

MEROMORPHIC LÉVY-KHINTCHINE EXPONENTS WITH POLES OF ORDER TWO

GUILLAUME COQUERET

ABSTRACT. Following the recent work of Kuznetsov, we propose the Wiener-Hopf factorization for a new class of Lévy processes. When all the poles of the Lévy exponent ϕ have an order equal to two and satisfy some additional conditions, we are able to locate, in the complex plane, the zeros of $q - \phi(z)$, for $q > 0$, thereby yielding the factorization. We also provide a detailed example and a set of conditions under which the factorization holds when the poles are allowed to have various orders.

1. Introduction

Fluctuations are one of the most interesting facets of Lévy processes (processes with stationary independent increments). They involve the running supremum $S_t = \sup_{0 \leq s \leq t} X_s$ and running infimum I_t of the Lévy process $(X_t)_{t \geq 0}$. The most important result in the theory is the well known Wiener-Hopf factorization which describes the distribution of S and I at an independent exponential random time. More precisely, if e_q is an independent exponential random variable with parameter q , then

$$\frac{q}{q - \psi(z)} = \mathbb{E}[e^{izS_{e_q}}] \mathbb{E}[e^{izI_{e_q}}] = \psi_q^+(z) \psi_q^-(z),$$

where

$$\psi(z) = \log(\mathbb{E}[e^{izX_1}]) = imz - \frac{\sigma^2 z^2}{2} + \int_{\mathbb{R}^*} (e^{izx} - 1 - izxh(x)) \nu(dx) \quad (1.1)$$

is the Lévy-Khintchine exponent of X (associated to the cut-off function h) and ψ_q^+, ψ_q^- are referred to as the Wiener-Hopf factors. For more details, see chapter VI of [3] or chapter 6 of [15].

Even though some old results exist ([4]), it is only recently that there has emerged a blossoming literature on closed forms of ψ_q^\pm : [2], [8], [9], [10], [13], [17].

The recent results rely on the fact that, in many cases, ψ can be extended to a meromorphic function on the complex plane and thus can be seen as the ratio of two analytic functions. Writing $q(\psi(z) - q)^{-1} = f_q(z)/g_q(z)$, the aim is then to obtain the Hadamard/Weierstrass factorization of f_q and g_q , which often yield formulae that are easily Laplace (or Fourier) inverted into the distributions of S_{e_q} and I_{e_q} . Until very recently, the models in the literature either proposed

Received 2012-7-25; Communicated by the editors.

2010 *Mathematics Subject Classification*. Primary 60G51; Secondary 30D30.

Key words and phrases. Wiener-Hopf factorization, Lévy processes, meromorphic function.

factorizations with finitely many (possibly) complex zeros ([17]), or factorizations with an infinite number of real zeros ([8],[9],[13]). The present paper, as well as the recent preprint [14], allows for a factorization with an infinite number of non-real zeros. The Lévy measure in [14] can have any general form but its support is bounded from above while our model allows unbounded positive jumps, but for a specific type of Lévy measure.

Obtaining closed-forms of Wiener-Hopf factorizations is very useful since, given the method developed in [12], it enables to efficiently simulate the couple (X_t, S_t) at fixed times. These results find applications in Quantitative Finance (pricing of barrier and lookback options) and Mathematical Insurance (ruin probabilities).

The paper is organized as follows: in section 2, we state our main result, that is, the Wiener-Hopf factorization when the Lévy density is an infinite sum of $\Gamma(2, \cdot)$ densities. Sections 3 and 4 are aimed at proving and discussing this result. Lastly, in section 5, we provide an example for which it is possible to asymptotically locate the terms in the factorization.

2. Notation and Main Result

Let $(X_t)_{t \geq 0}$ be a real-valued Lévy process starting at 0, with Lévy-Khintchine representation given by (1.1). The Lévy measure we are interested in is absolutely continuous and its density is given by

$$\pi(x) = \mathbf{1}_{(x>0)} \sum_{n=1}^{\infty} a_n \rho_n^2 x e^{-\rho_n x} + \mathbf{1}_{(x<0)} \sum_{n=1}^{\infty} \hat{a}_n \hat{\rho}_n^2 (-x) e^{\hat{\rho}_n x},$$

where the $(a_n, \hat{a}_n, \rho_n, \hat{\rho}_n)$ are positive, real numbers satisfying

$$\sum_{n=1}^{\infty} \frac{a_n}{\rho_n} < \infty, \quad \sum_{n=1}^{\infty} \frac{\hat{a}_n}{\hat{\rho}_n} < \infty, \tag{2.1}$$

which ensures that π is indeed a Lévy measure. Moreover, the sequences $\rho_n, \hat{\rho}_n$ are increasing and satisfy

$$\forall n \geq 1, \quad \min(\rho_n, \hat{\rho}_n) \geq cn^{1+\varepsilon}, \quad \max\left(\frac{\rho_{n+1}}{\rho_n}, \frac{\hat{\rho}_{n+1}}{\hat{\rho}_n}\right) \leq C, \tag{2.2}$$

for some strictly positive constants C, c, ε . It is easy to check that under (2.1), ν is indeed a Lévy measure since $\int_{\mathbb{R}^*} |x| \pi(x) dx < \infty$. In this case, the truncation function is not necessary and we set $h := 0$, which gives

$$\psi(z) = imz - \frac{\sigma^2 z^2}{2} + \sum_{n=1}^{\infty} a_n \left(\frac{\rho_n^2}{(\rho_n - iz)^2} - 1 \right) + \sum_{n=1}^{\infty} \hat{a}_n \left(\frac{\hat{\rho}_n^2}{(\hat{\rho}_n + iz)^2} - 1 \right).$$

In fact, it will be more convenient to work with the Laplace exponent:

$$\begin{aligned} \phi(z) &= \log(\mathbb{E}[e^{zX_1}]) \\ &= mz + \frac{\sigma^2 z^2}{2} + \sum_{n=1}^{\infty} a_n \frac{2\rho_n z - z^2}{(\rho_n - z)^2} - \sum_{n=1}^{\infty} \hat{a}_n \frac{2\hat{\rho}_n z + z^2}{(\hat{\rho}_n + z)^2}. \end{aligned} \tag{2.3}$$

This Laplace exponent is well defined on an open set containing zero because $c > 0$.

Before stating our main result, we introduce the following condition, which is discussed in the Appendix.

$$(*) \quad \begin{cases} \forall j \geq 1, \forall b^2 > 0, \\ \frac{\sigma^2}{2} + \sum_{n=1}^{\infty} a_n \frac{\rho_j^2 - 4\rho_j\rho_n + 3\rho_n^2 + b^2}{((\rho_n - \rho_j)^2 + b^2)^2} + \sum_{n=1}^{\infty} \hat{a}_n \frac{\rho_j^2 + 4\rho_j\hat{\rho}_n + 3\hat{\rho}_n^2 + b^2}{((\hat{\rho}_n + \rho_j)^2 + b^2)^2} > 0, \\ \frac{\sigma^2}{2} + \sum_{n=1}^{\infty} a_n \frac{\hat{\rho}_j^2 - 4\hat{\rho}_j\rho_n + 3\rho_n^2 + b^2}{((\rho_n - \hat{\rho}_j)^2 + b^2)^2} + \sum_{n=1}^{\infty} \hat{a}_n \frac{\hat{\rho}_j^2 + 4\hat{\rho}_j\hat{\rho}_n + 3\hat{\rho}_n^2 + b^2}{((\hat{\rho}_n + \hat{\rho}_j)^2 + b^2)^2} > 0. \end{cases}$$

We are now ready to proceed with our main result, which states that the Wiener-Hopf factorization in this setting has the same form as in [8] and the related literature. We formulate it in a probabilistic fashion using the Laplace transform of S_{e_q} and I_{e_q} .

Theorem 2.1. *For $q, z > 0$, under (2.1), (2.2) and (*),*

$$\mathbb{E}[e^{-zS_{e_q}}] = \frac{1}{1 + \frac{z}{\zeta_0^+}} \prod_{n=1}^{\infty} \frac{1 + \frac{z}{\rho_n}}{1 + \frac{z}{\zeta_n^+}} \frac{1 + \frac{z}{\rho_n}}{1 + \frac{z}{\zeta_n^-}}, \quad (2.4)$$

$$\mathbb{E}[e^{zI_{e_q}}] = \frac{1}{1 - \frac{z}{\zeta_0^-}} \prod_{n=1}^{\infty} \frac{1 + \frac{z}{\hat{\rho}_n}}{1 - \frac{z}{\zeta_n^+}} \frac{1 + \frac{z}{\hat{\rho}_n}}{1 - \frac{z}{\zeta_n^-}}, \quad (2.5)$$

where ζ_0^+ , ζ_n^+ and ζ_n^- (resp. ζ_0^- , $\hat{\zeta}_n^+$ and $\hat{\zeta}_n^-$) are the zeros of $\phi(z) - q$ with positive (resp. negative) real part.

The notation used for the zeros stems from the fact that they go by pairs since, as we shall see, each pole of order two ρ_n and $\hat{\rho}_n$ will engender two roots for $\phi(z) - q$. The proof of the theorem relies heavily on the location of the zeros of $\phi(z) - q$, a topic discussed in the next section.

