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A NEW MEHLER-FOCK WAVELET ON THE SPACE T_α

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Abstract

Translation associated with new Mehler-Fock transform studied by H. J. Glaeske and A. Hess [2] is used to define Mehler-Fock wavelet and its transform. The boundedness property of the new continuous Mehler-Fock wavelet and its wavelet transform are obtained. The new discrete Mehler-Fock wavelet and its wavelet transforms are defined and its boundedness property is studied.

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1. INTRODUCTION

The new Mehler-Fock transform (MMFT) of order $n \geq 0$ is defined by

$$\Im_n(\omega) = \hat{\Im}(\omega) = \int_1^\infty f(t)\, P^{-n}_{i\omega-1/2}(t)\, dt, \tag{1.1}$$

where the kernel $P^{-n}_{\nu}(t)$ denotes associated Legendre function [2] and it is also called cone function of order n given as

$$P^{-n}_{i\omega-1/2}(t) := \frac{1}{2^n.n!}\left(t^2-1\right)^{n/2} {}_2F_1(n+1/2+i\omega,\, n+1/2-i\omega;\, n+1;\, (1-t)/2)$$

with the Gaussian hypergeometric function ${}_2F_1(.,.;.;.)$.

Its applications may be possible from the ref [3] in dual integral equations and a few physical problems. Also, its distributional theory may be extended with the help of ref [1], [3], [4], [7].

Translation and convolution are defined with the help of new Mehler-Fock transform in [2] and these associated with generalized new Mehler-Fock transform have been introduced by the references [4], [6].

The new Mehler-Fock wavelet is different from [5] and its transform also.

THE TESTING FUNCTION SPACE T_α AND ITS DUAL

In this section, some definitions and results recall from [2] which will be needed in the following investigation.

Let $I=(1,\infty)$. For $t\in I$ let

$$\gamma_\alpha(t):=t^\alpha, \qquad 0\le\alpha<1/2,$$

and

$$C_t:=(t^2-1)^{-n/2}D_t(t^2-1)^{n+1}D_t(t^2-1)^{-n/2}.$$

Now, we recall the test function space T_α from [2] and defined as

$$\eta_{k,\alpha}(\phi):=\sup_{t\in I}\left|\gamma_\alpha(t)\,C_t^k\phi(t)\right|<\infty, \qquad \forall\, k=0,1,2,\ldots,$$

where $\phi\in C^\infty(I)$. The space T_α is a linear space with the family $\{\eta_{k,\alpha}\}$, $k\in\mathbf{N_0}$, of seminorms of which $\eta_{0,\alpha}$ is a norm. The dual of T_α is denoted by T'_α.

If $\phi\in T_\alpha$, then from [2, Prop.2.1, p.122] to get

$$D_t^k[(t^2-1)^{-n/2}\phi(t)]=O(1) \qquad \textit{for } t\to 1+ \tag{2.1}$$

$$t^k\,D_t^k\phi(t)=O(t^{-\alpha}) \qquad \textit{for } t\to\infty, \tag{2.2}$$

for all $k\in\mathbf{N_0}$.

We take the following definitions from [2], and it can be written as

Definition 2.2 For $f\in\mathrm{T}'_\alpha$ the generalized Mehler-Fock transform is given by

$$M_F(f)(\omega)=M(\omega):=<f(t),\,P^{-n}_{i\omega-1/2}(t)>, \qquad \omega>0. \tag{2.3}$$

This result coincides with equation (1.1) for the regular generalized function.

Definition 2.3 The subspace $\mathrm{T}'_{\alpha,\,reg.}$ of regular generalized functions of T'_α consists of all $f\in L_1^{loc}(I)$ such that

$$\int_1^\infty\frac{|f(t)|}{\gamma_\alpha(t)}dt<\infty. \tag{2.4}$$

in [2, p. 124].

