\alpha - \mathrm{Cl}(f(B))$ = $f(\theta\omega\alpha$ -Cl(B)).

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