3. The Location of the Zeros

In [8], Kuznetsov addresses the problem of root localization in two steps: first he finds obvious locations and then proves that they are sufficient (there are no other zeros) using an asymptotic result. Here, we proceed differently: we show that the function has exactly two zeros in an infinite number of well-defined zones, and none outside these zones.

More specifically, we define

$$R_0 = \{z \in \mathbb{C}, -\hat{\rho}_1 < \Re(z) < \rho_1, |\Im(z)| \leq \max(\rho_1, \hat{\rho}_1)\}$$

and the two series of rectangles: $\forall n \geq 1$,

$$R_n = \{z \in \mathbb{C}, \rho_n < \Re(z) < \rho_{n+1}, |\Im(z)| \leq \rho_{n+1}\}$$

$$\hat{R}_n = \{z \in \mathbb{C}, -\hat{\rho}_n > \Re(z) > -\hat{\rho}_{n+1}, |\Im(z)| \leq |\hat{\rho}_{n+1}|\}$$

The main result of the section follows.

Proposition 3.1. *For any $q > 0$, under $(*)$, all the zeros of $\phi(z) - q$ are located in $R_0 \cup \bigcup_{n \geq 1} (R_n \cup \hat{R}_n)$. Moreover, for any $n \geq 1$, R_n and \hat{R}_n both contain exactly 2 zeros or one double zero and so does R_0 .*

The proof of the proposition will require a few intermediate results. We first introduce two test functions, defined for $q_n, \hat{q}_n > 0$,

$$\Phi_n(z) = \frac{1}{(\rho_n - z)^2} + \frac{1}{(\rho_{n+1} - z)^2} - q_n, \quad \hat{\Phi}_n(z) = \frac{1}{(\hat{\rho}_n + z)^2} + \frac{1}{(\hat{\rho}_{n+1} + z)^2} - \hat{q}_n$$

and which have the following properties.

Lemma 3.2. *For all $n \geq 1$, Φ_n (resp $\hat{\Phi}_n$) has only two zeros in R_n (resp \hat{R}_n). Furthermore, the real part of Φ_n (resp $\hat{\Phi}_n$) is strictly negative on ∂R_n (resp $\partial \hat{R}_n$).*

Proof. We will often use the usual complex notation $z = a + ib$. We will prove the lemma for Φ_n since the transposition to $\hat{\Phi}_n$ will be straightforward. First notice that Φ_n has exactly four zeros:

$$z_q^{1,\pm} = \frac{\rho_n + \rho_{n+1}}{2} \pm \frac{\sqrt{q_n \left(4 + q_n(\rho_n - \rho_{n+1})^2 - 4\sqrt{1 + q_n(\rho_n - \rho_{n+1})^2} \right)}}{2q_n}$$

$$z_q^{2,\pm} = \frac{\rho_n + \rho_{n+1}}{2} \pm \frac{\sqrt{q_n \left(4 + q_n(\rho_n - \rho_{n+1})^2 + 4\sqrt{1 + q_n(\rho_n - \rho_{n+1})^2} \right)}}{2q_n}$$

The zeros $z_q^{2,\pm}$ are both real and outside R_n and since $|4 + x - 4\sqrt{1 + x}| \leq x$ for $x \geq 0$, it is obvious that

$$|4 + q_n(\rho_n - \rho_{n+1})^2 - 4\sqrt{1 + q_n(\rho_n - \rho_{n+1})^2}| \leq q_n(\rho_n - \rho_{n+1})^2,$$

hence, the $z_q^{1,\pm}$ belong to R_n and are either both real or both complex with $|\Im(z_q^{1,\pm})| \leq (\rho_{n+1} - \rho_n)/2$.

Moreover,

$$\Re(\Phi_n(a + ib)) = -q_n + \frac{(\rho_n - a)^2 - b^2}{((\rho_n - a)^2 + b^2)^2} + \frac{(\rho_{n+1} - a)^2 - b^2}{((\rho_{n+1} - a)^2 + b^2)^2},$$

so that for $a = \rho_n$ and denoting $\rho = (\rho_n - \rho_{n+1})^2$,

$$\Re(\Phi_n(\rho_n + ib)) = -q_n + \frac{-1}{b^2} + \frac{\rho - b^2}{(\rho + b^2)^2} = -q_n + \frac{-\rho^2 - b^2\rho - 2b^4}{b^2(\rho + b^2)^2} < 0, \quad \forall b \in \mathbb{R}.$$

The proof is the same for $a = \rho_{n+1}$. Lastly, for $b = \pm\rho_{n+1}$,

$$\Re(\Phi_n(a \pm i\rho_{n+1})) = -q_n + \frac{(\rho_n - a)^2 - \rho_{n+1}^2}{((\rho_n - a)^2 + \rho_{n+1}^2)^2} + \frac{(\rho_{n+1} - a)^2 - \rho_{n+1}^2}{((\rho_{n+1} - a)^2 + \rho_{n+1}^2)^2}$$

and both numerators are strictly negative for $a \in (\rho_n, \rho_{n+1})$. \square

The proof of Proposition 3.1 will rely on the following reinforcement, due to Estermann (see [5] p. 156), of Rouché's theorem.

Theorem 3.3 (Estermann-Rouché's Theorem). *Let f and g be two holomorphic functions inside and on some simple contour ∂K . If $|f(z) - g(z)| < |f(z)| + |g(z)|$ on ∂K , then f and g have the same number of zeros (counting multiplicities) inside K .*

Our objective is to apply this theorem to the functions ϕ and Φ_n . The technical assumption (*) will be required to ensure the strict inequality required by the theorem. This technique was already used in a similar context in the proof of the main theorem of [17]. However, we must choose proper contours K_n to proceed. To this purpose, we introduce two series of disks: given their radiuses $\varepsilon_{n,q}, \hat{\varepsilon}_{n,q} > 0$, they are defined for all $n \geq 1$ by

$$D_{n,q} = \{z \in \mathbb{C}, |z - \rho_n| \leq \varepsilon_{n,q}\}, \quad \hat{D}_{n,q} = \{z \in \mathbb{C}, |z - \hat{\rho}_n| \leq \hat{\varepsilon}_{n,q}\}$$

We will need the following result related to these disks.

Lemma 3.4. *For any $n \geq 1$ and $q > 0$, there exist $\varepsilon_{n,q}, \hat{\varepsilon}_{n,q} > 0$ such that*

- i) $\forall z \in D_{n,q}, |\phi(z) - q - \Phi_n(z)| < |\phi(z) - q| + |\Phi_n(z)|$*
- i') $\forall z \in \hat{D}_{n,q}, |\phi(z) - q - \hat{\Phi}_n(z)| < |\phi(z) - q| + |\hat{\Phi}_n(z)|$*
- ii) there are no zeros of $\phi(z) - q$ inside $D_{n,q}$*
- ii') there are no zeros of $\phi(z) - q$ inside $\hat{D}_{n,q}$*

Proof. The proof of both *i)* and *ii)* relies on the fact that as $\varepsilon_{n,q}$ decreases, both ϕ and Φ_n behave like $t(z) = (\rho_n - z)^{-2}$ inside $D_{n,q}$. Indeed, $\phi(z) - a_n \rho_n^2 t(z)$ is bounded inside $D_{n,q}$ and so is $\Phi_n(z) - t(z)$.

We divide $D_{n,q}$ into eight radial subsets of equal size (and angle), as shown in Figure 1. The function t has the following properties:

- in areas $\text{Im}+$, $\Im(t)$ can take arbitrarily large values for $\varepsilon_{n,q}$ small enough
- in areas $\text{Im}-$, $-\Im(t)$ can take arbitrarily large values for $\varepsilon_{n,q}$ small enough
- in areas $\text{Re}+$, $\Re(t)$ can take arbitrarily large values for $\varepsilon_{n,q}$ small enough
- in areas $\text{Re}-$, $-\Re(t)$ can take arbitrarily large values for $\varepsilon_{n,q}$ small enough

First, this means that for $\varepsilon_{n,q}$ small enough, $|\phi(z) - q| > 1$, yielding *ii)*. Moreover, for $\varepsilon_{n,q}$ small enough, there is either $\Im(\phi(z))\Im(\Phi_n(z)) > 0$ or $\Re(\phi(z) - q)\Re(\Phi_n(z)) > 0$ for any $z \in D_{n,q}$. This implies *i)* since $|x - y| = |x| + |y|$ if and only if 0 belongs to the segment $[x, y]$ in the complex plane (both imaginary and real parts must have opposite signs).

We prove *i')* and *ii')* likewise. \square

Lastly, to cover the whole complex plane, we need to show the following lemma.