Now, for all $\phi \in T_\alpha$ with

$$< f, \phi >= \int_1^\infty f(t)\ \phi(t)\ dt \tag{2.5}$$

the classical formula (1.1).

Next, let us consider the basic function $\Omega(x, y, z)$ from [2] such as

$$\Omega(x,y,z) = \begin{cases} \frac{1}{\pi(2n-1)!!}[\Gamma(x,y,z)]^{n-1/2}[(x^2-1)(y^2-1)(z^2-1)]^{-n/2}, & for\, \Gamma(x,y,z) > 0; \\ 0, & for\, \Gamma(x,y,z) \le 0, \end{cases} \tag{2.6}$$

where

$$\Gamma(x,y,z) = 2\,x\,y\,z - 1 - x^2 - y^2 - z^2\,.$$

Clearly, the basic function $\Omega(x, y, z)$ is symmetric in variables x, y, z.

Also, we have,

$$P^{-n}_{i\omega-1/2}(x)\ \ P^{-n}_{i\omega-1/2}(y) = \frac{1}{\pi(2n-1)!!}[(x^2-1)(y^2-1)]^{n/2}\int_0^\pi (z^2-1)^{-n/2} P^{-n}_{i\omega-1/2}(z)\ \sin^{2n}(\upsilon)d\upsilon,$$

is given in [2] with

$$z = z(x, y, \upsilon) = xy + [(x^2-1)(y^2-1)]^{1/2}\cos(\upsilon)\,. \tag{2.7}$$

Next, the generalized translation operator τ_x in [2] is defined by

$$\tau_x\phi(y) := \int_1^\infty \Omega(x,y,z)\ \phi(z)\ dz\,, \tag{2.8}$$

which is convergent.

Definition 2.4. For $\phi \in T_\alpha$ and $x, y \in I$ the generalized translation operator τ_x on T_α is defined by

$$\tau_x\phi(y) := < \Omega(x, y, .),\ \phi >= \int_1^\infty \Omega(x, y, z)\,\phi(z)\,dz\,. \tag{2.9}$$

Lemma 2.5. [2] If $\phi \in T_\alpha$, then $C_y \tau_x \phi(y) = \tau_x C_y \phi(y)$.

One estimates for $k \in N_0$; $x, y \in I$ via lemma 2.5 and [2, p. 124] to get

$$\left|C_y^k \tau_x \phi(y)\right| = \left|\tau_x C_y^k \phi(y)\right| \le \frac{2^n}{\pi(2n-1)!!}\int_0^{\pi}\left|C_z^k \phi(z)\right| d\upsilon\,, \tag{2.10}$$

where z is given in equation (2.7) , from [2, p. 124] we have

$$\left|C_y^k \tau_x \phi(y)\right| \le (xy)^{-\alpha} A_\alpha \eta_{k,\alpha}(\phi)\,. \tag{2.11}$$

with real number A_α.

The next section contributes wavelets and contains its important properties.

THE NEW CONTINUOUS MEHLER-FOCK WAVELETS

Let $\psi \in \mathrm{T}_\alpha$ be the mother wavelet. For $b \ge 0$ and $a > 0$ define new Mehler-Fock wavelets by

$$\psi_{b,a}(t) = [\tau_b D_a \psi](t)$$

$$= \int_1^\infty \psi(a\,z)\Omega(b,t,z)\ dz\,, \tag{3.1}$$

where $D_a\psi(z) = \psi(a\,z)$. In view of (2.1), (2.2) and (2.11), to easily seen that $\psi(a\,z) \in \mathrm{T}_\alpha$ and from the next theorem may be shown $\psi_{b,a}(t) \in \mathrm{T}_\alpha$.

Next, the new Mehler-Fock wavelet transform of $f \in \mathrm{T}'_{\alpha,\, reg.}$ define as

$$(M_\psi f)(b,a) = < f(t), \psi_{b,a}(t) > = \int_1^\infty f(t)\,\psi_{b,a}(t)\,dt\,, \tag{3.2}$$

where $\psi_{b,a}(t)$ is given in (3.1).

