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ON THE KOLMOGOROV-WIENER-MASANI SPECTRUM OF A MULTI-MODE WEAKLY STATIONARY QUANTUM PROCESS

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Abstract. We introduce the notion of a k-mode weakly stationary quantum process $\boldsymbol{\varrho}$ based on the canonical Schrödinger pairs of position and momentum observables in copies of $L^2(\mathbb{R}^k)$, indexed by an additive abelian group D of countable cardinality. Such observables admit an autocovariance map $\widetilde{K}$ from D into the space of real $2k \times 2k$ matrices. The map $\widetilde{K}$ admits a spectral representation as the Fourier transform of a $2k \times 2k$ complex Hermitian matrix-valued totally finite measure Φ on the compact character group $\widehat{D}$, called the Kolmogorov-Wiener-Masani (KWM) *spectrum* of the process $\boldsymbol{\varrho}$. Necessary and sufficient conditions on a $2k \times 2k$ complex Hermitian matrix-valued measure Φ on $\widehat{D}$ to be the KWM spectrum of a process $\boldsymbol{\varrho}$ are obtained. This enables the construction of examples. Our theorem reveals the dramatic influence of the uncertainty relations among the position and momentum observables on the KWM spectrum of the process $\boldsymbol{\varrho}$. In particular, the KWM spectrum cannot admit a gap of positive Haar measure in $\widehat{D}$.

1. Introduction

In his celebrated little book "Osnovnye ponyatiya teorii veroyatnostei" [5], A. N. Kolmogorov introduced the notion of a stochastic process as a consistent family of finite dimensional probability distributions in $\mathbb{R}^n$, $n = 1, 2, \cdots$. In the same spirit a quantum process can be described as a consistent family of density operators or, equivalently, states in tensor products $\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n$ of Hilbert spaces with $n = 1, 2, \cdots$. One can replace the 'time set' $\{1, 2, \cdots\}$ by an abstract countable set D with the discrete topology and a family $\{\mathcal{H}_a : a \in D\}$ of Hilbert spaces. Then a quantum process yields a density operator $\rho_{a_1,a_2,\cdots,a_n}$ in $\mathcal{H}_{a_1} \otimes \cdots \otimes \mathcal{H}_{a_n}$ for every finite sequence $(a_1, \cdots, a_n)$ with distinct elements from D. All these density operators will obey natural consistency conditions. For example, the relative trace of $\rho_{a_1,a_2,\cdots,a_n}$ over $\mathcal{H}_{a_n}$ is $\rho_{a_1,a_2,\cdots,a_{n-1}}$. If $(b_1, \cdots, b_n)$ is a permutation of $a_1, \cdots, a_n$ then $\rho_{b_1,\cdots,b_n} = U\rho_{a_1,\cdots,a_n}U^{-1}$ where U is the corresponding Hilbert space isomorphism from $\mathcal{H}_{a_1} \otimes \cdots \otimes \mathcal{H}_{a_n}$ onto $\mathcal{H}_{b_1} \otimes \cdots \otimes \mathcal{H}_{b_n}$ induced by the permutation. We denote the quantum process over D by

$$\boldsymbol{\varrho} = \{(\mathcal{H}_{a_1,\cdots,a_n}, \rho_{a_1,\cdots,a_n}) : (a_1, \cdots, a_n) \in \mathcal{S}_D\}, \tag{1.1}$$

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where $\mathcal{S}_D$ denotes the set of all finite-length sequences of distinct elements from the countable set D.

In this paper we are interested in the special case where $\mathcal{H}_a = L^2(\mathbb{R}^k)$ for all a in D, k being a fixed positive integer, called the *number of modes* of the process. Each $\mathcal{H}_a$ admits Schrödinger canonical pairs q_{ar}, p_{ar}, $r = 1, 2, \cdots, k$, of position and momentum observables obeying the Heisenberg canonical commutation relations (CCR). We can look upon q_{ar}, p_{ar}, $r = 1, 2, \cdots, k$, as observables in $\mathcal{H}_{a_1} \otimes \cdots \otimes \mathcal{H}_{a_n}$ whenever the sequence $(a_1, a_2, \cdots, a_n)$ from $\mathcal{S}_D$ contains the element a and denote such ampliated observables by the same respective symbols. With such a convention one obtains the algebra of all polynomials of all q_{ar}, p_{ar}, where $r = 1, 2, \cdots, k$, and $a \in D$. Using the finite-partite states $\rho_{a_1, \cdots, a_n}$, where $(a_1, \cdots, a_n) \in \mathcal{S}_D$, one can compute the expectations of the polynomials whenever they exist. Write

$$\left(X_{a\,1}, X_{a\,2}, \cdots, X_{a\,(2k-1)}, X_{a\,2k}\right) = (q_{a\,1}, p_{a\,1}, \cdots, q_{a\,k}, p_{a\,k})$$

and define the covariances

$$\kappa_{r\,s}(a,b) = \left\langle \frac{1}{2}\left(X_{a\,r}X_{b\,s} + X_{b\,s}X_{a\,r}\right)\right\rangle - \langle X_{a\,r}\rangle \langle X_{b\,s}\rangle \tag{1.2}$$

where $\langle , \rangle$ denotes expectation. To compute these quantities we need a knowledge of only the 'bipartite' states $\rho_{a,b}$ for all $(a, b) \in \mathcal{S}_D$. Thus we obtain a $2k \times 2k$ real matrix-valued covariance kernel $\mathcal{K} = [[K(a, b)]]$ defined by

$$K(a,b) = [[\kappa_{r\,s}(a,b)]], \quad r, s, \in \{1, 2, \cdots, 2k\} \tag{1.3}$$

for any a, $b \in D$.

Suppose D is a countable discrete additive abelian group with addition operation $+$ and null element 0. Let the covariance kernel $\mathcal{K}$ of a k-mode quantum process over D be translation invariant in the sense that

$$K(a+x, b+x) = K(a,b) \quad \forall a, b, x \in D. \tag{1.4}$$

Then we say that the quantum process is second order *weakly stationary*, or, simply, weakly stationary. For such a process there exists a map $\widetilde{K}$ from D into the space of $2k \times 2k$ real matrices such that

$$K(a,b) = \widetilde{K}(b-a) \quad \forall a,\, b \in D. \tag{1.5}$$

The map $\widetilde{K}$ is called the *autocovariance map* of the weakly stationary quantum process.

Owing to the matrix-positivity properties enjoyed by covariances between observables the autocovariance map $\widetilde{K}$ satisfies the matrix inequalities

$$\sum_{i,\,j} \alpha_i \alpha_j \widetilde{K}(a_j - a_i) \geq 0 \tag{1.6}$$

for any $a_1, a_2, \cdots, a_n \in D$ and real scalars $\alpha_1, \alpha_2, \cdots, \alpha_n$, $n = 1, 2, \cdots$. Thanks to Bochner's theorem for locally compact abelian groups there exists a complex Hermitian and positive $2k \times 2k$ matrix-valued measure Φ on the compact dual character group $\widehat{D}$ of D such that

$$\widetilde{K}(a) = \int_{\widehat{D}} \chi(a)\, \Phi(\mathrm{d}\chi) \quad \text{for all } a \text{ in } D. \tag{1.7}$$

The matrix-valued measure Φ satisfies the conjugate symmetry property

$$\Phi(S^{-1}) = \overline{\Phi(S)} \tag{1.8}$$

for any Borel set $S \subset \widehat{D}$. Furthermore, the Heisenberg uncertainty relations prevailing among the various position and momentum observables of the quantum process reveal their dramatic influence on the measure Φ through the matrix inequalities

$$\Phi(S) + \frac{\imath}{2}\lambda(S) J_{2k} \geq 0 \tag{1.9}$$

for all Borel sets $S \subset \widehat{D}$, where λ is the normalised Haar measure of the compact group $\widehat{D}$ and J_{2k} is the fundamental symplectic matrix given by

$$J_{2k} = \bigoplus_{k \text{ copies}} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \tag{1.10}$$

which is a diagonal block matrix with each diagonal block equal to J_2. The inequality (1.9) implies, in particular, that whenever $\Phi(S) = 0$, $\lambda(S)$ is also zero.