Lemma 3.5. *For $q > 0$, let $z^* = a + ib$ be a non-real zero of $\phi(z) - q$. Then $|a| \geq \sqrt{3}|b|$.*

Proof. First, we have the following identities

$$\begin{aligned} \frac{2\rho(a + bi) - (a + bi)^2}{(\rho - a - bi)^2} &= \frac{-a^4 + 4a^3\rho - 5a^2\rho^2 - 2a^2b^2 + 2a\rho^3 - 3b^2\rho^2 + 4ab^2\rho - b^4}{((\rho - a)^2 + b^2)^2} \\ &\quad + i \frac{2b\rho^2(\rho - a)}{((\rho - a)^2 + b^2)^2} \end{aligned} \quad (3.1)$$

GUILLAUME COQUERET

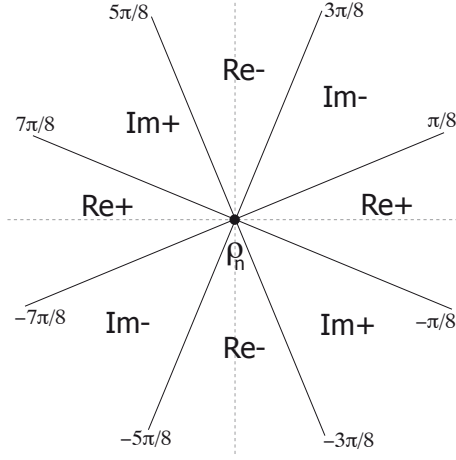


FIGURE 1. The subdivisions of $D_{n,q}$

$$\frac{2\rho(a+bi) + (a+bi)^2}{(\rho+a+bi)^2} = \frac{a^4 + 4a^3\rho + 5a^2\rho^2 + 2a^2b^2 + 2a\rho^3 + 3b^2\rho^2 + 4ab^2\rho + b^4}{((\rho-a)^2 + b^2)^2} + i \frac{2b\rho^2(\rho+a)}{((\rho+a)^2 + b^2)^2} \quad (3.2)$$

which leads to

$$\begin{aligned} \Re(\phi(z^*) - q) &= -q + \frac{\sigma^2}{2}(a^2 - b^2) + am \quad (3.3) \\ &+ \sum_{n=1}^{\infty} a_n \frac{-a^4 + 4a^3\rho_n - 5a^2\rho_n^2 - 2a^2b^2 + 2a\rho_n^3 - 3b^2\rho_n^2 + 4ab^2\rho_n - b^4}{((\rho_n - a)^2 + b^2)^2} \\ &- \sum_{n=1}^{\infty} \hat{a}_n \frac{a^4 + 4a^3\hat{\rho}_n + 5a^2\hat{\rho}_n^2 + 2a^2b^2 + 2a\hat{\rho}_n^3 + 3b^2\hat{\rho}_n^2 + 4ab^2\hat{\rho}_n + b^4}{((\hat{\rho}_n - a)^2 + b^2)^2} \end{aligned}$$

and

$$\Im(\phi(z^*)) = \sigma^2 ab + bm + \sum_{n=1}^{\infty} a_n \frac{2b\rho_n^2(\rho_n - a)}{((\rho_n - a)^2 + b^2)^2} - \sum_{n=1}^{\infty} \hat{a}_n \frac{2b\hat{\rho}_n^2(\hat{\rho}_n + a)}{((\hat{\rho}_n + a)^2 + b^2)^2} \quad (3.4)$$

so that as $b \neq 0$ and $\Im(\phi(z^*)) = \Re((\phi(z^*) - q) = 0$, after simplifications,

$$\begin{aligned} 0 &= \Re(\phi(z^*) - q) - \frac{a}{b} \Im(\phi(z^*)) \\ &= -q - (a^2 + b^2) \frac{\sigma^2}{2} \\ &- (a^2 + b^2) \left(\sum_{n=1}^{\infty} a_n \frac{a^2 - 4a\rho_n + 3\rho_n^2 + b^2}{((\rho_n - a)^2 + b^2)^2} + \sum_{n=1}^{\infty} \hat{a}_n \frac{a^2 + 4a\hat{\rho}_n + 3\hat{\rho}_n^2 + b^2}{((\hat{\rho}_n + a)^2 + b^2)^2} \right) \end{aligned} \quad (3.5)$$

If $a > 0$ and $|a| < \sqrt{3}|b|$, then $\forall n \geq 1$,

$$a^2 - 4a\rho_n + 3\rho_n^2 + b^2 > \frac{4}{3}(a + 3\rho_n/2)^2 \geq 0,$$

and $a^2 + 4a\hat{\rho}_n + 3\hat{\rho}_n^2 + b^2 > 0$ so that z^* cannot be a zero of $\phi(z) - q$. If $a < 0$ and $|a| < \sqrt{3}|b|$, then the contradiction is the same. Hence, for $a \neq 0$, the zero $z^* = a + ib$ must satisfy $|a| \geq \sqrt{3}|b|$.

Lastly, for $a = 0$,

$$\Re(\phi(z^*) - q) = -q - \frac{\sigma^2 b^2}{2} - \sum_{n=1}^{\infty} a_n \frac{3b^2 \rho_n^2 + b^4}{(\rho_n^2 + b^2)^2} - \sum_{n=1}^{\infty} \hat{a}_n \frac{3b^2 \hat{\rho}_n^2 + b^4}{(\hat{\rho}_n^2 + b^2)^2},$$

which is strictly negative for any real b , hence there are no purely imaginary zeros of $\phi(z) - q$. \square

The lemma tells us that there are no zeros of $\phi(z) - q$ in the angles $(\pi/6, 5\pi/6)$ and $(-5\pi/6, -\pi/6)$ of the complex plane. We are now ready to prove the proposition.

Proof of Proposition 3.1. We begin with R_n and \hat{R}_n . The aim of the proof is to show that there are $\varepsilon_{n,q}, \varepsilon_{n+1,q} > 0$ (resp $\hat{\varepsilon}_{n,q}, \hat{\varepsilon}_{n+1,q} > 0$) such that on the boundary of $K_n = R_n \setminus \{D_{n,q} \cup D_{n+1,q}\}$ (see Figure 2) (resp $\hat{K}_n = \hat{R}_n \setminus \{\hat{D}_{n,q} \cup \hat{D}_{n+1,q}\}$), the condition of Rouché's theorem applies, that is,

$$(**) \quad |\phi(z) - q - \Phi_n(z)| < |\phi(z) - q| + |\Phi_n(z)|$$

(resp $|\phi(z) - q - \hat{\Phi}_n(z)| < |\phi(z) - q| + |\hat{\Phi}_n(z)|$). To prove this, we will rely on the following equivalence, for $x, y \in \mathbb{C}$

$$\left. \begin{array}{l} |x - y| = |x| + |y| \\ \text{if and only if} \\ \Re(x)\Im(y) = \Im(x)\Re(y) \text{ and } \Im(x)\Im(y) \leq 0 \text{ and } \Re(x)\Re(y) \leq 0 \end{array} \right\} \quad (3.6)$$

We are first interested in the horizontal and vertical segments of ∂K_n , a set which we denote by S_n (see Figure 2). Recall the expression of $\Re(\phi(a + ib)) - \frac{a}{b}\Im(\phi(a + ib))$ given by (3.5). Because of (*) (vertical segments), and Lemma 3.5 (horizontal segments), we have, for $z = a + ib \in S_n$, $\Re(\phi(a + ib)) - \frac{a}{b}\Im(\phi(a + ib)) < 0$. Hence, near the zeros of $\Im(\phi)$, $\Re(\phi)$ is negative. More precisely, there are two cases involving $\varepsilon_n := \inf\{|\Im(\phi(z))|, z \in S_n\}$:

- either $\varepsilon_n > 0$ and $\Im(\phi)$ has no zero on S_n
- or $\varepsilon_n = 0$ and there exists $\epsilon > 0$ such that $V_{n,q,\epsilon} = \{z \in S_n, |\Im(\phi(z))| < \epsilon, \Re(\phi(z) - q) < 0\} \neq \emptyset$.

In either case, on $S_n \setminus V_{n,q,\epsilon}$, $|\Im(\phi)|$ is bounded from below by, say, $\kappa_{n,q,\epsilon} > 0$.

We want to prove (**) on the following three sets: $V_{n,q,\epsilon}$, $S_n \setminus V_{n,q,\epsilon}$ and $\partial K_n \setminus S_n$ (this last set consisting in the two semi-circles).

- by Lemma 3.2, $\Re(\Phi_n) < 0$ on S_n thus (3.6) ensures that (**) holds on $V_{n,q,\epsilon}$ if it is not empty.

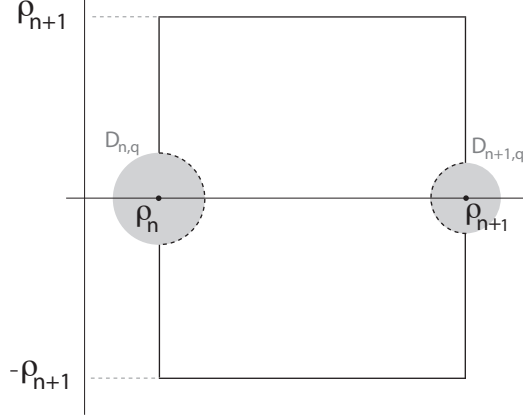


FIGURE 2. the contour K_n with S_n in hard line

- on $S_n \setminus V_{n,q,\epsilon}$, $|\Im(\phi)| \geq \kappa_{n,q,\epsilon} > 0$ and both $\Im(\Phi_n)$ and $\Re(\phi)$ are bounded, it is therefore possible to find a q_n (in the definition of Φ_n) such that

$$|\Re(\phi(z) - q)\Im(\Phi_n(z))| < |\Im(\phi(z))\Re(\Phi_n(z))|, \quad z \in S_n \setminus V_{n,q},$$

hence, by (3.6), $(**)$ holds on $S_n \setminus V_{n,q}$.

