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HOMEOMORPHIC PROPERTY OF THE STOCHASTIC FLOW OF A NATURAL EQUATION IN MULTI-DIMENSIONAL CASE

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Abstract. The one-default models are widely applied in modeling financial risk and price valuation of financial products such as credit default swap. In this paper, we are interested essentially in the so-called natural model. This model is expressed by a stochastic differential equation called ♮-equation introduced in [5]; this equation displays the evolution of the defaultable market. So, on the same model and with some assumptions, we will try to prove a few properties of the stochastic flow generated by a ♮-equation but in a multidimensional case and with some modifications. This is the main motivation of our research.

1. Introduction

In [5] a new class of random times has been introduced. Precisely, it is proved that, for any continuous increasing process Λ null at the origin, for any continuous non-negative local martingale N such that $Z_t = N_t e^{-\Lambda_t}$ with $0 < Z_t < 1$, $t > 0$ denotes the default intensity, for any continuous local martingale Y, and for any Lipschitz function f on $\mathbb{R}$ null at the origin, there exists a random variable τ such that the family of conditional expectations $X_t^u = \mathbb{Q}[\tau \leq u|\mathcal{F}_t]$, $u > 0$, $t < \infty$, satisfies the following stochastic differential equation :

$$(\natural_u) : \begin{cases} dX_t = X_t\left(- \dfrac{e^{-\Lambda_t}}{1 - Z_t} dN_t + f(X_t - (1 - Z_t))dY_t \right), & t \in [u, \infty), \\ X_u = x. \end{cases}$$

We call this setting a ♮-model, where the initial condition x can be any $\mathcal{F}_u$-measurable random variable.

We introduce the ♮-model in a multi-dimensional case. Let F be a continuous Lipschitz mapping from $\mathbb{R}^d$ into itself and $Y(t, \omega) = (Y_1(t, \omega), ..., Y_r(t, \omega))$ denote an r-dimensional continuous local martingale defined on a probability space $(\Omega, \mathbb{F} = (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$. We consider the stochastic differential equation $(\natural_u)$ on $\mathbb{R}^d$:

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$$(\natural_u) : \begin{cases} dX_t = X_t\Big(-\dfrac{e^{-\Lambda_t}}{1-Z_t}dN_t + \displaystyle\sum_{i=1}^{d}\sum_{j=1}^{r} F_j^i\,(X_t - (1-Z_t))dY_t^j \Big), \\ X_u = x, \end{cases}$$

for $t \in [u, \infty),\ 1 \leq j \leq r$.

The property appears important in recent study of stochastic differential geometry, and has been studied by several authors, e.g. Elworthy [3], Malliavin [7], Ikeda-Watanabe [4], Bismut [1]. We are inspired from the methods of proving the results obtained in [6] by Hiroshi Kunita. The main result of this paper is to prove the homeomorphism property of the stochastic flow generated by the stochastic flow associated with the $\natural$-equation based on Hiroshi Kunita theory, but we impose the following hypotheses:

Hypothesis 1.1. *We keep the same naturel model, but we assume that all the processes indicated in the $\natural$-equation (the multidimensional case) take real values. Thus, we impose that the coefficients of this equation are Lipschitz continuous.*

Hypothesis 1.2. *We always assume the hypothesis mentioned in* [5]*, which denoted that the stochastic integral* $\int_u^t \frac{e^{-\Lambda_s}}{1-Z_s}dN_s$*,* $u \leq t < \infty$*, exists and defines a local martingale. So called the hypothesis* $H_Y(C)$.

The paper is organized as follows. In section 2, we will prove the found theorems and lemmas motivated by T.Yamada and S.Varadhan, which will appear in [6]. Section 3 presents the main results of this paper.

2. The Stochastic Flow of a Stochastic Differential Equation

This section is borrowed from [6].

2.1. Flow of homeomorphisms for the solution of SDE. In this subsection, let $G_1(x), ..., G_r(x)$ be continuous mappings from $\mathbb{R}^d$ into itself and $M_t^1, ..., M_t^r$ continuous semimartingales defined on a probability space $(\Omega, \mathbb{F}, \mathbb{P}; \mathbb{F}_t)$. Here $\mathbb{F}_t$, $0 \leq t < \infty$ is an increasing family of sub σ-fields of $\mathbb{F}$ such that $\wedge_{\varepsilon>0}\mathbb{F}_{t+\varepsilon} = \mathbb{F}_t$ holds for each t. Consider an Itô stochastic differential equation (SDE) on $\mathbb{R}^d$:

$$d\xi_t = \sum_{j=1}^{r} G_j(\xi_t)dM_t^j. \tag{2.1}$$

A sample continuous $\mathbb{F}_t$-adapted stochastic process ξ_t with values in $\mathbb{R}^d$ is called a solution of (2.1), if it satisfies

$$\xi_t = \xi_0 + \sum_{j=1}^{d} \int_0^t G_j(\xi_s)dM_s^j, \tag{2.2}$$

where the right hand side is the Itô integral.

Concerning the coefficients of the equation, we will assume in this section that they are Lipschitz continuous, i.e., there is a positive constant L such that

$$|G_j^i(x) - G_j^i(y)| \leq L|x-y|, \qquad \forall x, y \in \mathbb{R}^d$$

holds for all indices i, j, where $G_j^i(x)$ is the i-th component of the vector function $G_j(x)$. Then for a given point x of $\mathbb{R}^d$, the equation has a unique solution such that $\xi_0(x) = x$. We denote it by $\xi_t(x)$ or $\xi_t(x, \omega)$. It is continuous in (t, x) a.s. In fact, the following proposition is well known.

Proposition 2.1 ([9]). *$\xi_t(x, \omega)$ is continuous in $[0, \infty) \times \mathbb{R}^d$ for almost all ω. Furthermore, for any $T > 0$ and $p \geq 2$, there is a positive constant $K_{p,T}^{(1)}$ such that*

$$\mathbb{E}|\xi_t(x) - \xi_s(y)|^p \leq K_{p,T}^{(1)} \left(|x - y|^p + |t - s|^{\frac{p}{2}}\right) \tag{2.3}$$

holds for all x, y of $\mathbb{R}^d$ and t, s of $[0, T]$.

We thus think that for a fixed t, $\xi_t(\cdot, \omega)$ is a continuous map from $\mathbb{R}^d$ into itself for almost all ω. The purpose of this section is to prove that the map $\xi_t(\cdot, \omega)$ is one to one and onto, and that the inverse map $\xi_t^{-1}(\cdot, \omega)$ is also continuous. Namely we will prove

Theorem 2.2. *Suppose that $G_1, ..., G_r$ of equation (2.1) are Lipschitz continuous. Then the solution map $\xi_t(\cdot, \omega)$ is a homeomorphism of $\mathbb{R}^d$ for all t, a.s. in ω.*

Before proving the theorem, we would like to mention a few remarks.

Remark 2.3. In the case of one dimensional SDE, Ogura and Yamada [8] have shown the same result under a weaker condition, using a strong comparison theorem of solutions. In fact, if coefficients are Lipschitz continuous on any finite interval (local Lipschitz) and if they are of linear growth, i.e., $|G_j(x)| \leq C(1 + |x|)$ holds for all x with some positive C, then the solution $\xi_t(\cdot, \omega)$ is a homeomorphism a.s. for any t.