Borrowing from the extensive theory of linear least square prediction of real valued weakly stationary processes pioneered by A. N. Kolmogorov [3, 4] and N. Wiener [10], and multivariate weakly stationary processes by N. Wiener and P. Masani [11, 12] we call (1.7) the spectral representation of $\widetilde{\mathcal{K}}$ in $\widehat{D}$ and the matrix-valued positive measure Φ the *Kolmogorov - Wiener - Masani spectrum* (or *KWM spectrum*) of the autocovariance map $\widetilde{\mathcal{K}}$ of the underlying quantum process.

As noted above, inequality (1.9) implies that whenever $\Phi(S) = 0$ for some Borel set $S \subset \widehat{D}$, then $\lambda(S) = 0$. In other words, the KWM spectrum does not admit a 'Haar gap'.

Conversely, given a complex Hermitian positive $2k \times 2k$ matrix-valued measure Φ on the Borel σ-algebra of $\widehat{D}$ satisfying the conjugate symmetry condition (1.8), the spectral uncertainty relations (1.9), and the condition $\Phi(\widehat{D}) = \widetilde{\mathcal{K}}(0)$, there exists a weakly stationary k-mode quantum process over D with KWM spectrum Φ. Indeed, such a process can be realized as a mean zero quantum Gaussian process in the sense that all its finite-partite states $\rho_{a_1,\cdots,a_n}$, where $(a_1, \cdots, a_n) \in \mathcal{S}_D$, are mean zero Gaussian states.

The spectral representation of the autocovariance function and its converse enable us to construct interesting examples of weakly stationary quantum processes.

2. Quantum Processes

A quantum system in its most elementary form is determined by a pair $(\mathcal{H}, \rho)$, where $\mathcal{H}$ is a complex separable Hilbert space and ρ is a density operator in $\mathcal{H}$, i.e., a positive operator with unit trace. The operator ρ is called the state of the system. We shall deal with several quantum systems and assume that all the Hilbert spaces in this paper are complex and separable. Scalar products in Hilbert spaces will be expressed in the Dirac notation and adjoints of operators as well as matrices will be indicated by the symbol †. By a positive operator X in a Hilbert space $\mathcal{H}$ we mean that $\langle u|X|u\rangle \geq 0$ for all $u \in \mathcal{H}$. By a positive $n \times n$ matrix we mean an $n \times n$ Hermitian matrix which is positive semidefinite.

If $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_n$ is the tensor product of Hilbert spaces $\mathcal{H}_i$ for $1 \leq i \leq n$, ρ is a state in $\mathcal{H}$ and $F \subset \{1, 2, \cdots, n\}$ is the subset $\{i_1 < i_2 < \cdots < i_k\}$ then we write $\mathcal{H}_F = \mathcal{H}_{i_1} \otimes \cdots \otimes \mathcal{H}_{i_k}$. One obtains a state ρ_F in $\mathcal{H}_F$ by taking the relative trace of ρ successively in $\mathcal{H}_i$, for all $i \notin F$ in some order. The resulting state ρ_F is independent of the order in which the traces are taken. The system $(\mathcal{H}_F, \rho_F)$ is called the F-marginal of $(\mathcal{H}, \rho)$.

In the Hilbert space of any quantum system a bounded or unbounded self-adjoint operator X is called an *observable* of the system. Suppose $\mathcal{F}_{\mathbb{R}}$ is the Borel σ-algebra of $\mathbb{R}$ and $P^X(\cdot)$ is the spectral measure of X on $\mathcal{F}_{\mathbb{R}}$. Then the quantity $\mathrm{Tr}\,\rho P^X(E)$, where $E \in \mathcal{F}_{\mathbb{R}}$, is interpreted as the probability that the observable takes a value in E in the state ρ. Thus $\mathrm{Tr}\,\rho P^X(\cdot)$ is the distribution of X in the state ρ. Such an interpretation enables the computation of all moments of X. Indeed, the n-th moment of X, if it exists, is denoted by $\langle X^n\rangle$ and is given by

$$\langle X^n\rangle = \mathrm{Tr}\, X^n \rho.$$

If X, Y are two observables such that $XY + YX$ is also an observable then the covariance between X and Y in the state ρ is denoted by $\mathrm{Cov}(X, Y)$ and is defined as

$$\mathrm{Cov}(X, Y) = \langle \frac{1}{2}(XY + YX)\rangle - \langle X\rangle \langle Y\rangle .$$

The quantity $\mathrm{Cov}(X, X)$ is called the variance of X. If X_1, X_2, $\cdots$, X_n are observables with well-defined covariance between X_i and X_j for all i, j then the $n\times n$ positive matrix

$$\Sigma_n = \Sigma_n(X_1, \cdots, X_n) = [[\mathrm{Cov}(X_i, X_j)]]$$

is called the *covariance matrix* of the observables $(X_1, X_2, \cdots, X_n)$ in the state ρ.

Consider a composite quantum system $(\mathcal{H}, \rho)$, where $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_n$. If the set $\{1, 2, \cdots, n\} = E \cup F$ with $E \cap F = \emptyset$, $E \neq \emptyset$, and $F \neq \emptyset$ then $\mathcal{H}$ can be viewed as the tensor product

$$\mathcal{H} = \mathcal{H}_E \otimes \mathcal{H}_F$$

and an observable in $\mathcal{H}_E$ can be looked upon as the observable $X_E \otimes I_F$ in $\mathcal{H}$ with I_F being the identity operator in $\mathcal{H}_F$. We call $X_E \otimes I_F$ the *ampliation* of X_E in $\mathcal{H}$ and denote it by the same symbol X_E. If ρ_E is the E-marginal of ρ in $\mathcal{H}_E$ then

$$\langle X_E\rangle = \mathrm{Tr}\, X_E \rho_E = \mathrm{Tr}\, X_E \rho = \langle X_E \otimes I_F\rangle .$$

We now introduce the notion of a quantum process over a countable index set D. Let $\{\mathcal{H}_a : a \in D\}$ be a family of Hilbert spaces. Denote by $\mathcal{S}_D$ the set of all finite sequences of distinct elements from D. Suppose $\rho_{a_1,a_2,\cdots,a_n}$ is a density operator in $\mathcal{H}_{a_1,a_2,\cdots,a_n}$ as in (1.1) for each $(a_1, a_2, \cdots, a_n)$ in $\mathcal{S}_D$, satisfying the following properties:

(1) If $\{a_1, a_2 \cdots, a_n\} = \{b_1, b_2, \cdots, b_n\}$ as sets and π is a permutation of $\{1, 2, \cdots, n\}$ such that $a_{\pi(j)} = b_j$, $\forall j$ and

$$U_\pi : \mathcal{H}_{a_1,a_2,\cdots,a_n} \to \mathcal{H}_{b_1,b_2,\cdots,b_n}$$

is the natural Hilbert space isomorphism induced by π then

$$\rho_{b_1,b_2,\cdots,b_n} = U_\pi \rho_{a_1,a_2,\cdots,a_n} U_\pi^{-1}.$$

(2) The $\{a_1, a_2, \cdots, a_n\}$-marginal of $\rho_{a_1,a_2,\cdots a_{n+1}}$ is equal to $\rho_{a_1,a_2,\cdots,a_n}$ for all $(a_1, a_2, \cdots, a_{n+1}) \in \mathcal{S}_D$, $n = 1, 2, \cdots$.

Then we say that $\{\rho_{a_1,a_2,\cdots,a_n} : (a_1, \cdots, a_n) \in \mathcal{S}_D\}$ is a consistent family of states. The family $\boldsymbol{\varrho} = \{(\mathcal{H}_{a_1,\cdots,a_n}, \rho_{a_1,\cdots,a_n}) : (a_1, \cdots, a_n) \in \mathcal{S}_D\}$ of finite-partite quantum systems is called a *quantum process* over D.