- for the two semi-circles of ∂K_n , we invoke Lemma 3.4. The radiuses must be chosen small enough for the zeros of Φ_n to be in K_n .

Therefore, by Rouché's theorem, for any $q > 0$, under $(*)$, the function $\phi(z) - q$ has exactly 2 zeros (or a double zero) in R_n .

The proof is identical for the sets \hat{R}_n . For R_0 , it is easy to see that the properties of Φ_n and $\hat{\Phi}_n$ described in Lemma 3.2 also hold for $\Phi_0 = \frac{1}{(\rho_1+z)^2} + \frac{1}{(\rho_1-z)^2} - q_0$ on ∂R_0 and hence the same reasoning applies (with the proper K_0, S_0 and $V_{0,q,\epsilon}$).

Lastly, the 3 sets were constructed so that with Lemma 3.5, the whole complex plane is covered. \square

We denote by ζ_n^+ and ζ_n^- the two roots in R_n . If they are complex then $\Im(\zeta_n^+) > \Im(\zeta_n^-)$, if not, then $\Re(\zeta_n^+) \geq \Re(\zeta_n^-)$. The equivalent notations hold for R_0 and \hat{R}_n .

4. Proof and Discussion of the Theorem

This section is divided into three parts. First, we prove Theorem 2.1, using Proposition 3.1 and Kuznetsov's paper [11]. Then we discuss a possible generalization when the poles of ϕ are allowed to have any finite order. Lastly, we introduce a simple condition which implies $(*)$.

4.1. Proof of Theorem 2.1. The proof will rely on the following lemma. We denote by \log the principal branch of the complex logarithm defined on $\mathbb{C} \setminus \mathbb{R}_-$.

Lemma 4.1. Define $A^\pm(z) = \left| \frac{1}{z} \log \left(\prod_{n=1}^{\infty} \frac{\zeta_n^\pm(\rho_n - z)}{\rho_n(\zeta_n^\pm - z)} \right) \right|$ and

$$\hat{A}^\pm(z) = \left| \frac{1}{z} \log \left(\prod_{n=1}^{\infty} \frac{\hat{\zeta}_n^\pm(\hat{\rho}_n + z)}{\hat{\rho}_n(\hat{\zeta}_n^\pm - z)} \right) \right|.$$

Then

$$A^\pm(z) \text{ (resp. } \hat{A}^\pm(z)) \rightarrow 0, \quad \text{as } |z| \rightarrow \infty, \quad \Re(z) \leq 0 \text{ (resp. } \Re(z) \geq 0).$$

Proof. First note that by Proposition 3.1 and (2.2),

$$\left| \frac{\zeta_n^\pm(\rho_n - z)}{\rho_n(\zeta_n^\pm - z)} - 1 \right| = \left| z \frac{\rho_n - \zeta_n^+}{\rho_n(\zeta_n^+ - z)} \right| = O(n^{-1-\varepsilon}),$$

so that the infinite product is well defined. Again by Proposition 3.1 and (2.2), $|\zeta_n^+/\rho_n| > 1$. Moreover, $\left| \frac{\rho_n - z}{\zeta_n^+ - z} \right| \rightarrow 1$ as $|z| \rightarrow \infty$, $\Re(z) \leq 0$. The function $z \mapsto \left| \frac{\zeta_n^+(\rho_n - z)}{\rho_n(\zeta_n^+ - z)} \right|$ is thus bounded from below for $\Re(z) \leq 0$ and there is $C > 0$ such that for $\Re(z) \leq 0$ and $n \geq 1$,

$$\left| \log \left(\frac{\zeta_n^+(\rho_n - z)}{\rho_n(\zeta_n^+ - z)} \right) \right| \leq C \left| \frac{\zeta_n^+(\rho_n - z)}{\rho_n(\zeta_n^+ - z)} - 1 \right|.$$

Hence,

$$A^+(z) \leq \left| \sum_{n=1}^{\infty} \frac{\rho_n - \zeta_n^+}{\rho_n(\zeta_n^+ - z)} \right| = O \left(\sum_{n=1}^{\infty} \frac{1}{|\zeta_n^+ - z|} \right).$$

Proposition 3.1 and (2.2) ensure that the infinite sum is finite for any z with negative real part and by dominated convergence, $A^+(z) \rightarrow 0$ as $|z| \rightarrow \infty$.

The proof for A^- , \hat{A}^\pm is achieved in a similar fashion. \square

Proof of Theorem 2.1. First note that due to condition (2.2) and Proposition 3.1,

$\sum_{n=1}^{\infty} \frac{1}{|\zeta_n^\pm|} < \infty$, so that the order of the entire function

$$(q - \phi(z)) \prod_{n=1}^{\infty} \left(1 - \frac{z}{\rho_n} \right)^2 \prod_{n=1}^{\infty} \left(1 + \frac{z}{\hat{\rho}_n} \right)^2$$

is less than one (see [16], lecture 5, for instance). Hence, the only possible Hadamard/Weierstrass representation ([16] page 26) for $\frac{q}{q - \phi(z)}$ is

$$\frac{q}{q - \phi(z)} = e^{cz} \frac{1}{1 - \frac{z}{\zeta_0^+}} \frac{1}{1 - \frac{z}{\zeta_0^-}} \prod_{n=1}^{\infty} \frac{1 - \frac{z}{\rho_n}}{1 - \frac{z}{\zeta_n^+}} \frac{1 - \frac{z}{\rho_n}}{1 - \frac{z}{\zeta_n^-}} \prod_{n=1}^{\infty} \frac{1 + \frac{z}{\hat{\rho}_n}}{1 - \frac{z}{\zeta_n^+}} \frac{1 + \frac{z}{\hat{\rho}_n}}{1 - \frac{z}{\zeta_n^-}}, \quad (4.1)$$

for some $c \in \mathbb{C}$ after arrangement.

We rely on Lemma 4.1 to prove that $c = 0$ (in the same way as in the end of the proof of Theorem 5 in [8], using the fact that $\phi(iz) = O(z^2)$ for $z \rightarrow \infty$, $z \in \mathbb{R}$). Lastly, all the conditions of Theorem 1 (f) from [11] are fulfilled, which completes the proof. \square

4.2. Towards a generalization. It is natural to ask what might happen if the density of the Lévy measure had the more general form

$$\begin{aligned} \pi(x) = & \mathbb{1}_{(x>0)} \sum_{n=1}^{\infty} a_n \frac{\rho_n^{m_n}}{(m_n - 1)!} x^{m_n-1} e^{-\rho_n x} \\ & + \mathbb{1}_{(x<0)} \sum_{n=1}^{\infty} \hat{a}_n \frac{\hat{\rho}_n^{\hat{m}_n}}{(\hat{m}_n - 1)!} (-x)^{\hat{m}_n-1} e^{\hat{\rho}_n x}, \end{aligned}$$

where $(a_n, \hat{a}_n, \rho_n, \hat{\rho}_n)$ satisfy the usual conditions and $(m_n, \hat{m}_n) \in \{1, \dots, M\}$, $(M < \infty)$, which would give

$$\begin{aligned} \phi(z) = & mz + \frac{\sigma^2 z^2}{2} + \sum_{n=1}^{\infty} a_n \left(\frac{\rho_n^{m_n}}{(\rho_n - z)^{m_n}} - 1 - \frac{zm_n}{\rho_n} \right) \\ & + \sum_{n=1}^{\infty} \hat{a}_n \left(\frac{\hat{\rho}_n^{\hat{m}_n}}{(\hat{\rho}_n + z)^{\hat{m}_n}} - 1 + \frac{z\hat{m}_n}{\hat{\rho}_n} \right). \end{aligned}$$

The case $M = 2$ can be treated using the ideas of the present paper (with a slightly different condition $(*)$ and four pairs of test functions). The central problem remains to get precise results on the location of the zeros of $\phi(z) - q$.

In the general case, finding a multiple zero is not easy and we cannot apply Theorem 2.1. However, heuristically, if we consider $q \rightarrow \infty$, we see that, asymptotically, the $(\rho_n, -\hat{\rho}_n)$ become zeros of order m_n and \hat{m}_n . Hence, when q decreases, the zeros should be located in the vicinity, in some sense, of the ρ_n and $-\hat{\rho}_n$. Empirically, many computations show that, in fact, each ρ_n engenders m_n zeros, all of which are located inside circles with center ρ_n , radius ρ_n and all the more close to ρ_n than q is large. However, this fact (which requires formal proof) does not suffice to show the convergence $A(z) \rightarrow 0$ in the proof of the theorem.

In this spirit, we would like expose a set of conditions under which the theorem remains valid. Namely,

Theorem 4.2. *If there is an ordering of the zeros and poles, repeated according to multiplicity, such that*

- i) $\phi(z) - q$ is meromorphic with real poles $\rho_n, -\hat{\rho}_n$ and zeros $\zeta_n, \hat{\zeta}_n$ satisfying $\Re(\zeta_n) > 0, \Re(\hat{\zeta}_n) < 0$,
 - ii) the series with terms $\frac{1}{\rho_n}, \frac{1}{\hat{\rho}_n}, \frac{1}{\zeta_n}, \frac{1}{\hat{\zeta}_n}$ are absolutely convergent,
 - iii) $\forall n > 0, |\zeta_n - \rho_n| < C|\rho_n|, |\hat{\zeta}_n - \hat{\rho}_n| < \hat{C}|\hat{\rho}_n|$ for some $C, \hat{C} > 0$,
- then (2.4) and (2.5) hold.