Remark 2.4. The (local) Lipschitz continuity of coefficients is crucial for the theorem. Ogura and Yamada [8] have given an example of a one dimensional SDE with α-Hölder continuous coefficients ($\frac{1}{2} < \alpha < 1$), which has a unique strong solution but does not have the "one to one" property.

Remark 2.5. It is enough to prove the theorem in the case that M_t^i, $i = 1, ..., r$, satisfy the properties below: Let $M_t^j = B_t^j + A_t^j$ be the decomposition of semimartingale such that B_t^j is a continuous local martingale and A_t^j is a continuous process of bounded variation. Let $\langle B^j \rangle_t$ be the quadratic variation of B_t^j. Then for each j and all $s < t$,

$$A_t^j - A_s^j \leq t - s \ , \ \langle B^j \rangle_t - \langle B^j \rangle_s \leq t - s, \qquad \forall s < t \tag{2.4}$$

In the following discussion, condition (2.4) is always assumed. We will first show the "one to one" property. Our approach is based on several elementary inequalities.

Lemma 2.6. *Let $T > 0$ and p be any real number. Then there is a positive constant $K_{p,T}^{(2)}$ such that $\forall x, y \in \mathbb{R}^d$ and $\forall t \in [0, T]$,*

$$\mathbb{E}|\xi_t(x) - \xi_s(y)|^p \leq K_{p,T}^{(2)} |x - y|^p. \tag{2.5}$$

Proof. If $x = y$, the inequality is clearly satisfied for any positive constant $K^{(2)}_{p,T}$. We shall assume $x \neq y$. Let ε be an arbitrary positive number and $\sigma_\varepsilon = \inf\{t > 0;\ |\xi_t(x) - \xi_t(y)| < \varepsilon\}$. We shall apply Itô's formula to $f(z) = |z|^p$. Then we have for $t < \sigma_\varepsilon$,

$$\begin{aligned}
&|\xi_t(x) - \xi_t(y)|^p - |x-y|^p \\
&= \sum_{i,j} \int_0^t \frac{\partial f}{\partial z_i}\left(\xi_s(x) - \xi_s(y)\right)\left(G^i_j(\xi_s(x)) - G^i_j(\xi_s(y))\right) dM^j_s \\
&\quad + \frac{1}{2}\sum_{i,j,k,l}\int_0^t \frac{\partial^2 f}{\partial z_i \partial z_j}\left(\xi_s(x) - \xi_s(y)\right)\left(G^i_k(\xi_s(x)) - G^i_k(\xi_s(y))\right) \\
&\qquad\qquad \times \left(G^j_l(\xi_s(x)) - G^j_l(\xi_s(y))\right)\, d\langle M^k, M^l\rangle_s \\
&= I_t + J_t.
\end{aligned}$$

Note $\dfrac{\partial f}{\partial z_i} = p|z|^{p-2} z_i$ and apply Lipschitz inequality. Then

$$\sum_i \left| \frac{\partial f}{\partial z_i}\left(\xi_s(x) - \xi_s(y)\right)\left(G^i_j(\xi_s(x)) - G^i_j(\xi_s(y))\right)\right| \leq |p|\sqrt{d}L|\xi_s(x) - \xi_s(y)|^p.$$

Therefore we have

$$|\mathbb{E}I_{t\wedge\sigma_\varepsilon}| \leq |p|r\sqrt{d}L\int_0^t \mathbb{E}|\xi_{s\wedge\sigma_\varepsilon}(x) - \xi_{s\wedge\sigma_\varepsilon}(y)|^p ds.$$

Next, note that

$$\frac{\partial^2 f}{\partial z_i \partial z_j} = p|z|^{p-2}\delta_{ij} + p(p-2)|z|^{p-4} z_i z_j,$$

where δ_{ij} is the Kronecker's delta. Then

$$\begin{aligned}
&\left|\sum_{i,j}\frac{\partial^2 f}{\partial z_i \partial z_j}(\xi_s(x) - \xi_s(y))(G^i_k(\xi_s(x)) - G^i_k(\xi_s(y)))(G^i_l(\xi_s(x)) - G^i_l(\xi_s(y)))\right| \\
&\leq |p|(|p-2| + d)L^2|\xi_s(x) - \xi_s(y)|^p|.
\end{aligned}$$

Therefore

$$|\mathbb{E}J_{t\wedge\sigma_\varepsilon}| \leq \frac{1}{2}r^2|p|(|p-2|+d)L^2\int_0^t \mathbb{E}|\xi_{s\wedge\sigma_\varepsilon}(x) - \xi_{s\wedge\sigma_\varepsilon}(y)|^p ds.$$

Summing up these two inequalities, we obtain

$$\mathbb{E}|\xi_{t\wedge\sigma_\varepsilon}(x) - \xi_{t\wedge\sigma_\varepsilon}(y)|^p \leq |x-y|^p + C_p\int_0^t \mathbb{E}|\xi_{t\wedge\sigma_\varepsilon}(x) - \xi_{t\wedge\sigma_\varepsilon}(y)|^p ds,$$

where C_p is a positive constant. By Grönwall's inequality,

$$\mathbb{E}|\xi_{t\wedge\sigma_\varepsilon}(x) - \xi_{t\wedge\sigma_\varepsilon}(y)|^p \leq K^{(2)}_{p,T}|x-y|^p, \quad \forall t \in [0,T],$$

where $K^{(2)}_{p,T} = \exp(C_p T)$. Letting ε tend to 0, we have

$$\mathbb{E}|\xi_{t\wedge\sigma}(x) - \xi_{t\wedge\sigma}(y)|^p \leq K^{(2)}_{p,T}|x-y|^p,$$

where σ is the first time t such that $\xi_t(x) = \xi_t(y)$. However, we have $\sigma = \infty$ a.s., since otherwise the left hand side would be infinity if $p < 0$. The proof is complete. □

The above lemma shows that if $x \neq y$ then $\xi_t(x) \neq \xi_t(y)$ holds a.s. for all t. But it does not conclude that $\xi_t(\cdot, \omega)$ is "one to one", since the exceptional null set $N_{x,y} = \{\omega\,; \xi_t(x) = \xi_t(y) \,\text{for some t}\}$ depends on the pair (x, y). To overcome this point, we shall prove the following lemma.

Lemma 2.7 (Varadhan). *Set*

$$\eta_t(x,y) = \frac{1}{|\xi_t(x) - \xi_t(y)|}. \tag{2.6}$$

Then $\eta_t(x, y)$ is continuous in $[0, \infty) \times \{(x, y) \in \mathbb{R}^{2d} | x \neq y\}$.