One obtains interesting examples of discrete 'time' quantum processes with D equal to $\mathbb{Z}$, $\mathbb{Z}^d$ or a general discrete abelian group. When D is $\mathbb{Z}$, the element a in $\mathbb{Z}$ can be interpreted as time. In general, a in D is interpreted as a site.

Suppose $D = \{0, 1, 2, \cdots\}$ and $\mathcal{H}_{n]} = \mathcal{H}_0 \otimes \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n$. Let $\rho_{n]}$ be a density operator in $\mathcal{H}_{n]}$ such that $\rho_{n-1]}$ is the marginal in $\mathcal{H}_{n-1]}$ obtained by tracing out $\rho_{n]}$ over $\mathcal{H}_n$ for each n. Then $\{(\mathcal{H}_{n]}, \rho_{n]}) : n = 0, 1, 2, \cdots\}$ yields a quantum process. Denote by $\mathcal{B}_{n]}$ the C* algebra of all bounded operators in $\mathcal{H}_{n]}$. Then there is a natural C* embedding $\phi_n : \mathcal{B}_{n]} \to \mathcal{B}_{n+1]}$ with the property

$$\phi_n(X) = X \otimes I \quad \forall X \in \mathcal{B}_{n]},$$

where I is the identity operator in $\mathcal{H}_{n+1}$. This enables the construction of an inductive limit C* algebra $\mathcal{B}_\infty$ with a C* embedding $\pi_n : \mathcal{B}_{n]} \to \mathcal{B}_\infty$ such that the sequence $\{\pi_n(\mathcal{B}_{n]})\}$ is increasing in n and $\bigcup_n \pi_n(\mathcal{B}_{n]})$ is dense in $\mathcal{B}_\infty$. This yields a normalized positive linear functional ω in $\mathcal{B}_\infty$ such that

$$\omega(\pi_n(X)) = \rho_{n]}(X) \quad \forall X \in \mathcal{B}_{n]} \text{ and } n = 0, 1, 2, , \cdots .$$

In other words $(\mathcal{B}_\infty, \omega)$ is a C* probability space which may be considered as the analogue of Kolmogorov's measure space constructed from a consistent family of finite dimensional probability distributions. However, there is no limiting Hilbert space in general with a density operator. A similar construction of a C* probability space is possible for a quantum process over any countable index set D.

Definition 2.1. Suppose D is a countable abelian group with addition operation $+$, $\mathcal{H}_a = \mathcal{H}$ for all $a \in D$, and $\boldsymbol{\varrho}$ is a quantum process over D. Then it is said to be *strictly stationary* or *translation invariant* if

$$\rho_{a_1+x,a_2+x,\cdots,a_n+x} = \rho_{a_1,a_2,\cdots,a_n} \quad \forall x \in D \text{ and } (a_1, \cdots, a_n) \in \mathcal{S}_D.$$

Let $\{\rho_{a_1,\cdots,a_n}\}$ and $\{\sigma_{a_1,\cdots,a_n}\}$, where $(a_1, \cdots, a_n) \in \mathcal{S}_D$, be a pair of consistent families of finite-partite states in $\{\mathcal{H}_{a_1,\cdots,a_n}\}$. Then, for any $0 < p < 1$, setting

$$\tau_{a_1,\cdots,a_n} = p\rho_{a_1,\cdots,a_n} + (1-p)\sigma_{a_1,\cdots,a_n} \quad \forall (a_1, \cdots, a_n) \in \mathcal{S}_D$$

yields a consistent family of finite-partite states.

Suppose, $a \mapsto U_a$ is any map on D where U_a is a unitary operator in $\mathcal{H}_a$ for every $a \in D$. Then setting

$$\rho'_{a_1,\cdots,a_n} = (U_{a_1} \otimes \cdots \otimes U_{a_n})\, \rho_{a_1,\cdots,a_n} \left(U_{a_1}^\dagger \otimes \cdots \otimes U_{a_n}^\dagger\right) \quad \forall (a_1, \cdots, a_n) \in \mathcal{S}_D$$

also yields a consistent family of states. Indeed, this is a consequence of the following proposition.

Proposition 2.2. *Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces, ρ a state in $\mathcal{H}\otimes\mathcal{K}$, and U and V unitary operators in $\mathcal{H}$ and $\mathcal{K}$ respectively. Then*

$$\mathrm{Tr}_{\mathcal{K}}(U\otimes V)\rho(U\otimes V)^{\dagger} = U\,(\mathrm{Tr}_{\mathcal{K}}\,\rho)\,U^{\dagger},$$

where $\mathrm{Tr}_{\mathcal{K}}$ *is relative trace over* $\mathcal{K}$.

Proof. This is immediate from the fact that the relative trace over $\mathcal{K}$ can be computed by using any orthonormal basis $\{\boldsymbol{e}_j\}$ in $\mathcal{K}$, and if $\{\boldsymbol{e}_j\}$ is one such basis so is $\{V^{\dagger}\boldsymbol{e}_j\}$. □

Combining the two elementary remarks above we can construct new quantum processes over D from a given quantum process $\{(\mathcal{H}_{a_1,\cdots,a_n},\rho_{a_1,\cdots,a_n}):(a_1,\cdots,a_n)\in\mathcal{S}_D\}$ as follows: Start with a probability space $(\Omega,\mathcal{F},\mathcal{P})$ and a random process $\{U_a(\omega):a\in D\}$ where $U_a(\omega)$ is a unitary operator in $\mathcal{H}_a$ for every a. Define

$$\rho'_{a_1,\cdots,a_n} = \int_{\Omega} P(\mathrm{d}\omega)\,(U_{a_1}\otimes\cdots\otimes U_{a_n})\,\rho_{a_1,\cdots,a_n}\,(U_{a_1}\otimes\cdots\otimes U_{a_n})^{\dagger}\,. \tag{2.1}$$

Then $\{\rho'_{a_1,\cdots,a_n}:(a_1,\cdots,a_n)\in\mathcal{S}_D\}$ is also a consistent family of finite-partite states.

Remark 2.3. When D is a countable additive abelian group and $\mathcal{H}_a=\mathcal{H}$ for all $a\in D$, $\boldsymbol{\varrho}$ is a strictly stationary quantum process and the random process $\{U_a(\omega):a\in D\}$ is also strictly stationary, then the quantum process $\boldsymbol{\varrho}'$ determined by equation (2.1) is also strictly stationary.

3. Multi-mode Processes and Their Covariance Kernels

We now pass on to the definition of a *k-mode quantum process* over a countable index set D. Let $\mathcal{H}_a=L^2(\mathbb{R}^k)$ for each $a\in D$, where k is a fixed positive integer called the number of modes. We view $\mathcal{H}_a$ as the a-th copy of $L^2(\mathbb{R}^k)$ and introduce the canonical Schrödinger pairs of position and momentum observables q_{aj}, p_{aj}, where $1\le j\le k$, given by

$$\begin{aligned}(q_{aj}f)(\boldsymbol{x}) &= x_j f(\boldsymbol{x}),\\ (p_{aj}f)(\boldsymbol{x}) &= \frac{1}{\imath}\frac{\partial}{\partial x_j}f(\boldsymbol{x})\end{aligned}$$

on their respective maximal domains in $L^2(\mathbb{R}^k)$, $\boldsymbol{x}$ denoting $(x_1,x_2,\cdots,x_k)\in\mathbb{R}^k$. We arrange these $2k$ observables as

$$(X_{a1},X_{a2},\cdots,X_{a\,2k-1},X_{a2k}) = (q_{a1},p_{a1},\cdots,q_{ak},p_{ak})\,.$$