Proof. Condition ii) ensures that $q/(\phi(z) - q)$ is the ratio of two entire functions of order less than one and has thus a similar form as (4.1), namely

$$\frac{q}{q - \phi(z)} = e^{cz} \frac{h_+(z)}{g_+(z)} \frac{h_-(z)}{g_-(z)},$$

where h_{\pm}, g_{\pm} are holomorphic in \mathbb{C} , with zeros in $\mathbb{C}^{\pm} = \{z \in \mathbb{C}, \pm \Re(z) > 0\}$ normalized so that $h_{\pm}(0) = g_{\pm}(0) = 1$. Then, as in the proof of Theorem 2.1,

condition *iii*) will ensure that $z^{-1} \log \left(\frac{h_{\pm}(z)}{g_{\pm}(z)} \right) \rightarrow 0$ as $z \rightarrow \mp\infty$, so that the factorization is indeed of Wiener-Hopf type (and $c = 0$). \square

Remark 4.3. We wish to show the connexion between the papers on meromorphic Lévy-Khintchine exponents and the ideas in [6]. For simplicity, we will consider $q \in \mathcal{Q} := \{q > 0, \phi(z) - q \text{ has only simple zeros}\}$. Once the zeros are located, Theorem 1 in [9] is a special case of Theorem 4.1 in [6]. It is however possible to consider complex zeros, as long as the proof of Lemma 3.1 in [6] remains valid. This can only happen if the arguments of the zeros stay away from $\pi/2$ and $-\pi/2$. Condition *iii*) in Theorem 4.2 implies this and it is verified in [8], [9] and [13], as long as the $(\rho_n, \hat{\rho}_n)$ in those papers have at most an exponential growth. It is thus possible to very briefly prove Theorem 4.2 using the fact that under *i*), *ii*) and *iii*),

$$\sum_{n=1}^{\infty} \frac{h_+(\zeta_n)}{g'_+(\zeta_n)} e^{-x\zeta_n}, \quad \sum_{n=1}^{\infty} \frac{h_-(\hat{\zeta}_n)}{g'_-(\hat{\zeta}_n)} e^{-x\hat{\zeta}_n}$$

are convergent for $x > 0$ and can then be proven to be the densities of S_{e_q} and I_{e_q} using dominated convergence, as in [6], Theorem 4.1 (using Lemma 3.2).

5. An Example

5.1. Introducing the Laplace Exponent. Our goal in this section is to be able to locate precisely the zeros of ϕ for one exponent involving known functions. We will show that even in this very simple case, this is not so easy to achieve. We set $\rho_n = \hat{\rho}_n = n^2$ and unit a_n and \hat{a}_n , and propose the following Lévy measure,

$$\pi(x) = \mathbb{1}_{\{x>0\}} \sum_{n=1}^{\infty} x n^4 e^{-n^2 x} + \mathbb{1}_{\{x<0\}} \sum_{n=1}^{\infty} (-x) n^4 e^{n^2 x}.$$

First note that in this case, as is shown at the very end of the Appendix, (*) holds. Moreover, it is easy to show (see Proposition 4 in [9]) that as $x \rightarrow 0^{\pm}$, $\pi(x) = O(|x|^{-3/2})$, which confirms that we can take a zero cut-off function ($h := 0$) and the following representation for ϕ :

$$\phi(z) = mz + \frac{\sigma^2 z^2}{2} + \sum_{n=1}^{\infty} \frac{2zn^2 - z^2}{(n^2 - z)^2} - \sum_{n=1}^{\infty} \frac{2zn^2 + z^2}{(n^2 + z)^2}.$$

This particular choice of ϕ was made to exhibit known functions. Using the fact that

$$\frac{2zn^2 - z^2}{(n^2 - z)^2} = \frac{1}{4} \left(\frac{-6z}{z - n^2} + 2 \frac{z^2 + zn^2}{(z - n^2)^2} \right)$$

and relations 1.421-3, 1.422-4 (second equality) in [7], we get

$$\sum_{n=1}^{\infty} \frac{2n^2 z - z^2}{(n^2 - z)^2} = \frac{1}{4} (2 - 3\pi\sqrt{z} \cot(\pi\sqrt{z}) + \pi^2 z \csc(\pi\sqrt{z})^2)$$

and substituting $i\sqrt{z}$ for \sqrt{z} ,

$$- \sum_{n=1}^{\infty} \frac{2n^2 z + z^2}{(n^2 + z)^2} = \frac{1}{4} (2 - i3\pi\sqrt{z} \cot(i\pi\sqrt{z}) - \pi^2 z \csc(i\pi\sqrt{z})^2),$$

where \cot and \csc are the usual cotangent and cosecant functions. Recalling

$$\csc(z)^2 = \sin(z)^{-2} = \frac{\cos(z)^2 + \sin(z)^2}{\sin(z)^2} = 1 + \cot(z)^2, \quad (5.1)$$

ϕ can in this case be expressed solely in terms of the cotangent function

$$\begin{aligned} \phi(z) &= mz + \frac{\sigma^2}{2}z^2 + 1 + \frac{1}{4} [\pi\sqrt{z} \cot(\pi\sqrt{z})(-3 + \pi\sqrt{z} \cot(\pi\sqrt{z}))] \\ &\quad + \frac{i}{4} [\pi\sqrt{z} \cot(i\pi\sqrt{z})(-3 + i\pi\sqrt{z} \cot(i\pi\sqrt{z}))] \\ &:= mz + \frac{\sigma^2 z^2}{2} + 1 + \frac{1}{4}(c_+(z) + c_-(z)). \end{aligned}$$

We refer to section 4.3 of [1] for the behavior of the cotangent function in the complex plane. One of its useful properties is that if z has a large imaginary part, then both the real and imaginary parts of $\cot(z)$ can be accurately estimated. More precisely, by Euler's formula and equation 4.3.58 in [1], for any $a_n > 0$ and for any $b_n > 0$ large enough,

$$\begin{aligned} \cot(a_n \pm ib_n) &= \cot(-a_n \pm ib_n) \\ &= 2 \sin(2a) e^{-2b_n} + i(\mp 1 + 2 \cos(2a) e^{-2b_n}) + o(e^{-2b_n})(1 + i). \end{aligned} \quad (5.2)$$

Lastly, recall that the zeros $\zeta_n^\pm = a_n \pm ib_n$ of $\phi(z) - q$ are ordered so that the series $(a_n)_{\{n \geq 1\}}$ is increasing.

Lemma 5.1. *There are, asymptotically, no real zeros of $\phi(z) - q$, except if $\sigma = 0$ and $m \geq \pi^2/4$ (resp $m \leq -\pi^2/4$), in which case, $\hat{\zeta}_n^\pm$ (resp ζ_n^\pm) are real.*

Proof. The proof lies in the fact that, by (5.2), for z real large enough, $c_-(z) = c_+(-z) \sim (-3\pi\sqrt{z} + \pi^2 z)/4$. Moreover, both c_+ and c_- are U -shaped between their poles with a negative local minimum which is close to -0.5 . The local minima of ϕ thus go to $+\infty$ (yielding only complex zeros) or to $-\infty$ (yielding only real zeros). \square

5.2. Locating the zeros. We will now focus on ζ_n^\pm , as the transposition to $\hat{\zeta}_n^\pm$ is straightforward. Note that ζ_n^\pm verifies

$$\begin{aligned} \frac{\sigma^2}{2}(\zeta_n^\pm)^2 + m\zeta_n^\pm + 1 - q + \frac{1}{4} \left[-3i\pi\sqrt{\zeta_n^\pm} \cot\left(i\pi\sqrt{\zeta_n^\pm}\right) - \pi^2\zeta_n^\pm \cot\left(i\pi\sqrt{\zeta_n^\pm}\right)^2 \right] \\ = -\frac{1}{4} \left[-3\pi\sqrt{\zeta_n^\pm} \cot\left(\pi\sqrt{\zeta_n^\pm}\right) + \pi^2\zeta_n^\pm \cot\left(\pi\sqrt{\zeta_n^\pm}\right)^2 \right]. \end{aligned} \quad (5.3)$$

For ζ_n^\pm away from the zeros of $\Re(\cot(\pi\sqrt{z}))$, dividing (5.3) by ζ_n^\pm yields

$$\Im \left(\frac{3\pi \cot\left(\pi\sqrt{\zeta_n^\pm}\right)}{4\sqrt{\zeta_n^\pm}} - \frac{\pi^2}{4} \cot\left(\pi\sqrt{\zeta_n^\pm}\right)^2 \right) = \pm \frac{\sigma^2}{2} b_n + O(\Im((\zeta_n^\pm)^{-1})), \quad n \rightarrow +\infty \quad (5.4)$$

and

$$\Re \left(-\frac{\pi^2}{4} \cot \left(\pi \sqrt{\zeta_n^\pm} \right)^2 \right) = \frac{\sigma^2}{2} a_n + m + \frac{\pi^2}{4} + R(n) + o(R(n)), \quad n \rightarrow +\infty \quad (5.5)$$

where $R(n) = -\frac{3\pi}{4} \Re \left((\zeta_n^\pm)^{-1/2} \left[\cot \left(\pi \sqrt{\zeta_n^\pm} \right) + i \cot \left(i\pi \sqrt{\zeta_n^\pm} \right) \right] \right)$.