Proof. Suppose $p > 2(2d + 1)$. We have

$$|\eta_t(x,y) - \eta_{t'}(x',y')|^p \le 2^p \eta_t(x,y)^p \eta_{t'}(x',y')^p \{|\xi_t(x) - \xi_{t'}(x')|^p + |\xi_t(y) - \xi_{t'}(y')|^p\}.$$

By Hölder's inequality,

$$\begin{aligned}
&\mathbb{E}|\eta_t(x,y) - \eta_{t'}(x',y')|^p \\
&\le 2^p \{\mathbb{E}(\eta_t(x,y)^{4p})\mathbb{E}(\eta_{t'}(x',y')^{4p})\}^{\frac14} \\
&\quad \times \{(\mathbb{E}|\xi_t(x) - \xi_{t'}(x')|^{2p})^{\frac12} + (\mathbb{E}|\xi_t(y) - \xi_{t'}(y')|^{2p})^{\frac12}\}.
\end{aligned}$$

By Lemma 2.6 and Proposition 2.1, we have

$$\begin{aligned}
&\mathbb{E}|\eta_t(x,y) - \eta_{t'}(x',y')|^p \\
&\le C_{p,T}|x-y|^{-p}|x'-y'|^{-p}\{|x-x'|^p + |y-y'|^p + 2|t-t'|^{\frac{p}{2}}\},
\end{aligned}$$

$$\mathbb{E}|\eta_t(x,y) - \eta_{t'}(x',y')|^p \le C_{p,T}\delta^{-2p}\{|x-x'|^p + |y-y'|^p + 2|t-t'|^{\frac{p}{2}}\}, \tag{2.7}$$

if $|x - y| \ge \delta$ and $|x' - y'| \ge \delta$, where $C_{p,T}$ is a positive constant. Then by Kolmogorov's theorem, $\eta_t(x, y)$ is continuous in $[0, T] \times \{(x, y); |x-y| \ge \delta\}$. Since T and δ are arbitrary positive numbers, we get the assertion. The proof is complete. □

The above lemma leads immediately to the "one to one" property of the map $\xi_t(\cdot, \omega)$ a.s. for all t. We shall consider next the onto property. We first establish

Lemma 2.8. *Let $T > 0$ and let p be any real number. Then there is a positive constant $K^{(3)}_{p,T}$ such that*

$$\mathbb{E}(1 + |\xi_t(x)|^2)^p \le K^{(3)}_{p,T}(1 + |x|^2)^p, \quad \forall x \in \mathbb{R}^d, \ \forall t \in [0, T]. \tag{2.8}$$

Proof. We shall apply Itô's formula to the function $f(z) = (1 + |z|^2)^p$. We have

$$
\begin{aligned}
f(\xi_t(x)) - f(x) &= \sum_{i,j} \int_0^t \frac{\partial f}{\partial z_i}(\xi_s(x)) G^i_j(\xi_s(x)) dM^j_s \\
&\quad + \frac{1}{2} \sum_{i,j,k,l} \int_0^t \frac{\partial^2 f}{\partial z_i \partial z_j}(\xi_s(x)) G^i_k(\xi_s(x)) G^j_l(\xi_s(y))\ d\langle M^k, M^l \rangle_s \\
&= I_t + J_t.
\end{aligned}
$$

Let K be a positive constant such that

$$|G^i_j(x)| \leq K(1+|x|^2)^{\frac{1}{2}}.$$

holds for all i and j. Then,

$$\left| \sum_i \frac{\partial f}{\partial z_i}(\xi_s(x)) G^i_j(\xi_s(x)) \right| \leq 2\sqrt{d}\,|p| K (1+|\xi_s(x)|^2)^p.$$

Therefore,

$$|\mathbb{E}\, I_t| \leq 2r\sqrt{d}\,|p| K \int_0^t \mathbb{E}(1+|\xi_s(x)|^2)^p ds.$$

Similarly,

$$\left| \frac{1}{2} \sum_{i,j} \frac{\partial^2 f}{\partial z_i \partial z_j}(\xi_s(x)) G^i_k(\xi_s(x)) G^j_l(\xi_s(x)) \right| \leq |p|(d+2|p-1|)K^2(1+|\xi_s(x)|^2)^p,$$

so that

$$|\mathbb{E}\, J_t| \leq |p| r^2 (d+2|p-1|) K^2 \int_0^t \mathbb{E}(1+|\xi_s(x)|^2)^p ds.$$

Therefore we have

$$\mathbb{E}(1+|\xi_t(x)|^2)^p \leq (1+|x|^2)^p + \text{ const.} \int_0^t \mathbb{E}(1+|\xi_s(x)|^2)^p ds.$$

By Grönwall's inequality, we get the inequality of the lemma. □

Remark 2.9. We have $(1+|x|^2) \leq (1+|x|)^2 \leq 2(1+|x|^2)$. Therefore, inequality (2.8) implies

$$\mathbb{E}(1+|\xi_t(x)|)^{2p} \leq 2^{|p|} K^{(3)}_{p,T} (1+|x|)^{2p}. \tag{2.9}$$

Now taking negative p in the above lemma, we see that $|\xi_t(x)|$ tends to infinity in probability as x tends sequencially to infinity. We shall prove a stronger convergence. We claim

Lemma 2.10. *Let $\overline{\mathbb{R}^d} = \mathbb{R}^d \cup \{\infty\}$ be the one point compactification of $\mathbb{R}^d$. Set*

$$\eta_t(x) = \begin{cases} \dfrac{1}{1+|\xi_t(x)|}, & \text{if } x \in \mathbb{R}^d, \\ 0, & \text{if } x = \infty. \end{cases}$$

Then $\eta_t(x,\omega)$ is a continuous map from $[0,\infty) \times \overline{\mathbb{R}^d}$ into $\mathbb{R}$ a.s..

Proof. Obviously $\eta_t(x)$ is continuous in $[0,\infty)\times\mathbb{R}^d$. Hence it is enough to prove the continuity in the neighborhood of infinity. Suppose $p > 2(2d+1)$. We have

$$|\eta_t(x)-\eta_s(y)|^p \le \eta_t(x)^p\eta_s(y)^p|\xi_t(x)-\xi_s(y)|^p.$$

By Hölder's inequality, Proposition 2.1 and lemma 2.8, we have

$$\begin{aligned}\mathbb{E}|\eta_t(x)-\eta_s(y)|^p &\le (\mathbb{E}\eta_t(x)^{4p})^{\frac14}(\mathbb{E}\eta_s(y)^{4p})^{\frac14}(\mathbb{E}|\xi_t(x)-\xi_s(y)|^{2p})^{\frac12}\\ &\le C_{p,T}(1+|x|)^{-p}(1+|y|)^{-p}(|x-y|^p+|t-s|^{\frac{p}{2}}),\end{aligned}$$

if $t,s\in[0,T]$ and $x,y\in\mathbb{R}^d$, where $C_{p,T}$ is a positive constant. Set $\dfrac{1}{x} = (x_1^{-1},...,x_d^{-1})$. Since

$$\frac{|x-y|}{(1+|x|)(1+|y|)} \le \left|\frac{1}{x}-\frac{1}{y}\right|,$$

we get the inequality

$$\mathbb{E}|\eta_t(x)-\eta_s(y)|^p \le C_{p,T}\left(\left|\frac{1}{x}-\frac{1}{y}\right|^p + |t-s|^{\frac{p}{2}}\right).$$

Define

$$\tilde{\eta}_t(x) = \begin{cases} \eta_t(\frac{1}{x}), & \text{if } x\neq 0,\\ 0, & \text{if } x = 0.\end{cases}$$

Then the above inequality implies

$$\mathbb{E}|\tilde{\eta}_t(x)-\tilde{\eta}_s(y)|^p \le C_{p,T}\left(|x-y|^p + |t-s|^{\frac{p}{2}}\right), \quad x\neq 0\,,\, y\neq 0.$$

In the case $y=0$, we have

$$\mathbb{E}|\tilde{\eta}_t(x)|^p \le C_{p,T}|x|^p.$$

Therefore, $\tilde{\eta}_t(x)$ is continuous in $[0,\infty)\times\mathbb{R}^d$ by Kolmogorov's theorem. This proves that $\tilde{\eta}_t(x)$ is continuous in $[0,\infty)\times$neighborhood of infinity. □

Lemma 2.11. *Define a stochastic process* $\bar{\xi}_t$ *on* $\overline{\mathbb{R}^d} = \mathbb{R}^d\cup\{\infty\}$ *by*

$$\bar{\xi}_t(x) = \begin{cases} \xi_t(x), & if\ x\in\mathbb{R}^d,\\ \infty, & if\ x=\infty.\end{cases}$$

Then $\bar{\xi}_t(x)$ *is continuous in* $[0,\infty)\times\overline{\mathbb{R}^d}$.