Theorem 3.1 Let $\psi \in \mathrm{T}_\alpha$; for $0 \le \alpha < 1/2$, then

$$\left|\psi_{b,\,a}(t)\right| \le (b\,t)^{-\alpha} A_\alpha\, \gamma_{0,\,\alpha}[\psi(a.)],$$

where A_α be the positive real number.

Proof In view of (3.1) for $k \in N_0$; $z \in I$, we get

$$C_t^k \psi_{b,a}(t) = C_t^k \tau_b[\psi(a.)](t)$$

i.e.,

$$\left|C_t^k \tau_b[\psi(a.)](t)\right| = \left|\int_1^\infty \psi(a z)\, C_t^k[\Omega(b,t,z)]\, dz\right|$$

using result (2.11), we get

$$\left|C_t^k \tau_b[\psi(a.)](t)\right| \le (bt)^{-\alpha} A_\alpha\, \eta_{k,\alpha}[\psi(a.)]$$

gives the desire result for $k = 0$.

Theorem 3.2 Let $f \in \mathrm{T}'_{\alpha,\, reg.}$ and $\psi \in \mathrm{T}_\alpha$, then

$$\left|< f(t), \psi_{b,a}(t) >\right| \le b^{-\alpha} A_\alpha\, \gamma_{0,\,\alpha}[\psi(a.)]\beta_f\,,$$

where $\psi_{b,a}(t)$ in (3.1) and

$$\beta_f = \int_1^\infty t^{-\alpha}\left|f(t)\right| dt < \infty\,.$$

Proof We consider the equation (3.2) such that

$$\left|< f(t), \psi_{b,a}(t) >\right| \le \int_1^\infty \frac{\left|f(t)\right|}{\gamma_\alpha(t)} \left|t^{-\alpha}\, \psi_{b,a}(t)\right| dt\,.$$

Using equation (2.4) and theorem 3.1 we get

$$\le b^{-\alpha} A_\alpha\, \gamma_{0,\,\alpha}[\psi(a.)]\int_1^\infty t^{-\alpha}\left|f(t)\right| dt$$

it is required result.

NEW DISCRETE MEHLER-FOCK WAVELET TRANSFORMS

In the equation (3.1), if we take discrete form of the dilation parameter a with $a = 2^{-j} a_0; j \in Z$, and the translation parameter b is allowed to vary over all of [1,∞], then the transform so obtained is called new semi-discrete Mehler-Fock wavelet transform. Also, if discretize the translation parameter b by restricting it to the discrete set of points $b = \frac{\mathrm{k}}{2^{\mathrm{j}}}\mathrm{b}_0$, $\mathrm{j} \in Z; k \in N_0$, where $b_0 > 0$ is a fixed constant, then get following new discrete Mehler-Fock wavelet and its transform. We shall use the notation

$$\psi_{j,k}(\mathrm{t}) = \psi(2^{\mathrm{j}} t,\ 2^{\mathrm{j}}\, \mathrm{k}\, \mathrm{b}_0)\,. \tag{4.1}$$

and discretize version of wavelet transform for $f \in \mathrm{T}'_{\alpha,\, reg.}$ such as

$$(M_{\psi} f)(j,k) = \int_{1}^{\infty} f(t)\, \psi_{j,k}(t)\, dt \tag{4.2}$$

Theorem 4.1 Let $f \in U'_{\alpha,\, reg.}$, then

$$\left| < f, \psi_{j,k} > \right| < \infty,$$

where $\psi_{j,k}$ is known as discrete wavelets in (4.1).

Proof In view of (4.2) we have

$$\left| < f, \psi_{b_0;j,k} > \right| = \left| \int_{1}^{\infty} f(t)\ \ \psi_{j,k}(t) dt \right|$$

with the help of theorem 3.2 we have

$$\left| < f, \psi_{b_0;j,k} > \right| < \infty .$$

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