We are now able to accurately locate the ζ_n^\pm , asymptotically, in all possible cases.

Proposition 5.2. *As $n \rightarrow +\infty$,*

- *if $\sigma > 0$, $\zeta_n^\pm = n^2 \pm i\sqrt{2/\sigma^2} + O(n^{-2})(1+i)$*
- *if $\sigma = 0$ and $m > 0$, then*

$$\zeta_n^\pm = n^2 + c + O(n^{-1}) \pm i\frac{n}{\pi} \left(\cosh^{-1} \left(1 + \frac{\pi^2}{2m} \right) + O(n^{-1}) \right)$$

- *if $\sigma = 0$ and $m \in (-\pi^2/4, 0)$, then*

$$\zeta_n^\pm = n^2 + n + c + O(n^{-1}) + \pm i\frac{n}{\pi} \left(\cosh^{-1} \left(-1 - \frac{\pi^2}{2m} \right) + O(n^{-1}) \right)$$

- *if $\sigma = 0$ and $m = 0$,*

$$\Re(\sqrt{\zeta_n^\pm}) = n + 3/8 + O(\log(n)n^{-2}) \text{ and } \frac{2\pi \Im(\sqrt{\zeta_n^\pm})}{\log(\frac{4\pi}{3\sqrt{2}}n)} \rightarrow 1$$

- *if $\sigma = 0$ and $m \leq -\pi^2/4$, then*

$$\zeta_n^- = n^2 + \frac{n}{\pi} \left(\cos^{-1} \left(1 + \frac{\pi^2}{2m} \right) + cn^{-1} + o(n^{-1}) \right)$$

and

$$\zeta_n^+ = (n+1)^2 - \frac{n}{\pi} \left(\cos^{-1} \left(1 + \frac{\pi^2}{2m} \right) + cn^{-1} + o(n^{-1}) \right),$$

for some irrelevant constants c , and where \cos^{-1} and \cosh^{-1} are defined on $[-1, 1]$ and $[1, +\infty)$ respectively.

Proof. The proof being quite lengthy, some minor steps and details will be omitted.

Since the zeros will be located in the vicinity, in some sense, of the poles of ϕ , we will keep the following notation throughout the proof

$$\sqrt{\zeta_n^\pm} = n + d_n \pm i\frac{b_n}{2(n+d_n)}.$$

The periodicity of the cotangent function in the real variable yields

$$\begin{aligned}\cot\left(\pi\sqrt{\zeta_n^\pm}\right) &= \cot\left(\pi d_n \pm i\frac{\pi b_n}{2(n+d_n)}\right) \\ &= \cot\left(\pi(2d_n^2 + 2nd_n \pm ib_n)\frac{1}{2(n+d_n)}\right).\end{aligned}\quad (5.6)$$

If $\sigma > 0$, then by (5.5), $b_n/n \rightarrow 0$ and $d_n \rightarrow 0$ (if not the real part on the l.h.s. would be bounded).

Using the following series expansions for $z \rightarrow 0$, (4.3.70 in [1])

$$\cot(\pi(a+ib)z)^2 = (\pi z(a+ib))^{-2} - 2/3 + O(((a+ib)z)^2),$$

we have the following asymptotics as $n \rightarrow +\infty$,

$$\begin{aligned}-\frac{\pi^2}{4}\cot\left(\pi\sqrt{\zeta_n^\pm}\right)^2 &= -\frac{(n+d_n)^2}{(2d_n^2 + 2nd_n \pm ib_n)^2} + \pi^2/6 + o(1)(1+i) \\ &= -(n+d_n)^2\frac{4(d_n^2 + nd_n)^2 - b_n^2}{(4(nd_n + d_n^2)^2 + b_n^2)^2} + \pi^2/6 \\ &\quad \pm i(n+d_n)^2\frac{4(nd_n + d_n^2)b_n}{(4(nd_n + d_n^2)^2 + b_n^2)^2} + o(1)(1+i) \\ &= n^2\frac{b_n^2 - 4n^2d_n^2}{(4n^2d_n^2 + b_n^2)^2} + \frac{\pi^2}{6} \\ &\quad \pm in^2\frac{4nd_nb_n}{(4n^2d_n^2 + b_n^2)^2} + o(1)(1+i).\end{aligned}\quad (5.7)$$

From (5.5) and (5.4), it follows that

$$n^2\frac{b_n^2 - 4n^2d_n^2}{(4n^2d_n^2 + b_n^2)^2} = \frac{\sigma^2}{2}n^2 + o(n^2), \quad n \rightarrow +\infty \quad (5.8)$$

$$\pm n^2\frac{4nd_nb_n}{(4nd_n^2 + b_n^2)^2} = \pm\frac{\sigma^2}{2}b_n + o(1), \quad n \rightarrow +\infty. \quad (5.9)$$

Because $\sigma > 0$, (5.8) imposes that both b_n and nd_n do not diverge. Therefore, the l.h.s. of (5.9) implies that either b_n or nd_n goes to 0 and because the r.h.s. in (5.8) is positive, then it must be $nd_n \rightarrow 0$, which gives $b_n \rightarrow \sqrt{2/\sigma^2}$. With (5.9), this yields $d_n = O(n^{-3})$. All these facts imply that the $o(n^2)$ in (5.8) (which stems from (5.5)) is in fact a $O(1)$, which completes the proof.

If $\sigma = 0$ and $m \neq 0$, then using the eulerian representation of the cotangent function (see 4.3.58 in [1] for instance), (5.4) and (5.5) are rewritten into

$$\frac{\sinh\left(\frac{\pi b_n}{n+d_n}\right)^2 - \sin(2\pi d_n)^2}{\left(\cosh\left(\frac{\pi b_n}{n+d_n}\right) - \cos(2\pi d_n)\right)^2} = 1 + \frac{4m}{\pi^2} + \frac{4}{\pi^2}R(n) + o(R(n)) \quad (5.10)$$

and

$$\begin{aligned}
 & \pm \frac{2\pi \sinh\left(\frac{\pi b_n}{n+d_n}\right) \sin(2\pi d_n)}{\left(\cosh\left(\frac{\pi b_n}{n+d_n}\right) - \cos(2\pi d_n)\right)^2} \\
 & - \frac{3\left(\frac{b_n}{2(n+d_n)} \sin(2\pi d_n) \pm (n+d_n) \sinh\left(\frac{\pi b_n}{n+d_n}\right)\right)}{\left((n+d_n)^2 + \frac{b_n^2}{4(n+d_n)^2}\right)\left(\cosh\left(\frac{\pi b_n}{n+d_n}\right) - \cos(2\pi d_n)\right)} \\
 & = O(\Im((\zeta_n^\pm)^{-1})), \tag{5.11}
 \end{aligned}$$

where both $R(n)$ and $O(\Im((\zeta_n^\pm)^{-1}))$ converge to 0.

Notice that for $b_n/n \rightarrow +\infty$, the l.h.s. of (5.10) converges to 1, thus b_n/n must be bounded (since $m \neq 0$). Hence, because 0 is a pole for $\Im(\cot(z))$, (5.11) imposes that either $\sin(2\pi d_n)$ or b_n/n goes to 0. In fact, because of the positivity of (5.10), $\sin(2\pi d_n) \rightarrow 0$ while $b_n = O(n)$. Note that this gives $\Im((\zeta_n^\pm)^{-1}) = O(n^{-3})$ and $R(n) = O(n^{-1})$. Equation (5.10) then imposes $d_n \rightarrow 0$ if $m > 0$ and $d_n \rightarrow 1/2$ if $m < 0$. For $m > 0$, it can then be rewritten into

$$\frac{\cosh\left(\frac{\pi b_n}{n+d_n}\right)^2 - 1}{\left(\cosh\left(\frac{\pi b_n}{n+d_n}\right) - 1\right)^2} = \frac{\cosh\left(\frac{\pi b_n}{n+d_n}\right) + 1}{\cosh\left(\frac{\pi b_n}{n+d_n}\right) - 1} = 1 + \frac{4m}{\pi^2} + O(n^{-1}),$$

thereby yielding the constant in the imaginary part. Using the Taylor expansion of the sinus function at 0, (5.11) implies $d_n = (d + \delta_n)/n$ and is simplified into

$$n^{-1} \left(\pm \frac{4\pi^2(d + \delta_n)}{\cosh\left(\frac{\pi b_n}{n+d_n}\right) - 1} \mp 3 + O(n^{-2}) \right) = O(n^{-3}),$$

from which d and $\delta_n = O(n^{-2})$ can be inferred. Writing $b_n = (b + \beta_n)n$ for $\beta_n \rightarrow 0$, and recalling the expansion

$$\sinh(\pi(b_n + \beta_n)) = \sinh(\pi b_n) + \pi\beta_n \cosh(\pi b_n) + o(\beta_n)$$

implies that $\beta_n = O(n^{-1})$, by (5.10). Note that in this case, the constant c in the proposition depends on d and b . The case $m \in (-\pi^2/4, 0)$ is treated similarly.