Let now $\boldsymbol{\varrho}$ be a quantum process over D. Then $X_{a\,r}$ can be viewed as an ampliated observable in $\mathcal{H}_{a_1,a_2,\cdots,a_n}$ whenever the element a occurs in the sequence $a_1,a_2,\cdots,a_n$. We assume that all observables which are closures of polynomials of degree not exceeding 2 in $\{X_{a\,r}:\,a\in D,\,r=1,2,\cdots,2k\}$ have finite expectations under the process so that $X_{a\,r}$ and $X_{b\,s}$ have a well-defined covariance for any $a,b\in D$, and $r,s\in\{1,2,\cdots,2k\}$. We write for any $a,b\in D$,

$$\kappa_{r,s}(a,b) = \mathrm{Cov}\,(X_{a\,r},X_{b\,s}) \quad \forall r,\,s\in\{1,2,\cdots,2k\},$$

where the covariances can be evaluated in any state $\rho_{a_1,a_2,\cdots,a_n}$ when both a and b occur in $(a_1,\cdots,a_n)\in\mathcal{S}_D$. Indeed, this follows from consistency of the states occurring in the quantum process. We call $\mathcal{K}=[[K(a,b)]]$, where $a,b\in D$, the *covariance kernel* of the k-mode quantum process $\boldsymbol{\varrho}$, where $K(a,b)$ is the $2k\times 2k$ matrix $[[k_{r,s}(a,b)]]$ with $r,\ s\in\{1,2,\cdots,2k\}$.

If D is an additive abelian group, $\langle X_{aj}\rangle=0$ for all $a\in D$, $1\le j\le 2k$, and $K(a,b)=K(a+c,b+c)$ for all $a,\ b,\ c\in D$ we then say that $\boldsymbol{\varrho}$ is a mean zero *second order weakly stationary* or simply *weakly stationary* k-mode quantum process. In such a case, there exists a map $\widetilde{K}$ from D into the space of $2k\times 2k$ real matrices, such that $K(a,b)=\widetilde{K}(b-a)$. This map $\widetilde{K}$ is called the *autocovariance map* of the weakly stationary process.

Theorem 3.1. *Let $K(a,b)=[[\kappa_{r\,s}(a,b)]]$ where $a,b\in D$, be a family of $2k\times 2k$ real matrices satisfying the following conditions:*

$$\kappa_{r\,s}(a,b)=\kappa_{s\,r}(b,a)\quad\forall r,\ s\in\{1,2,\cdots,2k\}\ \text{and}\ a,b\in D.$$

Then there exists a k-mode quantum process $\boldsymbol{\varrho}$ with covariance kernel $K(\cdot,\cdot)$ if and only if for any sequence $(a_1,\cdots,a_n)\in\mathcal{S}_D$ the block matrix $[[K(a_i,a_j)]]$ satisfies the matrix inequality

$$[[K(a_i,a_j)]]+\frac{\imath}{2}J_{2kn}\ge 0. \tag{3.1}$$

Proof. Since $\rho_{a_1,\cdots,a_n}$ is a kn-mode state and $[[K(a_i,a_j)]]$ is the covariance matrix of the position-momentum observables $(X_{a_1\,1},\cdots,X_{a_1\,2k},X_{a_2\,1},\cdots,X_{a_2\,2k},\cdots,X_{a_n\,1},\cdots,X_{a_n\,2k})$ in $L^2(\mathbb{R}^{kn})$, necessity is immediate from the uncertainty relation fulfilled by such a covariance matrix [2, 1, 7]. From Theorem 3.1 of [7] and inequality (3.1), it follows that thre exists a Gaussian state $\{\rho_{a_1,\cdots,a_n}\}$ with covariance matrix $[[K(a_i,a_j)]]$. Then $\{\rho_{a_1,\cdots,a_n}\}$ is a consistent family of Gaussian states constituting the required quantum process. □

Definition 3.2. A kernel

$$\mathcal{K}=[[K(a,b)]]\quad\forall a,b\in D,$$

where $K(a,b)$ are real $2k\times 2k$ matrices satisfying the conditions

(1) $K(a,b)^T=K(b,a)$ for all $a,\ b\in D$,
(2) $[[K(a_i,a_j)]]\ge 0$ for all $(a_1,a_2,\cdots,a_n)\in\mathcal{S}_D$,

is called a k-mode *classical covariance* kernel.

If, in addition, the inequality (3.1) is fulfilled, then it is called a (k-mode) *quantum covariance kernel*.

Corollary 3.3. *If $\mathcal{K}$ is a k-mode quantum covariance kernel and $\mathcal{C}$ is a k-mode classical covariance kernel then $\mathcal{K}+\mathcal{C}$ is a k-mode quantum covariance kernel.*

Proof. Immediate. □

Let $\boldsymbol{\varrho}$ be a k-mode quantum process over D with quantum covariance kernel $\mathcal{K}=[[K(a,b)]]$. Suppose $\mathcal{C}=[[C(a,b)]]$ is the covariance kernel of a real $2k$-variate classical stochastic process with index set D so that the matrix inequalities

$$\sum_{i,j}\alpha_i\alpha_j C(a_i,a_j)\ge 0$$

for all real scalars $\alpha_1, \cdots, \alpha_n$ and all elements $a_1, \cdots, a_n \in D$. Then the sum

$$\mathcal{K} + \mathcal{C} = [[K(a,b) + C(a,b)]]$$

is the covariance kernel of a k-mode quantum process $\boldsymbol{\sigma}$. We shall now realise such a process $\boldsymbol{\sigma}$ by an explicit construction which is an interaction between the quantum process $\boldsymbol{\varrho}$ and a family of unitary conjugations mediated by a classical process with covariance kernel $\mathcal{C}$.

To this end we start with the 1-mode Hilbert space $L^2(\mathbb{R})$, its Schrödinger position-momentum pair q, p, the associated annihilation-creation pair $\hat{a}$, $\hat{a}^\dagger$ given by $\hat{a} = 2^{-\frac{1}{2}}(q + \imath p)$, $\hat{a}^\dagger = 2^{-\frac{1}{2}}(q - \imath p)$ and the unitary Weyl (displacement) operators $W(z) = \exp(z\hat{a}^\dagger - \bar{z}\hat{a})$, where $z \in \mathbb{C}$. These satisfy the relations

$$W(z)\hat{a}W(z)^\dagger = \hat{a} - z \quad \forall z \in \mathbb{C},$$

with the convention that z denotes the scalar as well as the operator zI. This leads to the relations

$$W(2^{-\frac{1}{2}}z)qW(2^{-\frac{1}{2}}z)^\dagger = q - x, \tag{3.2}$$

$$W(2^{-\frac{1}{2}}z)pW(2^{-\frac{1}{2}}z)^\dagger = p - y, \tag{3.3}$$

where $x = \operatorname{Re} z$ and $y = \operatorname{Im} z$.

Now for $a \in D$, let

$$\begin{aligned} \boldsymbol{z}_a &= (z_{a\,1}, z_{a\,2}, \cdots, z_{a\,k})^T, \\ z_{a\,r} &= x_{a\,r} + \imath y_{a\,r}, \end{aligned}$$

where $x_{a\,r} = \operatorname{Re} z_{a\,r}$ and $y_{a\,r} = \operatorname{Im} z_{a\,r}$. Viewing $\mathcal{H} = L^2(\mathbb{R}^k)$ as the k-fold product $L^2(\mathbb{R}) \otimes L^2(\mathbb{R}) \otimes \cdots \otimes L^2(\mathbb{R})$, introduce the k-mode Weyl operators

$$W(\boldsymbol{z}_a) = W(z_{a\,1}) \otimes \cdots \otimes W(z_{a\,k}).$$

Then the relations (3.2, 3.3) yield the relations for the operators $X_{a\,1}, \cdots, X_{a\,2k}$ defined above as

$$W(2^{-\frac{1}{2}}\boldsymbol{z}_a) \begin{bmatrix} X_{a\,1} \\ X_{a\,2} \\ \vdots \\ X_{a\,2k} \end{bmatrix} W(2^{-\frac{1}{2}}\boldsymbol{z}_a)^\dagger = \begin{bmatrix} X_{a\,1} - \alpha_{a\,1} \\ X_{a\,2} - \alpha_{a\,2} \\ \vdots \\ X_{a\,2k} - \alpha_{a\,2k} \end{bmatrix} \tag{3.4}$$

where

$$(\alpha_{a\,1}, \alpha_{a\,2}, \cdots, \alpha_{a\,2k}) = (x_{a\,1}, y_{a\,1}, x_{a\,2}, y_{a\,2}, \cdots, x_{a\,k}, y_{a\,k}). \tag{3.5}$$