Proof. We have the proof by the previous lemma. Thus for each $t>0$, the map $\bar{\xi}_t(\cdot,\omega)$ is homotopic to the identity map on $\overline{\mathbb{R}^d}$, which is homeomorphic to d-dimensional sphere $\mathcal{S}^d$. Then $\bar{\xi}_t(\cdot,\omega)$ is an onto map of $\overline{\mathbb{R}^d}$ by a well known homotopic theory. □

Now the map $\bar{\xi}_t$ is a homeomorphism of $\overline{\mathbb{R}^d}$, since it is one to one, onto and continuous. Since ∞ is the invariant point of the map $\bar{\xi}_t$, we see that ξ_t is a homeomorphism of $\mathbb{R}^d$. This completes the proof of Theorem 2.2.

3. Main Result

We now turn to the $\natural$-equation in higher dimensions. Let $(\Lambda_1, ..., \Lambda_d)$ be a d-dimensional continuous increasing process null at the origin, and a d-dimensional continuous non-negative local martingale N such that $Z = N e^{-\Lambda}$ with $0 < Z < 1$, $t > 0$ and $Z(t,\omega) = (Z_1(t,\omega), ..., Z_d(t,\omega))$ denotes the default intensity. Let F be continuous, Lipschitz mapping from $\mathbb{R}^d$ into itself and $Y(t,\omega) = (Y_1(t,\omega), ..., Y_r(t,\omega))$ denote a r-dimensional continuous local martingale defined on a probability space $(\Omega, \mathbb{F} = (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$. We consider the $\natural$-equation in multi-dimensional case :

$$(\natural_u) : \begin{cases} dX_1(t) = X_1(t)\left(-\dfrac{e^{-\Lambda_1(t)}}{1-Z_1(t)}dN_1(t) + F_{11}dY_1 + \cdots + F_{1d}dY_r\right), \\ \vdots \qquad\qquad \vdots \qquad\qquad\qquad\qquad \vdots \\ dX_d(t) = X_d(t)\left(-\dfrac{e^{-\Lambda_d(t)}}{1-Z_d(t)}dN_d(t) + F_{r1}dY_1 + \cdots + F_{rd}dY_r\right), \end{cases}$$

with $X(u) = (x_1, ..., x_d)^T$ as the initial condition. Or, in matrix notation simply:

$$(\natural_u) : \begin{cases} dX_t = X_t\left(-\dfrac{e^{-\Lambda_t}}{1-Z_t}dN_t + F\left(X_t - (1-Z_t)\right)dY_t\right), \quad t \in [u, \infty), \\ X_u = x, \end{cases}$$

where $X(t) = (X_1(t), ..., X_d(t))^{\mathbf{T}}$, $-\dfrac{e^{-\Lambda_t}}{1-Z_t} = \left(-\dfrac{e^{-\Lambda_1}(t)}{1-Z_1(t)}, ..., -\dfrac{e^{-\Lambda_d}(t)}{1-Z_d(t)}\right)^{\mathbf{T}}$, $dN(t) = (dN_1(t), ..., dN_d(t))^{\mathbf{T}}$, $dY(t) = (dY_1(t), ..., dY_r(t))^{\mathbf{T}}$, where $\mathbf{T}$ denotes the transpose of a vector, and

$$F = \begin{pmatrix} F_{11} & . & . & . & F_{1d} \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ F_{r1} & . & . & . & F_{rd} \end{pmatrix}.$$

Concerning coefficients of our equation, we will assume in this section that they are Lipschitz continuous, i.e., there is a positive constant $\tilde{L}$ such that:

$$|F_j^i(x) - F_j^i(y)| \leq \tilde{L}|x-y|, \quad \forall\, x, y \in \mathbb{R}^d,\ 1 \leq i \leq d,\ 1 \leq j \leq r,$$

holds for all indices i, j, where $F_j^i(x)$ is the i-th component of the vector function $F_j(x)$. Then for a given point x of $\mathbb{R}^d$, the $(\natural_u)$-equation has a unique solution such that $X_u = x$. We denote it as $X_t(x)$ or $X_t(x,\omega)$. It is continuous in (t,x) a.s. applying proposition 2.1 [9].

$$X_t^u(x) = x + \int_u^t X_s\left(-\frac{e^{-\Lambda_s}}{1-Z_s}\right)dN_s + \int_u^t \sum_{i=1}^d \sum_{j=1}^r X_s F_j^i\left(X_s - (1-Z_s)\right)dY_s^j.$$

We know that the quantity $F_j^i(X_s - (1-Z_s))$ is bounded because F is a Lipschitz function, but we do not know a priori if the quantity $\left(-\dfrac{e^{-\Lambda_s}}{1-Z_s}\right)$ is finite or not;

we introduce the stopping time $\tau_n = \inf\{t,\, 1 - Z_t < \frac{1}{n}\}$, therefore, we assume the process $\tilde{X}$ instead of X:

$$d\tilde{X}_t = \tilde{X}_t\Big(- \frac{e^{-\Lambda_t}}{1 - Z_{t\wedge\tau_n}} dN_t + \sum_{i=1}^{d}\sum_{j=1}^{r} F_j^i\,(\tilde{X}_t - (1 - Z_t))dY_t^j\Big),$$

such as $\tilde{X}_t = X_t$, $\forall t \le \tau_n$, $n \in \mathbb{N}$.

3.1. Proof of the one to one property. In this part we will apply lemma 2.6 to our model. So if $x = y$ the inequality is clearly satisfied for any constant $\tilde{K}^{(2)}_{p,T}$. We shall assume $x \neq y$. Let $\tilde{\varepsilon}$ be an arbitrary positive number and:

$$\sigma_{\tilde{\varepsilon}} = \inf\{t > 0;\, |\tilde{X}_t^u(x) - \tilde{X}_t^u(y)| < \tilde{\varepsilon}\}.$$