The case $\sigma = 0$ and $m = 0$ is very special as, by (5.10), it is in fact necessary that $b_n/n \rightarrow +\infty$. More precisely, (5.11) yields $\frac{\pi b_n}{n \log(\delta n)} \rightarrow 1$ for some constant $\delta > 0$ which verifies

$$\frac{4\pi \sin(2\pi d_n)}{\delta n} - 3 \left(\frac{\log(\delta n) \sin(2\pi d_n)}{\delta n^3} + \frac{1}{n} \right) = O(\log(n)n^{-3}).$$

From the Taylor expansion of the sinus function at 0 and the fact that $\sin(a + b) = \sin(a)\cos(b) + \cos(a)\sin(b)$, we can deduce that $d_n = d + O(\log(n)n^{-2})$ with $4\pi \sin(2\pi d) = 3\delta$. Because $b_n/n \rightarrow +\infty$, $|\Im((\zeta_n^\pm)^{1/2})| \rightarrow +\infty$ and hence, by (5.2), $R(n) = -3\pi/4n^{-1} + o(n^{-1})$. As $\sinh\left(\frac{\pi b_n}{n+d_n}\right)^2 \sim \cosh\left(\frac{\pi b_n}{n+d_n}\right)^2 \sim \delta n/2$, $n \rightarrow +\infty$,

it can be deduced from (5.10) that

$$\frac{4 \cos(2\pi d_n)}{\delta n} + O(n^{-2}) = -\frac{3}{\pi n} + o(n^{-1}),$$

from which we infer $d = 3/8$ and $\delta = \frac{4\pi}{3} \sin(3\pi/4)$.

Lastly, **if** $\sigma = 0$ **and** $m < \pi^2/4$, writing $\sqrt{\zeta_n^-} = n + d_n$ and again using the eulerian form of the cotangent function and (5.5),

$$\begin{aligned} -\frac{\sin(2\pi d_n)^2}{(\cos(2\pi d_n) - 1)^2} &= \frac{\cos(2\pi d_n) + 1}{\cos(2\pi d_n) - 1} \\ &= 1 + \frac{4m}{\pi^2} + \frac{3}{\pi} \left[1 + \frac{\sin(2\pi d_n)}{\cos(2\pi d_n) - 1} \right] n^{-1} + o(n^{-1}), \end{aligned}$$

which gives $d_n \rightarrow d = \cos^{-1} \left(1 + \frac{\pi^2}{2m} \right) / 2\pi$. Recalling $\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$ yields $d_n = d + cn^{-1} + o(n^{-1})$. The same method holds for $\sqrt{\zeta_{n-1}^+} = n - d_n$. \square

5.3. Laplace transform of the first passage time. This result enables us to compute interesting fluctuations quantities, such as the Laplace transform of the first passage time of X : $T_x = \inf\{t \geq 0, X_t \geq x\}$. Indeed, since $P[S_t \geq x] = P[T_x \leq t]$ it is easy to show that

$$P[S_{e_q} \geq x] = \int_0^1 P[e^{-qT_x} \geq y] dy = \mathbb{E}[e^{-qT_x}] := h_q(x).$$

We aim at providing a graph of h_q for some fixed values of q , m and σ^2 . Using residues, it is possible to perform a Laplace transform inversion on the Wiener-Hopf factors to get a series representation of the law of S_{e_q} (see Theorem 1 in [9] for instance):

$$\frac{d}{dx} P[S_{e_q} \leq x] = \sum_{n=1}^{\infty} \left[c_n^+ \zeta_n^+ e^{-\zeta_n^+ x} + c_n^- \zeta_n^- e^{-\zeta_n^- x} \right] + c_0 \zeta_0^+ e^{-\zeta_0^+ x}, \quad x > 0, \quad (5.12)$$

$$\text{where } c_n^\pm = \frac{1 - \frac{\zeta_n^\pm}{n^2}}{1 - \frac{\zeta_n^\pm}{\zeta_0^\pm}} \prod_{\substack{k \geq 1 \\ k \neq n}} \frac{1 - \frac{\zeta_k^\pm}{k^2}}{1 - \frac{\zeta_k^\pm}{\zeta_k^\pm}}, \quad c_0 = \prod_{k \geq 1} \frac{1 - \frac{\zeta_0^+}{k^2}}{1 - \frac{\zeta_0^+}{\zeta_k^+}} \frac{1 - \frac{\zeta_0^-}{k^2}}{1 - \frac{\zeta_0^-}{\zeta_k^-}},$$

$$\text{and } P[S_{e_q} = 0] = \zeta_0^+ \prod_{k \geq 1} \frac{\zeta_k^+}{k^2} \frac{\zeta_k^-}{(k+1)^2} = \mathbb{E}[e^{-qT_{0+}}],$$

with $T_{0+} = \inf\{t \geq 0, X_t > 0\}$. The Laplace transform of this latter random variable is not equal to 1 whenever 0 is not regular for $(0, \infty)$ (see chapter 6 in [15] for further details).

We start by providing a sample of the the locations of the roots with positive real and imaginary parts for the following parametrization: $\sigma^2/2 \in \{0, 1\}$, $m = 0$ and $q = 1$:

| | Case $\sigma^2/2 = 1$ | Case $\sigma^2 = 0$ |
|----|-----------------------|---------------------|
| n | ζ_n^+ | |
| 0 | 0.4431 | 0.4596 |
| 1 | 1.5284+0.4173i | 1.800+0.273i |
| 2 | 4.2785+0.9257i | 5.4705+1.2401i |
| 3 | 9.1176+0.9813i | 11.1627+2.2600i |
| 4 | 16.0638+0.9884i | 18.866+3.3642i |
| 5 | 25.0403+0.9914i | 28.5772+4.5378i |
| 6 | 36.0278+0.9933i | 40.2934+5.7690i |
| 7 | 49.0204+0.9947i | 54.0134+7.0492i |
| 8 | 64.0156+0.9957i | 69.7364+8.3720i |
| 9 | 81.0123+0.9964i | 87.4620+9.7321i |
| 10 | 100.0100+0.9971i | 107.1894+11.1263i |
| ⋮ | ⋮ | ⋮ |
| 50 | 2500.0004+0.9998i | 2536.771+80.085i |

The last line of the table is coherent with the asymptotic results of Proposition 5.2. Integrating (5.12) and taking $x = 0$ yields that

$$\sum_{n \geq 0} c_n = 1 - P[S_{e_q}],$$

from which we infer that the c_n are bounded (in fact , their moduli decrease very rapidly). Hence, it is possible to compute h_q using a finite number of terms, even for x close to zero. We provide below the graphs for the cases $\sigma^2/2 \in \{0, 1\}$ and $q \in \{1, 2, 3\}$ in figures 3 and 4.

Appendix A. Some Remarks on the Condition of the Theorem

It is easy to find examples for which (*) fails. However, for many cases when a_n, \hat{a}_n and $\rho_n, \hat{\rho}_n$ are expressed using basic functions (exponential, power), (*) will in fact hold. We provide an example below.

We split (*) into its two inequalities: the upper (*₁) and the lower (*₂), and we will only comment on (*₁) because the transposition to (*₂) will be immediate. We

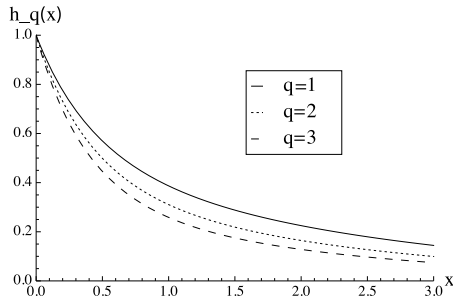


FIGURE 3. Plot of the function h_q for $\sigma^2 = 2$ and $m = 0$

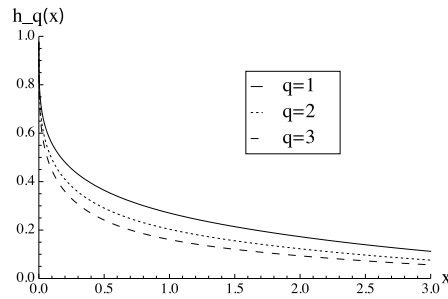


FIGURE 4. Plot of the function h_q for $\sigma^2 = 0$ and $m = 0$

are in fact only interested in the term $T(\rho_j, b) = \sum_{n=1}^{\infty} a_n \frac{(\rho_j - 3\rho_n)(\rho_j - \rho_n) + b^2}{((\rho_n - \rho_j)^2 + b^2)^2}$, since it is the only one which can be negative. Furthermore, if we consider $l(j) = \sup\{k \geq 1, 3\rho_{j-k} \geq \rho_j\}$ (with $l(j) = 0$ if $3\rho_{j-1} \leq \rho_j$), we have

$$\begin{aligned} T(\rho_j, b) &> \sum_{k=1}^{l(j)} \left[\frac{a_j}{b^2 l(j)} + a_{j+k} \frac{(\rho_{j+k} - \rho_j)(3\rho_{j+k} - \rho_j) + b^2}{((\rho_{j+k} - \rho_j)^2 + b^2)^2} \right. \\ &\quad \left. + a_{j-k} \frac{(\rho_{j-k} - \rho_j)(3\rho_{j-k} - \rho_j) + b^2}{((\rho_{j-k} - \rho_j)^2 + b^2)^2} \right] \quad (\text{A.1}) \\ &:= \sum_{k=1}^{l(j)} t_k(\rho_j, b). \end{aligned}$$

Now, $t_k(\rho_j, b)$ can only be negative if $3\rho_{j-k} - \rho_j > 0$. This seldom happens if ρ_n increases quickly. For instance, if $\rho_n = c^n$ for $c > 1$, then $3\rho_{j-k} - \rho_j > 0 \iff k < \frac{\ln(3)}{\ln(c)}$. In this case, if $(a_n)_{n \geq 1}$ is smooth enough (increasing, for instance), it is not hard to show that $(*_1)$ holds since for any $j \geq 1$, there are only a fixed number of negative terms. However, if ρ_n increases at a slower rate (typically of power type), then the number of negative terms increases with j . To show that $(*_1)$ holds then requires some additional conditions, a set of which is detailed below.