Let $(\xi_{a\,1}, \eta_{a\,1}, \xi_{a\,2}, \eta_{a\,2}, \cdots, \xi_{a\,k}, \eta_{a\,k})(\omega)$, where $\omega \in \Omega$, and $a \in D$, be a $2k$ real variate stochastic process defined on a probability space $(\Omega, \mathcal{F}, P)$ with zero mean and covariance kernel $\mathcal{C} = [[C(a,b)]]$, where

$$C(a,b) = \mathbb{E} \begin{bmatrix} \xi_{a\,1} \\ \eta_{a\,1} \\ \xi_{a\,2} \\ \eta_{a\,2} \\ \vdots \\ \xi_{a\,k} \\ \eta_{a\,k} \end{bmatrix} \left[\xi_{a\,1}, \eta_{a\,1}, \xi_{a\,2}, \eta_{a\,2}, \cdots, \xi_{a\,k}, \eta_{a\,k}\right] \quad \forall a,\, b \in D. \tag{3.6}$$

Define the random unitary operators $U_a(\omega)$ in $\mathcal{H}_a$, where $a \in D$, by putting

$$\zeta_{a\,r} = \xi_{a\,r} + \imath\eta_{a\,r} \quad \text{for } r = 1, 2, \cdots, k, \tag{3.7}$$
$$U_a(\omega) = W(2^{-\frac{1}{2}}\boldsymbol{\zeta}_a(\omega))^\dagger, \tag{3.8}$$

where $\boldsymbol{\zeta}_a = (\zeta_{a\,1} \cdots, \zeta_{a\,k}) \in \mathbb{C}^k$. By following the remarks in §2 around equation (2.1), define the k-mode quantum process $\boldsymbol{\sigma}$ by

$$\sigma_{a_1,\cdots,a_n} = \int_\Omega P(\mathrm{d}\omega) U_{a_1}(\omega) \otimes \cdots \otimes U_{a_n}(\omega) \rho_{a_1,\cdots,a_n} U_{a_1}(\omega)^\dagger \otimes \cdots \otimes U_{a_n}(\omega)^\dagger. \tag{3.9}$$

Then we have the following theorem:

Theorem 3.4. *The covariance kernel of the $\boldsymbol{\sigma}$ process determined by the finite-partite states (3.9) is equal to $\mathcal{K} + \mathcal{C}$.*

Proof. Consider the observable $X_{a\,j}$. Its expectation under the $\boldsymbol{\sigma}$ process is given by

$$\begin{aligned} \mathrm{Tr}\, X_{a\,r}\sigma_a &= \int P(\mathrm{d}\omega) \mathrm{Tr}\, U_a(\omega)^\dagger X_{a\,r} U_a(\omega) \rho_a \\ &= \int P(\mathrm{d}\omega) \mathrm{Tr}\, (X_{a\,r} - \gamma_{a\,r})\, \rho_a, \end{aligned}$$

where

$$(\gamma_{a\,1}(\omega), \cdots, \gamma_{a\,2k}(\omega)) = (\xi_{a\,1}(\omega), \eta_{a\,1}(\omega), \cdots, \xi_{a\,k}(\omega), \eta_{a\,k}(\omega)).$$

Since the classical $\boldsymbol{\gamma}$ process has mean $\mathbf{0}$ we have

$$\langle X_{a\,r}\rangle_{\boldsymbol{\sigma}} = \langle X_{a\,r}\rangle_{\boldsymbol{\rho}}. \tag{3.10}$$

Going to second order moments

$$\begin{aligned} &\mathrm{Tr}\, X_{a\,r} X_{b\,s}\, \sigma_{a\,b} \\ &= \int P(\mathrm{d}\omega) \mathrm{Tr}\, X_{a\,r} X_{b\,s} U_a(\omega) \otimes U_b(\omega) \rho_{a\,b} U_a(\omega)^\dagger \otimes U_b(\omega)^\dagger \\ &= \int P(\mathrm{d}\omega) W(2^{-\frac{1}{2}}\boldsymbol{\zeta}_a) \otimes W(2^{-\frac{1}{2}}\boldsymbol{\zeta}_b) X_{a\,r} X_{b\,s} W(2^{-\frac{1}{2}}\boldsymbol{\zeta}_a)^\dagger \otimes W(2^{-\frac{1}{2}}\boldsymbol{\zeta}_b)^\dagger \rho_{a\,b} \\ &= \int P(\mathrm{d}\omega) \mathrm{Tr}\, (X_{a\,r} - \gamma_{a\,r})(X_{b\,s} - \gamma_{b\,s})\, \rho_{a\,b} \\ &= \int P(\mathrm{d}\omega) \left[\langle X_{a\,r} X_{b\,s}\rangle_{\boldsymbol{\varrho}} + \gamma_{a\,r}(\omega)\gamma_{b\,s}(\omega) - \gamma_{a\,r}(\omega) \langle X_{b\,s}\rangle_{\boldsymbol{\varrho}} - \gamma_{b\,s}(\omega) \langle X_{a\,r}\rangle_{\boldsymbol{\varrho}} \right] \\ &= \langle X_{a\,r} X_{b\,s}\rangle_{\boldsymbol{\varrho}} + C_{r\,s}(a, b). \end{aligned}$$

Let $a \neq b$. Then

$$\begin{aligned} \mathrm{Cov}_{\boldsymbol{\sigma}}(X_{a\,r}, X_{b\,s}) &= \langle X_{a\,r} X_{b\,s}\rangle_{\boldsymbol{\sigma}} - \langle X_{a\,r}\rangle_{\boldsymbol{\sigma}} \langle X_{b\,s}\rangle_{\boldsymbol{\sigma}} \\ &= \langle X_{a\,r} X_{b\,s}\rangle_{\boldsymbol{\varrho}} + C_{r\,s}(a, b) - \langle X_{a\,r}\rangle_{\boldsymbol{\varrho}} \langle X_{b\,s}\rangle_{\boldsymbol{\varrho}} \\ &= K_{r\,s}(a, b) + C_{r\,s}(a, b). \end{aligned}$$

Let $a = b$. Then $C_{r\,s}(a,b) = C_{r\,s}(a,a) = C_{s\,r}(a,a)$, and so

$$\begin{aligned}
\mathrm{Cov}_{\boldsymbol{\sigma}}(X_{a\,r}, X_{a\,s}) &= \left\langle \frac{1}{2}(X_{a\,r}X_{a\,s} + X_{a\,s}X_{a\,r}) \right\rangle_{\boldsymbol{\sigma}} - \langle X_{a\,r}\rangle_{\boldsymbol{\varrho}} \langle X_{a\,s}\rangle_{\boldsymbol{\varrho}} \\
&= \left\langle \frac{1}{2}(X_{a\,r}X_{a\,s} + X_{a\,s}X_{a\,r}) \right\rangle_{\boldsymbol{\varrho}} + C_{r\,s}(a,a) - \langle X_{a\,r}\rangle_{\boldsymbol{\varrho}} \langle X_{a\,s}\rangle_{\boldsymbol{\varrho}} \\
&= \mathrm{Cov}_{\boldsymbol{\varrho}}(X_{a\,r}, X_{a\,s}) + C_{r\,s}(a,a) \\
&= K_{r\,s}(a,a) + C_{r\,s}(a,a).
\end{aligned}$$

□

Remark 3.5. If $\boldsymbol{\varrho}$ is a weakly stationary quantum process and $(\xi_{a\,1}, \eta_{a\,1}, \cdots, \xi_{a\,k}, \eta_{a\,k})$ is a weakly stationary classical process with mean $\mathbf{0}$ such that

$$\begin{aligned}
K(a,b) &= \widetilde{K}(b-a) \quad \text{and} \\
C(a,b) &= \widetilde{C}(b-a) \quad \forall a,\, b \in D,
\end{aligned}$$

then $\boldsymbol{\sigma}$ is also a weakly stationary quantum process. If in addition, $\boldsymbol{\varrho}$ is Gaussian then so is $\boldsymbol{\sigma}$.