We denote $A_t = \tilde{X}_t^u(x) - \tilde{X}_t^u(y)$, and we shall apply Itô's formula to the function $f(z) = |z|^p$. Then we have for $t < \tilde{\varepsilon}$,

$$\tilde{X}_t^u(x) = x + \int_u^t \tilde{X}_s\left(-\frac{e^{-\Lambda_s}}{1 - Z_{s\wedge\tau_n}}\right) dN_s + \int_u^t \sum_{i=1}^{d}\sum_{j=1}^{r} \tilde{X}_s F_j^i\,\left(\tilde{X}_s - (1 - Z_s)\right) dY_s^j,$$

$$d\tilde{X}_t = \tilde{X}_t\left(-\frac{e^{-\Lambda_t}}{1 - Z_{t\wedge\tau_n}}\right) dN_t + \sum_{i=1}^{d}\sum_{j=1}^{r} \tilde{X}_t F_j^i\,\left(\tilde{X}_t - (1 - Z_t)\right) dY_t^j,$$

$$\begin{aligned}
&\left|\tilde{X}_t^u(x) - \tilde{X}_t^u(y)\right|^p - |x-y|^p \\
&= \sum_{i,j}\int_u^t \frac{\partial f}{\partial z_i}\left(\tilde{X}_t^u(x) - \tilde{X}_t^u(y)\right) \\
&\qquad \times \Bigg[\tilde{X}_s(x)\left(-\frac{e^{-\Lambda_s}}{1 - Z_{s\wedge\tau_n}}\right) dN_s + \tilde{X}_s(x)F_j^i\,\left(\tilde{X}_s(x) - (1 - Z_s)\right) dY_s^j \\
&\qquad\quad - \tilde{X}_s(y)\left(-\frac{e^{-\Lambda_s}}{1 - Z_{s\wedge\tau_n}}\right) dN_s + \tilde{X}_s(y)F_j^i\,\left(\tilde{X}_s(y) - (1 - Z_s)\right) dY_s^j\Bigg] \\
&\quad + \frac{1}{2}\sum_{i,j,k,l}\int_u^t \frac{\partial^2 f}{\partial z_i \partial z_j}\left(\tilde{X}_s^u(x) - \tilde{X}_s^u(y)\right) \\
&\qquad \times \Bigg[\tilde{X}_s(x)\left(-\frac{e^{-\Lambda_s}}{1 - Z_{s\wedge\tau_n}}\right) dN_s + \tilde{X}_s(x)F_k^i\,\left(\tilde{X}_s(x) - (1 - Z_s)\right) dY_s^k \\
&\qquad\quad - \tilde{X}_s(y)\left(-\frac{e^{-\Lambda_s}}{1 - Z_{s\wedge\tau_n}}\right) dN_s + \tilde{X}_s(y)F_k^i\,\left(\tilde{X}_s(y) - (1 - Z_s)\right) dY_s^k\Bigg] \\
&\qquad \times \Bigg[\tilde{X}_s(x)\left(-\frac{e^{-\Lambda_s}}{1 - Z_{s\wedge\tau_n}}\right) dN_s + \tilde{X}_s(x)F_l^j\,\left(\tilde{X}_s(x) - (1 - Z_s)\right) dY_s^l \\
&\qquad\quad - \tilde{X}_s(y)\left(-\frac{e^{-\Lambda_s}}{1 - Z_{s\wedge\tau_n}}\right) dN_s + \tilde{X}_s(y)F_l^j\,\left(\tilde{X}_s(y) - (1 - Z_s)\right) dY_s^l\Bigg],
\end{aligned}$$

$$
\begin{aligned}
&\left|\tilde{X}_t^u(x)-\tilde{X}_t^u(y)\right|^p-|x-y|^p\\
&=\sum_{i,j}\int_u^t\frac{\partial f}{\partial z_i}\left(\tilde{X}_s^u(x)-\tilde{X}_s^u(y)\right)\\
&\qquad\times\Bigg[\left(\tilde{X}_s(x)-\tilde{X}_s(y)\right)\left(-\frac{e^{-\Lambda_s}}{1-Z_{s\wedge\tau_n}}\right)dN_s\\
&\qquad\quad+\left(\tilde{X}_s(x)F_j^i\left(\tilde{X}_s(x)-(1-Z_s)\right)-\tilde{X}_s(y)F_j^i\left(\tilde{X}_s(y)-(1-Z_s)\right)\right)dY_s^j\Bigg]\\
&\quad+\frac{1}{2}\sum_{i,j,k,l}\int_u^t\frac{\partial^2 f}{\partial z_i\partial z_j}\left(\tilde{X}_s^u(x)-\tilde{X}_s^u(y)\right)\\
&\qquad\times\Bigg[\left(\tilde{X}_s(x)-\tilde{X}_s(y)\right)\left(-\frac{e^{-\Lambda_s}}{1-Z_{s\wedge\tau_n}}\right)dN_s\\
&\qquad\quad+\left(\tilde{X}_s(x)F_k^i\left(\tilde{X}_s(x)-(1-Z_s)\right)-\tilde{X}_s(y)F_k^i\left(\tilde{X}_s(y)-(1-Z_s)\right)\right)dY_s^k\Bigg]\\
&\qquad\times\Bigg[\left(\tilde{X}_s(x)-\tilde{X}_s(y)\right)\left(-\frac{e^{-\Lambda_s}}{1-Z_{s\wedge\tau_n}}\right)dN_s\\
&\qquad\quad+\left(\tilde{X}_s(x)F_l^j\left(\tilde{X}_s(x)-(1-Z_s)\right)-\tilde{X}_s(y)F_l^j\left(\tilde{X}_s(y)-(1-Z_s)\right)\right)dY_s^l\Bigg]\\
&=\tilde{I}_t+\tilde{J}_t,
\end{aligned}
$$

$$
\begin{aligned}
\tilde{I}_t=&\sum_{i,j}\int_u^t\frac{\partial f}{\partial z_i}\left(\tilde{X}_s^u(x)-\tilde{X}_s^u(y)\right)\\
&\times\Bigg[\left(\tilde{X}_s(x)-\tilde{X}_s(y)\right)\left(-\frac{e^{-\Lambda_s}}{1-Z_{s\wedge\tau_n}}\right)dN_s\\
&\quad+\left(\tilde{X}_s(x)F_j^i\left(\tilde{X}_s(x)-(1-Z_s)\right)-\tilde{X}_s(y)F_j^i\left(\tilde{X}_s(y)-(1-Z_s)\right)\right)dY_s^j\Bigg].
\end{aligned}
$$

Noting

$$
\begin{aligned}
V_j^i(\tilde{X}_s^x)&=\tilde{X}_s(x)F_j^i\left(\tilde{X}_s(x)-(1-Z_s)\right),\\
V_j^i(\tilde{X}_s^y)&=\tilde{X}_s(y)F_j^i\left(\tilde{X}_s(y)-(1-Z_s)\right),
\end{aligned}
$$

such that

$$
\left|V_j^i(\tilde{X}_s^x)-V_j^i(\tilde{X}_s^y)\right|\le\tilde{L}\left|\tilde{X}_s^x-\tilde{X}_s^y\right|,
$$

and

$$
\frac{\partial f}{\partial z_i}=p|z|^{p-2}z_i,
$$

we put

$$
\tilde{I}_t=\tilde{I}_t^1+\tilde{I}_t^2,
$$

such that

$$\tilde{I}_t^1 = \sum_{i,j} \int_u^t \frac{\partial f}{\partial z_i} \left(\tilde{X}_s^u(x) - \tilde{X}_s^u(y)\right) \left(\tilde{X}_s(x) - \tilde{X}_s(y)\right) \left(-\frac{e^{-\Lambda_s}}{1 - Z_{s\wedge\tau_n}}\right) dN_s,$$

$$\tilde{I}_t^2 = \sum_{i,j} \int_u^t \frac{\partial f}{\partial z_i} \left(\tilde{X}_s^u(x) - \tilde{X}_s^u(y)\right) \left(V_j^i(\tilde{X}_s^x) - V_j^i(\tilde{X}_s^y)\right) dY_s^j.$$