Proposition A.1. *If the following conditions are fulfilled,*

- i) $(a_n)_{n \geq 1}$ is increasing*
- ii) $\forall n \geq 2, \rho_{n+1} - \rho_n \geq (\rho_n - \rho_{n-1})$*
- iii) $\forall j > k \geq 1, 3\rho_{j-k} - \rho_j \geq 0 \implies 2(\rho_j - \rho_{j-k})^3 \geq (3\rho_{j-k} - \rho_j)(\rho_{j+k} - \rho_j)(\rho_{j+k} - 2\rho_j + \rho_{j-k})$,*

*then $(*_1)$ holds.*

Proof. We want to study $t_k(\rho_j, b)$ as a function of b . The idea is to show that any possible negative term indexed by $j-k$ in (A.1) is absolutely smaller than its $j+k$ counterpart. It is obvious that, by *i)*, it is sufficient to prove this for a constant sequence $(a_n)_{n \geq 1}$; hence we set $a_j := 1$ for all $j \geq 1$. For notational convenience, we denote

$$A_1 = \rho_{j+k} - \rho_j, \quad A_2 = 3\rho_{j+k} - \rho_j, \quad B_1 = \rho_j - \rho_{j-k}, \quad B_2 = 3\rho_{j-k} - \rho_j,$$

which are all positive (the case $B_2 < 0$ is irrelevant) and satisfy $A_2 = 3A_1 + 3B_1 + B_2$. Omitting the constant term in (A.1),

$$\begin{aligned} t_k(\rho_j, b) &\geq \frac{A_1 A_2 + b^2}{(A_1^2 + b^2)^2} - \frac{B_1 B_2 - b^2}{(B_1^2 + b^2)^2} \\ &= \frac{A_1(3A_1 + 3B_1 + B_2) + b^2}{(A_1^2 + b^2)^2} - \frac{B_1 B_2 - b^2}{(B_1^2 + b^2)^2} \\ &\geq \frac{c_0 + c_2 b^2 + c_4 b^4 + 2b^6}{(A_1^2 + b^2)^2 (B_1^2 + b^2)^2}, \end{aligned}$$

where

$$\begin{aligned} c_0 &= -A_1^4 B_1 B_2 + A_1 B_1^4 B_2 + 3A_1^2 B_1^4 + 3A_1 B_1^5 \\ c_2 &= -2A_1^2 B_1 B_2 + 2A_1 B_1^2 B_2 + 6A_1^2 B_1^2 + 6A_1 B_1^3 + A_1^4 + B_1^4 \\ c_4 &= -B_1 B_2 + A_1 B_2 + 5A_1^2 + 3A_1 B_1 + 2B_1^2 \end{aligned}$$

Note that by *ii*), $A_1 \geq B_1$, hence $c_4 > 0$. Condition *iii*) : $2B_1^3 \geq A_1 B_2 (A_1 - B_1)$ implies

$$c_0 \geq -2B_1^6 + A_1^2 B_1^4 + A_1 B_1^5 \quad c_2 \geq -4B_1^4 + 6A_1^2 B_1^2 + 6A_1 B_1^3 + A_1^4 + B_1^4,$$

which are both positive, by *ii*). This leads to $t_k(\rho_j, b) \geq 0$; the sum in (A.1) is therefore positive and $(*_1)$ holds. \square

This result calls for a few comments. First of all, many positive terms have been left out in the proof, thus $(*_1)$ holds under much weaker conditions. Furthermore, for any explicit formulation of a_n and ρ_n , *i*) is usually easily verified, and so is the convexity condition *ii*) which is in fact not too restrictive, given (2.2). However, *iii*) is much harder to prove. If we consider $\rho_n = n^\alpha$ for $\alpha > 1$, then we are interested in

$$2(j^\alpha - (j-k)^\alpha)^3 - (3(j-k)^\alpha - j^\alpha)((j+k)^\alpha - j^\alpha)((j+k)^\alpha - 2j^\alpha + (j-k)^\alpha)$$

and denoting k as a proportion of j : $k = cj$, this becomes

$$[2(1 - (1-c)^\alpha)^3 - (3(1-c)^\alpha - 1)((1+c)^\alpha - 1)((1+c)^\alpha - 2 + (1-c)^\alpha)]j^{3\alpha}.$$

Using numerical softwares, it is possible to show that this function of the variable c is increasing and positive on $(0, 1)$ for $1 < \alpha < 15$ (and in fact positive for $1 < \alpha \leq 15.87$). Note that $3\rho_{j-k} - \rho_j \geq 0 \iff k \leq \frac{3^{1/\alpha}-1}{3^{1/\alpha}}j$, hence the positivity criterion should only be checked for $c \in \left(0, \frac{3^{1/\alpha}-1}{3^{1/\alpha}}\right)$.

Other techniques can be used to show that $(*_1)$ also holds for $\alpha > 15.87$ when a_n is increasing. They rely on the fact that, as in the exponential case, there are, proportionally, very few negative terms in $T(\rho_j, b)$.

Acknowledgment. The author wishes to thank Thomas Simon for his numerous remarks, especially with respect to the proof of Proposition 3.1.

References

1. Abramowitz, M., Stegun, I.A.: *Handbook of mathematical functions*, Tenth printing, Dover Publications, 1972
2. Asmussen, S., Avram, F., Pistorius, M.R.: Russian and American put options under exponential phase-type Lévy models, *Stoch. Proc. Appl.*, Vol. 14, No. 1 (2004) 79–111
3. Bertoin, J.: *Lévy Processes*, Cambridge University Press, 1996
4. Doney, R.: On Wiener-Hopf factorization and the distribution of extrema for certain stable processes, *Ann. Prob.*, Vol. 15, No. 4 (1987) 1352–1362
5. Estermann, E.: *Complex Numbers and Functions*, Oxford University Press, 1962
6. Ferreiro-Castilla, A., Utzet, F.: Inversion of analytic characteristic functions and infinite convolutions of exponential and Laplace densities, *Journal of Theoretical Probability*, Vol. 25, No. 1 (2012) 205–230
7. Gradshteyn, I.S, Ryzhik, I.M.: *Tables of Integrals, Series and Products*, 7th Ed., Academic Press, 2007

GUILLAUME COQUERET

8. Kuznetsov, A.: Wiener-Hopf factorization and distribution of extrema for a family of Lévy processes, *Ann. Appl. Probab.*, Vol. 20, No. 5 (2010) 1801–1830
9. Kuznetsov, A.: Wiener-Hopf factorization for a family of Lévy processes related to theta functions, *J. Appl. Probab.*, Vol. 47, No. 4 (2010) 1023–1033
10. Kuznetsov, A.: On extrema of stable processes, *Ann. Probab.*, Vol. 39, No. 3 (2011) 1027–1060
11. Kuznetsov, A.: Analytic proof of Pecherskii-Rogozin identity and Wiener-Hopf factorization, *Theory Probab. Appl.*, Vol. 55, No. 3 (2011) 432–443
12. Kuznetsov, A., Kyprianou, A.E., Pardo, J.C., van Schaik, K.: A Wiener-Hopf Monte-Carlo simulation technique for Levy processes, *Ann. Appl. Probab.*, Vol. 21, No. 6 (2011) 2171–2190
13. Kuznetsov, A., Kyprianou, A.E., Pardo, J.C.: Meromorphic Lévy processes and their fluctuation identities, *Ann. Appl. Probab.*, Vol. 22, No. 3 (2012) 1101–1135
14. Kuznetsov, A., Peng, X.: On the Wiener-Hopf factorization for Lévy processes with bounded positive jumps, *Stoch. Proc. Appl.*, Vol. 122, No. 7 (2012) 2610–2638
15. Kyprianou, A.E.: *Introductory Lectures on Fluctuations of Levy Processes with Applications*, Springer, 2006
16. Levin, B. Ya.: *Lecture on Entire Functions*, Translations of Mathematical Monographs, 150, Amer. Math. Soc, 1996
17. Lewis, A.L., Mordecki, E.: Wiener-Hopf factorization for Lévy processes having positive jumps with rational transforms, *J. Appl. Probab.*, Vol. 45, No. 1 (2008) 118–134

GUILLAUME COQUERET: PHD PROGRAM - ESSEC BUSINESS SCHOOL, AVENUE B. HIRSCH
95000 CERGY-PONTOISE FRANCE

E-mail address: guillaume.coqueret@essec.edu