4. The KWM Spectrum of a Weakly Stationary k-mode Quantum Process

Let $\boldsymbol{\varrho}$ be a weakly stationary k-mode quantum process over a countable discrete additive abelian group D, with autocovariance map $\widetilde{K}$. Let $\widehat{D}$ be the compact dual multiplicative group of all characters of D. Denote by $\mathcal{F}$ the Borel σ-algebra on $\widehat{D}$.

Define

$$L(a) = \widetilde{K}(a) + \frac{\imath}{2}\mathbb{1}_{\{0\}}(a)J_{2k} \quad \forall a \in D. \tag{4.1}$$

Then Theorem 3.1 yields the following proposition.

Proposition 4.1. *A real $2k \times 2k$ matrix-valued map $\widetilde{K}$ is the autocovariance map of a second order weakly stationary k-mode quantum process on D if and only if the associated map L defined by (4.1) satisfies the following matrix inequalities*

$$[[L(a_s - a_r)]] \geq 0, \quad \text{where } r,\, s \in \{1, 2, \cdots, n\},$$

for all $(a_1, a_2, \cdots, a_n) \in \mathcal{S}_D$, and $n = 1, 2, \cdots$.

Proof. Immediate. □

Theorem 4.2. *A real $2k \times 2k$ matrix-valued map $\widetilde{K}$ on D is the autocovariance map of a second order weakly stationary k-mode quantum process on D if and only if there exists a $2k \times 2k$ Hermitian positive matrix-valued measure Φ on $(\widehat{D}, \mathcal{F})$ satisfying the following conditions:*

1. $\Phi(\widehat{D}) = \widetilde{K}(0)$.
2. $\widetilde{K}(a) = \int_{\widehat{D}} \chi(a)\Phi(\mathrm{d}\chi)$.
3. $\Phi(S) + \frac{\imath}{2}\lambda(S)J_{2k} \geq 0, \forall S \in \mathcal{F}$, *where λ is the normalized Haar measure of the compact group $\widehat{D}$. In particular, λ is absolutely continuous with respect to* $\mathrm{Tr}\,\Phi$.

(4) $\Phi(S^{-1}) = \overline{\Phi(S)} = \Phi(S)^T$, $\forall S \in \mathcal{F}$.

Under the conditions (1)-(4) the underlying quantum process can be chosen to be a strictly stationary k-mode quantum Gaussian process with mean zero.

Proof. Let $\widetilde{K}$ be the autocovariance map of a weakly stationary k-mode process. Define L by (4.1). By Proposition 4.1 the matrices $[[L(a_s - a_r)]]$, where $r, s \in \{1, 2, \cdots, n\}$, are positive for all $(a_1, a_2, \cdots, a_n) \in \mathcal{S}_D$. Hence for any vector $\boldsymbol{u} \in \mathbb{C}^{2k}$, the function

$$a \mapsto \psi_{\boldsymbol{u}}(a) = \boldsymbol{u}^\dagger L(a) \boldsymbol{u} \tag{4.2}$$

is positive definite on the abelian group D in the sense of Bochner. By Bochner's theorem there exists a totally finite measure $\nu_{\boldsymbol{u}}$ on $\mathcal{F}$ satisfying the relations

$$\begin{aligned} \psi_{\boldsymbol{u}}(a) &= \int_{\widehat{D}} \chi(a)\, \nu_{\boldsymbol{u}}(\mathrm{d}\chi) \quad \forall a \in D, && (4.3) \\ \psi_{\boldsymbol{u}}(0) &= \boldsymbol{u}^\dagger L(0) \boldsymbol{u} \\ &= \boldsymbol{u}^\dagger \left(\widetilde{K}(0) + \frac{\imath}{2} J_{2k} \right) \boldsymbol{u}. && (4.4) \end{aligned}$$

By (4.2) the left hand side of (4.3) is a quadratic form in $\boldsymbol{u}$ for each fixed a in D. By the bijective correspondence between totally finite measures on $\widehat{D}$ and their Fourier transforms on D it follows that there exists a $2k \times 2k$ Hermitian positive matrix-valued measure Ψ on $\mathcal{F}$ such that

$$\begin{aligned} L(a) &= \int_{\widehat{D}} \chi(a)\, \Psi(\mathrm{d}\chi) \quad \forall a \in D, && (4.5) \\ L(0) &= \widetilde{K}(0) + \frac{\imath}{2} J_{2k} = \Psi(\widehat{D}) \geq 0. && (4.6) \end{aligned}$$

Now define

$$\phi_{\boldsymbol{u}}(a) = \boldsymbol{u}^\dagger \widetilde{K}(a) \boldsymbol{u} \quad \forall a \in D \text{ and } \boldsymbol{u} \in \mathbb{C}^{2k}.$$

By (4.1), $\widetilde{K}(a) = \operatorname{Re} L(a)$, where the real part is taken entry-by-entry, and hence $[[\widetilde{K}(a_s - a_r)]] \geq 0$ for any $(a_1, a_2, \cdots, a_n) \in \mathcal{S}_D$. In other words, $\phi_{\boldsymbol{u}}$ is also a positive definite function on D and by the same arguments as employed for L we have the relations

$$\begin{aligned} \widetilde{K}(a) &= \int_{\widehat{D}} \chi(a)\, \Phi(\mathrm{d}\chi), && (4.7) \\ \widetilde{K}(0) &= \Phi(\widehat{D}), && (4.8) \end{aligned}$$

where Φ is again a $2k \times 2k$ Hermitian positive matrix-valued measure on $\mathcal{F}$.

By (4.1)

$$\begin{aligned} L(a) - \widetilde{K}(a) &= \frac{\imath}{2} \mathbb{1}_{\{0\}}(a) J_{2k} \\ &= \left[\frac{\imath}{2} \int_{\widehat{D}} \chi(a) \lambda(\mathrm{d}\chi) \right] J_{2k} \quad \forall a \in D. && (4.9) \end{aligned}$$

Subtracting (4.7) from (4.5) and using (4.9) we have

$$\int_{\widehat{D}} \chi(a)(\Psi - \Phi)(\mathrm{d}\chi) = \left[\frac{\imath}{2} \int_{\widehat{D}} \chi(a) \lambda(\mathrm{d}\chi) \right] J_{2k}$$

for all $a \in D$. Thus by uniqueness of Fourier transforms we have

$$\Psi(S) - \Phi(S) = \frac{\imath}{2}\lambda(S)J_{2k}, \quad \forall S \in \mathcal{F}.$$

Thus

$$\Phi(S) + \frac{\imath}{2}\lambda(S)J_{2k} \geq 0, \quad \forall S \in \mathcal{F}. \tag{4.10}$$

Now property (1) follows from (4.8), property (2) from (4.7), and property (3) from (4.10). If $\operatorname{Tr}\Phi(S) = 0$ then $\Phi(S) = 0$, by positivity, and (4.10) implies

$$\frac{\imath}{2}\lambda(S)J_{2k} \geq 0$$

which is positive only if $\lambda(S) = 0$, as may be seen by taking the determinant. In other words $\lambda \ll \operatorname{Tr}\Phi$.