For $\tilde{I}_t^1$, we have:

$$\begin{aligned}
&\sum_i \left| \frac{\partial f}{\partial z_i} \left(\tilde{X}_s^u(x) - \tilde{X}_s^u(y)\right) \left(\tilde{X}_s(x) - \tilde{X}_s(y)\right) \right| \\
&\leq |p||z|^{p-2}|z_i|\sqrt{d} \left|\tilde{X}_s(x) - \tilde{X}_s(y)\right| \\
&\leq |p|\sqrt{d} \left|\tilde{X}_s(x) - \tilde{X}_s(y)\right|^p.
\end{aligned}$$

Therefore,

$$\tilde{I}_t^1 \leq |p|\sqrt{d} \int_u^t \left|\tilde{X}_s(x) - \tilde{X}_s(y)\right|^p ds \times \int_u^t -\frac{e^{-\Lambda_s}}{1 - Z_{s\wedge\tau_n}} dN_s.$$

Note that $Q_t = \displaystyle\int_u^t -\frac{e^{-\Lambda_s}}{1 - Z_{s\wedge\tau_n}} dN_s$, it is a local martingale (the so called hypothesis $H_Y(C)$ [5]). So

$$\tilde{I}_t^1 \leq |p| r\sqrt{d}\; Q_t \int_u^t \left|\tilde{X}_s(x) - \tilde{X}_s(y)\right|^p ds.$$

For $\tilde{I}_t^2$, we have

$$\begin{aligned}
&\sum_i \left| \frac{\partial f}{\partial z_i} \left(\tilde{X}_s^u(x) - \tilde{X}_s^u(y)\right) \left(V_j^i(\tilde{X}_s^x) - V_j^i(\tilde{X}_s^y)\right) \right| \\
&\leq |p||z|^{p-2}|z_i|\sqrt{d}\, \tilde{L} \left|\tilde{X}_s(x) - \tilde{X}_s(y)\right| \\
&\leq |p|\sqrt{d}\, \tilde{L} \left|\tilde{X}_s(x) - \tilde{X}_s(y)\right|^p.
\end{aligned}$$

Therefore,

$$\tilde{I}_t^2 \leq |p|\sqrt{d}\; r\, \tilde{L} \int_u^t \left|\tilde{X}_s(x) - \tilde{X}_s(y)\right|^p ds.$$

So, we have

$$\begin{aligned}
\tilde{I}_t =& \tilde{I}_t^1 + \tilde{I}_t^2 \\
\leq& |p| r\sqrt{d}\; Q_t \int_u^t \left|\tilde{X}_s(x) - \tilde{X}_s(y)\right|^p ds + |p|\sqrt{d}\; r\, \tilde{L} \int_u^t \left|\tilde{X}_s(x) - \tilde{X}_s(y)\right|^p ds \\
\leq& |p| r\sqrt{d} \int_u^t \left|\tilde{X}_s(x) - \tilde{X}_s(y)\right|^p ds\, (Q_t + \tilde{L}).
\end{aligned}$$

Therefore, we have

$$\left|\mathbb{E}\, \tilde{I}_{t\wedge\sigma_{\tilde{\varepsilon}}}\right| \leq |p| r\sqrt{d}\, (Q_{t\wedge\sigma_{\tilde{\varepsilon}}} + \tilde{L}) \int_u^t \mathbb{E} \left|\tilde{X}_{s\wedge\sigma_{\tilde{\varepsilon}}}(x) - \tilde{X}_{s\wedge\sigma_{\tilde{\varepsilon}}}(y)\right|^p ds. \tag{3.1}$$

Next,

$$\begin{aligned}\tilde{J}_t =&\frac{1}{2}\sum_{i,j,k,l}\int_u^t \frac{\partial^2 f}{\partial z_i \partial z_j}\left(\tilde{X}_s^u(x)-\tilde{X}_s^u(y)\right)\\
&\times\left[\left(\tilde{X}_s(x)-\tilde{X}_s(y)\right)\left(-\frac{e^{-\Lambda_s}}{1-Z_{s\wedge\tau_n}}\right)dN_s+\left(V_k^i(\tilde{X}_s^x)-V_k^i(\tilde{X}_s^y)\right)dY_s^k\right]\\
&\times\left[\left(\tilde{X}_s(x)-\tilde{X}_s(y)\right)\left(-\frac{e^{-\Lambda_s}}{1-Z_{s\wedge\tau_n}}\right)dN_s+\left(V_l^j(\tilde{X}_s^x)-V_l^j(\tilde{X}_s^y)\right)dY_s^l\right].\end{aligned}$$

$$\begin{aligned}\tilde{J}_t =&\frac{1}{2}\sum_{i,j,k,l}\int_u^t \frac{\partial^2 f}{\partial z_i \partial z_j}\left(\tilde{X}_s^u(x)-\tilde{X}_s^u(y)\right)\\
&\times\Bigg[\left(\tilde{X}_s(x)-\tilde{X}_s(y)\right)^2\left(-\frac{e^{-\Lambda_s}}{1-Z_{s\wedge\tau_n}}\right)^2 dN_s dN_s+\left(V_k^i(\tilde{X}_s^x)-V_k^i(\tilde{X}_s^y)\right)\\
&\times\left(V_l^j(\tilde{X}_s^x)-V_l^j(\tilde{X}_s^y)\right)dY_s^k dY_s^l\ +\left(\tilde{X}_s(x)-\tilde{X}_s(y)\right)\left(-\frac{e^{-\Lambda_s}}{1-Z_{s\wedge\tau_n}}\right)\\
&\times\left(V_l^j(\tilde{X}_s^x)-V_l^j(\tilde{X}_s^y)\right)\times dN_s\,dY_s^l+\left(\tilde{X}_s(x)-\tilde{X}_s(y)\right)\left(-\frac{e^{-\Lambda_s}}{1-Z_{s\wedge\tau_n}}\right)\\
&\times\left(V_k^i(\tilde{X}_s^x)-V_k^i(\tilde{X}_s^y)\right)dN_s\,dY_s^k\Bigg].\end{aligned}$$

Note that $\tilde{J}_t=\frac{1}{2}\left[\tilde{J}_t^1+\tilde{J}_t^2+\tilde{J}_t^3+\tilde{J}_t^4\right]$ such that

$$\begin{aligned}\tilde{J}_t^1 =&\sum_{i,j,k,l}\int_u^t \frac{\partial^2 f}{\partial z_i \partial z_j}\left(\tilde{X}_s^u(x)-\tilde{X}_s^u(y)\right)\times\left(\tilde{X}_s(x)-\tilde{X}_s(y)\right)^2\\
&\times\left(-\frac{e^{-\Lambda_s}}{1-Z_{s\wedge\tau_n}}\right)^2 dN_s dN_s,\end{aligned}$$

$$\begin{aligned}\tilde{J}_t^2 =&\sum_{i,j,k,l}\int_u^t \frac{\partial^2 f}{\partial z_i \partial z_j}\left(\tilde{X}_s^u(x)-\tilde{X}_s^u(y)\right)\times\left(V_k^i(\tilde{X}_s^x)-V_k^i(\tilde{X}_s^y)\right)\\
&\times\left(V_l^j(\tilde{X}_s^x)-V_l^j(\tilde{X}_s^y)\right)dY_s^k dY_s^l,\end{aligned}$$

$$\begin{aligned}\tilde{J}_t^3 =&\sum_{i,j,k,l}\int_u^t \frac{\partial^2 f}{\partial z_i \partial z_j}\left(\tilde{X}_s^u(x)-\tilde{X}_s^u(y)\right)\times\left(\tilde{X}_s(x)-\tilde{X}_s(y)\right)\left(-\frac{e^{-\Lambda_s}}{1-Z_{s\wedge\tau_n}}\right)\\
&\times\left(V_l^j(\tilde{X}_s^x)-V_l^j(\tilde{X}_s^y)\right)dN_s\,dY_s^l,\end{aligned}$$