To prove property (4) of Φ we introduce the map $\tau : \widehat{D} \to \widehat{D}$, $\tau(\chi) = \overline{\chi} = \chi^{-1}$ and observe that

$$\begin{aligned} \overline{\widetilde{K}(a)} &= \int_{\widehat{D}} \overline{\chi(a)}\, \overline{\Phi}(\mathrm{d}\chi) \\ &= \int_{\widehat{D}} \chi(a)\, \overline{\Phi}\tau^{-1}(\mathrm{d}\chi). \end{aligned}$$

Since $\widetilde{K}(a)$ has real entries, property (2) implies

$$\overline{\Phi}\tau^{-1} = \Phi,$$

or

$$\Phi(S^{-1}) = \overline{\Phi(S)} = \Phi(S)^T, \quad \forall S \in \mathcal{F}.$$

This completes the proof of necessity. To prove sufficiency consider a $2k \times 2k$ Hermitian positive matrix-valued measure Φ satisfying properties (3) and (4) of the theorem. Define

$$\widetilde{K}(a) = \int_{\widehat{D}} \chi(a)\, \Phi(\mathrm{d}\chi).$$

Then properties (2) and (1) hold. Property (3) implies that the function $L(a)$, on D defined by setting

$$\begin{aligned} L(a) &= \widetilde{K}(a) + \frac{\imath}{2}\mathbb{1}_{\{0\}}(a)J_{2k} \\ &= \int_{\widehat{D}} \chi(a)\, \left(\Phi + \frac{\imath}{2}\lambda J_{2k}\right)(\mathrm{d}\chi) \end{aligned}$$

satisfies the matrix inequalities $[[L(a_s - a_r)]] \geq 0$ for any sequence $(a_1, \cdots, a_n) \in \mathcal{S}_D$. By Proposition 4.1, $\widetilde{K}$ is the autocovariance function of a second order weakly stationary k-mode quantum process which, indeed, can be chosen to be a strictly stationary Gaussian process of mean zero. □

Remark 4.3. As already described in the introduction, we call equation (2) in Theorem 4.2 the *spectral representation* of the autocovariance map $\widetilde{K}$ and say that Φ is the *Kolmogorov-Wiener-Masani (KWM) spectrum* of the k-mode weakly stationary quantum process. Theorem 4.2 enables us to construct a whole class of examples of KWM spectra and hence autocovariance maps as follows. Choose and fix any Borel map $\chi \mapsto M(\chi)$, on $\widehat{D}$ where $M(\chi)$ is a k-mode quantum covariance matrix of order $2k$, so that

$$M(\chi) + \frac{\imath}{2} J_{2k} \geq 0 \quad \text{for every } \chi \in \widehat{D}.$$

Assume that $M(\cdot)$ is integrable with respect to the normalised Haar measure λ on $\widehat{D}$. Let Ψ be any totally finite positive Hermitian $2k \times 2k$ matrix-valued measure on $(\widehat{D}, \mathcal{F})$ satisfying the conjugate symmetry condition $\Psi(S^{-1}) = \overline{\Psi(S)}$ for any Borel set $S \in \mathcal{F}$. Define

$$\Phi(S) = \int_{\widehat{D}} M(\chi)\lambda(\mathrm{d}\chi) + \Psi(S), \quad S \in \mathcal{F}.$$

Then by Theorem 4.2, Φ is the KWM spectrum of a stationary quantum process over D with autocovariance map $\widetilde{K}$ given by equation (2) of the theorem.

Remark 4.4. The second part of property (3) of Φ in Theorem 4.2 implies that $\lambda(S) = 0$ whenever $\operatorname{Tr} \Phi(S) = 0$. In other words the KWM spectrum of a weakly stationary k-mode quantum process over D cannot admit a gap of positive Haar measure in $\widehat{D}$. For example, when $D = \mathbb{Z}$ and $\widehat{D}$ is identified with $[0, 2\pi]$, the KWM spectrum of a stationary k-mode quantum process over $\mathbb{Z}$ cannot admit an interval gap.

Remark 4.5. In Theorem 4.2, express the KWM spectrum Φ as

$$\Phi = [[\phi_{rs}]], \quad \text{where } r, s \in \{1, 2, \cdots, 2k\},$$

and write

$$\begin{aligned} \Phi_q &= [[\phi_{2i-1,2j-1}]], \quad \text{where } i, j \in \{1, 2, \cdots, k\}, \\ \Phi_p &= [[\phi_{2i,2j}]], \quad \text{where } i, j \in \{1, 2, \cdots, k\}. \end{aligned}$$

In the inductive limit C* probability space $(\mathcal{B}_\infty, \omega)$ associated with the process $\boldsymbol{\varrho}$ described in §2 the commuting family of position observables $\{q_{a\,r} : a \in D, r \in \{1, 2, \cdots, k\}\}$ affiliated to $\mathcal{B}_\infty$ execute a classical weakly stationary process with spectrum Φ_q. A similar property holds for the family $\{p_{a\,r} : a \in D, r \in \{1, 2, \cdots, k\}\}$.

In the quantum Gaussian case this raises the question that under what conditions on the spectrum Φ do these processes enjoy properties like ergodicity, weak mixing, strong mixing etc. The results of G. Maruyama [6] suggest that a minimum requirement would be the absence of atoms in the spectrum Φ.

Suppose Φ has no atoms. For arbitrary real scalars c_r, where $1 \leq r \leq 2k$, consider the associated observables

$$Z_a = \sum_{r=1}^{2k} (c_{2r-1} q_{ar} + c_{2r} p_{ar}) \quad \forall a \in D.$$

Then $\{Z_a : a \in D\}$ executes a classical Gaussian process with values in the real line and autocovariance function $\boldsymbol{c}^T \widetilde{K}(\cdot)\boldsymbol{c}$ with $\boldsymbol{c}^T = (c_1, c_2, \cdots, c_{2k})$ and spectrum equal to $\boldsymbol{c}^T \Phi(\cdot)\boldsymbol{c}$, a measure in $\widehat{D}$. Now let $D = \mathbb{Z}$ be the integer group. Then Maruyama's theorem implies that this scalar-valued process is, indeed, weakly mixing, and in particular ergodic. If $\lim_{a\to\infty} \boldsymbol{c}^T \widetilde{K}(a)\boldsymbol{c} = 0$, then this scalar-valued process is also strongly mixing.

Remark 4.6. Following [9] one can introduce the observable

$$N_{a\,j} = \frac{1}{2}\left(q_{a\,j}^2 + p_{a\,j}^2 - 1\right), \qquad \text{where } 1 \le j \le k,$$

which is the number of particles (photons) in the j-th mode at the site a. If the underlying process $\boldsymbol{\varrho}$ is Gaussian with mean $\mathbf{0}$ then

$$\langle N_{a\,j}\rangle = \frac{1}{2}\left\{\phi_{2j-1,2j-1}(\widehat{D}) + \phi_{2j,2j}(\widehat{D}) - \frac{1}{2}\right\}.$$

This is a consequence of property (1) of Φ in Theorem 4.2 and equation (3.4) from Corollary 3.1 in [9].

This shows that the relationships between photon numbers and KWM spectrum need a deeper exploration.

Theorem 4.2 can be strengthened as follows:

Theorem 4.7. *Let D be a countable, discrete, additive abelian group with its compact character group $\widehat{D}$ and let λ be the normalized Haar measure of $\widehat{D}$ on its Borel σ-algebra $\mathcal{F}$. Suppose Φ is a complex, totally finite $2k \times 2k$ positive matrix-valued measure on $\mathcal{F}$. Then Φ is the KWM spectrum of a k-mode weakly stationary quantum process $\boldsymbol{\varrho}$ over D if and only if Φ admits the representation*

$$\Phi(S) = \int_S F(\chi)\lambda(\mathrm{d}\chi) + \Psi(S) \quad \forall S \in \mathcal{F}, \tag{4.11}$$

where F is a $2k \times 2k$ positive Hermitian matrix-valued Borel function satisfying the matrix inequality

$$F(\chi) + \frac{\imath}{2}J_{2k} \ge 0 \quad \forall \chi \in D, \tag{4.12}$$

and

$$S \mapsto [[\psi_{rs}(S)]] \tag{4.13}$$

is a $2k\times 2k$ positive matrix-valued measure on $\mathcal{F}$ with each ψ_{rs} being singular with respect to λ.

In particular, F can be chosen to satisfy

$$\det \operatorname{Re} F(\chi) \ge \frac{1}{4^k} \quad \forall \chi \in \widehat{D} \tag{4.14}$$

and the absolutely continuous part of $\boldsymbol{u}^T\Phi(\cdot)\boldsymbol{u}$ is equivalent to λ for every nonzero element $\boldsymbol{u}$ of $\mathbb{C}^{2k}$.

Proof. Let $\Phi = [[\phi_{rs}]]$, with $r, s \in \{1, 2, \cdots, 2k\}$, be the KWM spectrum of a k-mode weakly stationary quantum process $\boldsymbol{\varrho}$ over D. By property (3) of Theorem 4.2 it follows that

$$\begin{bmatrix} \phi_{2r-1,2r-1}(S) & \phi_{2r-1,2r}(S) + \frac{\imath}{2}\lambda(S) \\ \phi_{2r,2r-1}(S) - \frac{\imath}{2}\lambda(S) & \phi_{2r,2r}(S) \end{bmatrix} \ge 0$$

for any $S \in \mathcal{F}$. Suppose $\phi_{2r-1,2r-1}(S) = 0$. Then positivity of Φ implies that $\phi_{2r-1,2r}(S) = \phi_{2r,2r-1}(S) = 0$ and hence

$$\begin{bmatrix} 0 & \frac{\imath}{2}\lambda(S) \\ -\frac{\imath}{2}\lambda(S) & \phi_{2r,2r}(S) \end{bmatrix} \geq 0.$$

Since the determinant of the left hand side in the inequality is nonnegative we conclude that $\lambda(S) = 0$. In other words $\lambda \ll \phi_{2r-1,2r-1}$. By the same argument $\lambda \ll \phi_{2r,2r}$. In other words λ is absolutely continuous with respect to every diagonal entry of Φ.

Choose and fix an $S_0 \in \mathcal{F}$ such that $S_0 = S_0^{-1}$, $\lambda(S_0) = 1$, the measure μ_{rr} defined by setting

$$\mu_{rr}(S) = \phi_{rr}(S \cap S_0) \quad \forall S \in \mathcal{F}$$

is the part of ϕ_{rr} equivalent to λ and the measure ψ_{rr} defined by setting

$$\psi_{rr}(S) = \phi_{rr}(S \cap (\widehat{D} \backslash S_0)) \quad \forall S \in \mathcal{F}$$

is the part of ϕ_{rr} singular with respect to λ, so that

$$\phi_{rr} = \mu_{rr} + \psi_{rr} \quad \forall r = 1, 2, \cdots, 2k.$$

Now define

$$\begin{aligned} \mu_{rs}(S) &= \phi_{rs}(S \cap S_0) \quad \text{and} \\ \psi_{rs}(S) &= \phi_{rs}(S \cap (\widehat{D} \backslash S_0)) \quad \forall S \in \mathcal{F}. \end{aligned}$$

If $\lambda(S) = 0$ then $\mu_{rr}(S) = 0$, so $\phi_{rr}(S \cap S_0) = 0$, whence $\phi_{rs}(S \cap S_0) = 0$ and therefore $\mu_{rs}(S) = 0$. In other words $\mu_{rs} \ll \lambda$. By definition all the measures ψ_{rs} are singular with respect to λ and

$$\phi_{rs} = \mu_{rs} + \psi_{rs} \quad \forall r, s \in \{1, 2, \cdots, 2k\}.$$

Define f_{rs} to be the Radon Nykodym derivative of μ_{rs} with respect to λ and put

$$F(\chi) = [[f_{rs}(\chi)]] \quad \forall r, s \in \{1, 2, \cdots, 2k\}.$$

Now F is defined a.e. with respect to λ and

$$\Phi(S) = \int_S F(\chi)\lambda(\mathrm{d}\chi) + \Psi(S) \quad \forall S \in \mathcal{F}, \tag{4.15}$$

where every entry ψ_{rs} of Ψ is singular with respect to λ. By the choice of S_0 it follows that

$$\int_S F(\chi)\lambda(\mathrm{d}\chi) + \frac{\imath}{2}\lambda(S)J_{2k} \geq 0$$

for all $S \subset S_0$, such that $S \in \mathcal{F}$, so that

$$\int_S \left(F(\chi) + \frac{\imath}{2}J_{2k}\right)\lambda(\mathrm{d}\chi) \geq 0.$$

Thus

$$F(\chi) + \frac{\imath}{2}J_{2k} \geq 0, \quad \lambda \text{ a.e.}, \tag{4.16}$$

and also, by the symmetry of the Haar measure,

$$F(\chi^{-1}) + \frac{\imath}{2}J_{2k} \geq 0, \quad \lambda \text{ a.e..} \tag{4.17}$$

On the other hand

$$\begin{aligned}\Phi(S^{-1}) &= \int_S F(\chi^{-1})\lambda(\mathrm{d}\chi) + \Psi(S^{-1}),\\ \overline{\Phi(S)} &= \int_S \overline{F(\chi)}\lambda(\mathrm{d}\chi) + \overline{\Psi(S)}.\end{aligned}$$

The conjugate symmetry of Φ, property (4) of Theorem 4.2, and the fact that $\Psi(S) = 0$ if $S \subseteq S_0 = S_0^{-1}$ imply

$$F(\chi^{-1}) = \overline{F(\chi)}, \quad \lambda \text{ a.e.}$$

Choosing $F(\chi) = \frac{1}{2}I_{2k}$ whenever this fails we may assume that $F(\chi^{-1}) = \overline{F(\chi)}$ for all χ. Now (4.16) and (4.17) imply that F can be altered on a set of λ-measure zero so that

$$\operatorname{Re} F(\chi) = \frac{F(\chi) + F(\chi^{-1})}{2} \geq -\frac{\imath}{2} J_{2k}$$

holds for every χ. In other words F is a complex positive $2k \times 2k$ matrix whose real part is the quantum covariance matrix of position and momentum observables in a k-mode state in $L^2(\mathbb{R}^k)$. It follows from [8] that

$$\det \operatorname{Re} F(\chi) \geq \frac{1}{4^k} \quad \forall \chi \in \widehat{D}$$

and the representation (4.11) holds.

The converse is already a part of Theorem 4.2. □

Remark 4.8. Suppose the quantum process $\boldsymbol{\varrho}$ of Theorem 4.7 is Gaussian and symmetric under the reflection transformation $a \mapsto -a$ in D. Then the autocovariance function K of the process $\boldsymbol{\varrho}$ satisfies the condition $K(a) = K(-a)$, $a \in D$ and the KWM spectrum Φ is real, i.e., $\Phi = \overline{\Phi}$. Then (4.14) implies

$$\int_{\widehat{D}} \log\det \Phi(\chi)\lambda(\mathrm{d}\chi) > -\infty.$$

When $D = \mathbb{Z}$ is the integer group it follows from the Wiener Masani theorem, in particular, that the position observables $(q_{n\,1}, \cdots, q_{n\,k})$ execute a purely indeterministic shift invariant k-variate Gaussian process. So do the momentum observables $(p_{n\,1}, \cdots, p_{n\,k})$. It may be recalled that a d-variate, mean zero shift invariant Gaussian process of random variables $\{\boldsymbol{\xi}_n = (\xi_{n\,1}, \xi_{n\,2}, \cdots, \xi_{n\,d}), -\infty < n < \infty\}$ defined on a probability space $(\Omega, \widetilde{\mathcal{F}}, P)$ is a purely indeterministic if, for any n,

$$\bigcap_{-\infty}^{n} \mathcal{H}_m = \{0\}$$

where $\mathcal{H}_m$ denotes the closed subspace spanned by the random variables $\{\xi_{r\,j},\, -\infty < r \leq m,\, 1 \leq j \leq d\}$ in $L^2(P)$.

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