$$\begin{aligned}\tilde{J}_t^4 =&\sum_{i,j,k,l}\int_u^t \frac{\partial^2 f}{\partial z_i \partial z_j}\left(\tilde{X}_s^u(x)-\tilde{X}_s^u(y)\right)\times\left(\tilde{X}_s(x)-\tilde{X}_s(y)\right)\left(-\frac{e^{-\Lambda_s}}{1-Z_{s\wedge\tau_n}}\right)\\
&\times\left(V_k^i(\tilde{X}_s^x)-V_k^i(\tilde{X}_s^y)\right)dN_s\,dY_s^k,\end{aligned}$$

and note that

$$\frac{\partial^2 f}{\partial z_i \partial z_j} = p|z|^{p-2}\delta_{ij} + p(p-2)|z|^{p-4} z_i z_j,$$

where δ_{ij} is the Kronecker's delta. Then for $\tilde{J}_t^1$, we have

$$\begin{aligned}
&\left| \sum_{i,j} \frac{\partial^2 f}{\partial z_i \partial z_j} \left(\tilde{X}_s^u(x) - \tilde{X}_s^u(y) \right) \times \left(\tilde{X}_s(x) - \tilde{X}_s(y) \right)^2 \right| \\
&\leq \left| \left(p|z|^{p-2}\delta_{ij} d + p(p-2)|z|^{p-4} z_i z_j \right) \left(\tilde{X}_s(x) - \tilde{X}_s(y) \right)^2 \right| \\
&\leq |p| \left(|p-2| + d \right) \left| \tilde{X}_s(x) - \tilde{X}_s(y) \right|^p .
\end{aligned}$$

Therefore,

$$\tilde{J}_t^1 \leq |p| \left(|p-2| + d \right) \int_u^t \left| \tilde{X}_s(x) - \tilde{X}_s(y) \right|^p ds \int_u^t \left(-\frac{e^{-\Lambda_s}}{1 - Z_{s \wedge \tau_n}} \right)^2 dN_s dN_s.$$

The hypothesis $H_Y(C)$ is always assumed, so

$$\tilde{J}_t^1 \leq |p| \left(|p-2| + d \right) Q_t^2 \int_u^t \left| \tilde{X}_s(x) - \tilde{X}_s(y) \right|^p ds.$$

For $\tilde{J}_t^2$, we have

$$\begin{aligned}
&\left| \sum_{i,j} \frac{\partial^2 f}{\partial z_i \partial z_j} \left(\tilde{X}_s^u(x) - \tilde{X}_s^u(y) \right) \times \left(V_k^i(\tilde{X}_s^x) - V_k^i(\tilde{X}_s^y) \right) \left(V_l^j(\tilde{X}_s^x) - V_l^j(\tilde{X}_s^y) \right) \right| \\
&\leq \left| \left(p|z|^{p-2}\delta_{ij} d + p(p-2)|z|^{p-4} z_i z_j \right) \tilde{L}^2 \left(\tilde{X}_s(x) - \tilde{X}_s(y) \right)^2 \right|,
\end{aligned}$$

$$\tilde{J}_t^2 \leq |p| \left(|p-2| + d \right) \tilde{L}^2 \left| \tilde{X}_s(x) - \tilde{X}_s(y) \right|^p .$$

So

$$\tilde{J}_t^2 \leq |p| \left(|p-2| + d \right) \tilde{L}^2 r^2 \int_u^t \left| \tilde{X}_s(x) - \tilde{X}_s(y) \right|^p ds.$$

For $\tilde{J}_t^3$, we have

$$\begin{aligned}
&\left| \sum_{i,j} \frac{\partial^2 f}{\partial z_i \partial z_j} \left(\tilde{X}_s^u(x) - \tilde{X}_s^u(y) \right) \times \left(\tilde{X}_s(x) - \tilde{X}_s(y) \right) \left(V_l^j(\tilde{X}_s^x) - V_l^j(\tilde{X}_s^y) \right) \right| \\
&\leq \left| \left(p|z|^{p-2}\delta_{ij} d + p(p-2)|z|^{p-4} z_i z_j \right) \left(\tilde{X}_s(x) - \tilde{X}_s(y) \right) \tilde{L}\, r \left(\tilde{X}_s(x) - \tilde{X}_s(y) \right) \right| \\
&\leq |p| \left(|p-2| + d \right) \tilde{L}\, r \left| \tilde{X}_s(x) - \tilde{X}_s(y) \right|^p .
\end{aligned}$$

The hypothesis $H_Y(C)$ is always assumed, so

$$\tilde{J}_t^3 \leq |p| \left(|p-2| + d \right) \tilde{L}\, r\, Q_t \int_u^t \left| \tilde{X}_s(x) - \tilde{X}_s(y) \right|^p ds.$$

For $\tilde{J}^4_t$, we have also

$$\tilde{J}^4_t \leq |p|\,(|p-2|+d)\,\tilde{L}\,r\,Q_t \int_u^t \left|\tilde{X}_s(x)-\tilde{X}_s(y)\right|^p ds,$$

$$\begin{aligned}
\tilde{J}_t =& \frac{1}{2}\left[\tilde{J}^1_t+\tilde{J}^2_t+\tilde{J}^3_t+\tilde{J}^4_t\right]\\
\leq& \frac{1}{2}\Bigg[|p|\,(|p-2|+d)\;Q^2_t \int_u^t \left|\tilde{X}_s(x)-\tilde{X}_s(y)\right|^p ds\\
&\quad + |p|\,(|p-2|+d)\,\tilde{L}^2\,r^2 \int_u^t \left|\tilde{X}_s(x)-\tilde{X}_s(y)\right|^p ds\\
&\quad + 2|p|\,(|p-2|+d)\,\tilde{L}\,r\,Q_t \int_u^t \left|\tilde{X}_s(x)-\tilde{X}_s(y)\right|^p ds\Bigg]\\
\leq& \frac{1}{2}\left[|p|\,(|p-2|+d)\int_u^t \left|\tilde{X}_s(x)-\tilde{X}_s(y)\right|^p ds\left(Q^2_t+\tilde{L}^2\,r^2+2\tilde{L}\,r\,Q_t\right)\right].
\end{aligned}$$

Therefore,

$$\left|\mathbb{E}\,\tilde{J}_{t\wedge\sigma_{\tilde{\varepsilon}}}\right| \leq \frac{1}{2}|p|\,(|p-2|+d)\left(Q_t+r\,\tilde{L}\right)^2\int_u^t \mathbb{E}\left|\tilde{X}_{s\wedge\sigma_{\tilde{\varepsilon}}}(x)-\tilde{X}_{s\wedge\sigma_{\tilde{\varepsilon}}}(y)\right|^p ds. \quad (3.2)$$

Summing up these two inequalities 3.1 and 3.2, we obtain

$$\mathbb{E}\left|\tilde{X}^u_{t\wedge\sigma_{\tilde{\varepsilon}}}(x)-\tilde{X}^u_{t\wedge\sigma_{\tilde{\varepsilon}}}(y)\right|^p \leq |x-y|^p+\;\tilde{C}_p\int_u^t \mathbb{E}\left|\tilde{X}_{s\wedge\sigma_{\tilde{\varepsilon}}}(x)-\tilde{X}_{s\wedge\sigma_{\tilde{\varepsilon}}}(y)\right|^p ds,$$

where $\tilde{C}_p$ is a positive constant.

By Grönwall's inequality we have

$$\mathbb{E}\left|\tilde{X}^u_{t\wedge\sigma_{\tilde{\varepsilon}}}(x)-\tilde{X}^u_{t\wedge\sigma_{\tilde{\varepsilon}}}(y)\right|^p \leq K^{(2)}_{p,u}\;|x-y|^p, \quad u\leq t\leq\infty,$$

such that

$$K^{(2)}_{p,u}\;|x-y|^p=\exp(\tilde{C}_p u).$$

Letting $\tilde{\varepsilon}$ tend to 0, we have

$$\mathbb{E}\left|\tilde{X}^u_{t\wedge\sigma}(x)-\tilde{X}^u_{t\wedge\sigma}(y)\right|^p \leq K^{(2)}_{p,u}\;|x-y|^p,$$

where σ is the first time such that $\tilde{X}^u_t(x)=\tilde{X}^u_t(y)$. However, we have $\sigma=\infty$ a.s., since otherwise the left hand side would be infinity if $p<0$. The proof is complete.

The above lemma shows that if $x\neq y$ then $\tilde{X}^u_t(x)\neq\tilde{X}^u_t(y)$ holds a.s. for all t. But it does not conclude that $\tilde{X}_t(.,\omega)$ is one to one, since the exceptional null set $\tilde{N}_{x,y}=\{\omega;\tilde{X}^u_t(x)=\tilde{X}^u_t(y)\,\mathrm{for\,some\,t}\}$ depends on the pair (x,y). To overcome this point, we shall apply lemma 2.7.

In this case, suppose $p>2(2d+1)$. We have

$$\tilde{X}^u_t(x)=x+\int_u^t \tilde{X}_s\left(-\frac{e^{-\Lambda_s}}{1-Z_{s\wedge\tau_n}}\right)dN_s+\int_u^t\sum_{i=1}^d\sum_{j=1}^r \tilde{X}_s F^i_j\;\left(\tilde{X}_s-(1-Z_s)\right)dY^j_s,$$

$$\tilde{X}^u_{t'}(x') = x' + \int_u^{t'} \tilde{X}_s \left(-\frac{e^{-\Lambda_s}}{1 - Z_{s\wedge\tau_n}}\right) dN_s + \int_u^{t'} \sum_{i=1}^{d}\sum_{j=1}^{r} \tilde{X}_s F^i_j \left(\tilde{X}_s - (1 - Z_s)\right) dY^j_s,$$

$$\tilde{X}^u_t(y) = y + \int_u^{t} \tilde{X}_s \left(-\frac{e^{-\Lambda_s}}{1 - Z_{s\wedge\tau_n}}\right) dN_s + \int_u^{t} \sum_{i=1}^{d}\sum_{j=1}^{r} \tilde{X}_s F^i_j \left(\tilde{X}_s - (1 - Z_s)\right) dY^j_s,$$

$$\tilde{X}^u_{t'}(y') = y' + \int_u^{t'} \tilde{X}_s \left(-\frac{e^{-\Lambda_s}}{1 - Z_{s\wedge\tau_n}}\right) dN_s + \int_u^{t'} \sum_{i=d}^{r}\sum_{j=1}^{r} \tilde{X}_s F^i_j \left(\tilde{X}_s - (1 - Z_s)\right) dY^j_s.$$

Put

$$\tilde{\eta}_t(x,y) = \frac{1}{|\tilde{X}^u_t(x) - \tilde{X}^u_t(y)|},$$

$$\tilde{\eta}_{t'}(x',y') = \frac{1}{|\tilde{X}^u_{t'}(x') - \tilde{X}^u_{t'}(y')|}.$$

So

$$\begin{aligned}
&|\tilde{\eta}_t(x,y) - \tilde{\eta}_{t'}(x',y')|^p \\
&= \left| \frac{1}{|\tilde{X}^u_t(x) - \tilde{X}^u_t(y)|} - \frac{1}{|\tilde{X}^u_{t'}(x') - \tilde{X}^u_{t'}(y')|} \right|^p \\
&\le 2^p \left(\frac{1}{|\tilde{X}^u_t(x) - \tilde{X}^u_t(y)|}\right)^p \left(\frac{1}{|\tilde{X}^u_{t'}(x') - \tilde{X}^u_{t'}(y')|}\right)^p \\
&\quad \times \left[|\tilde{X}^u_t(x) - \tilde{X}^u_{t'}(x')|^p + |\tilde{X}^u_t(y) - \tilde{X}^u_{t'}(y')|^p\right].
\end{aligned}$$

By Hölder's inequality,

$$\begin{aligned}
&\mathbb{E}\,|\tilde{\eta}_t(x,y) - \tilde{\eta}_{t'}(x',y')|^p \\
&\le 2^p \left(\mathbb{E}(\tilde{\eta}_t(x,y)^{4p})\mathbb{E}(\tilde{\eta}_{t'}(x',y')^{4p})\right)^{\frac{1}{4}} \\
&\quad \times \left[\left(\mathbb{E}|\tilde{X}^u_t(x) - \tilde{X}^u_{t'}(x')|^{2p}\right)^{\frac{1}{2}} + \left(\mathbb{E}|\tilde{X}^u_t(y) - \tilde{X}^u_{t'}(y')|^{2p}\right)^{\frac{1}{2}}\right].
\end{aligned}$$

By lemma 2.6 and proposition 2.1, we have

$$\begin{aligned}
&\mathbb{E}\,|\tilde{\eta}_t(x,y) - \tilde{\eta}_{t'}(x',y')|^p \\
&\le \tilde{C}_{p,T}|x-y|^{-p}|x'-y'|^{-p}\left(|x-x'|^p + |y-y'|^p + 2|t-t'|^{\frac{p}{2}}\right) \\
&\le \tilde{C}_{p,T}\tilde{\delta}^{-2p}\left(|x-x'|^p + |y-y'|^p + 2|t-t'|^{\frac{p}{2}}\right),
\end{aligned}$$

if $|x-y| \ge \tilde{\delta}$ and $|x'-y'| \ge \tilde{\delta}$, where $\tilde{C}_{p,T}$ is a positive constant. Then by Kolmogorov Theorem 2.2, $\tilde{\eta}_t(x,y)$ is continuous in $[0,T] \times \{(x,y)/|x-y| \ge \tilde{\delta}\}$. Since T and $\tilde{\delta}$ are arbitrary positive numbers, we get the assertion. The proof is complete.

The above calculus leads immediately to the one to one property of the map $\tilde{X}^u_t(.,\omega)$ a.s. for all t. We shall next consider the onto property.

3.2. Proof of the onto property. In this part we will apply lemmas 2.8, 2.10, and 2.11 to our model.

Let $T>0$ and p any real number:

$$\tilde{X}_t^u(x)=x+\int_u^t \tilde{X}_s\left(-\frac{e^{-\Lambda_s}}{1-Z_{s\wedge\tau_n}}\right)dN_s+\int_u^t\sum_{i=1}^d\sum_{j=1}^r \tilde{X}_sF_j^i\left(\tilde{X}_s-(1-Z_s)\right)dY_s^j,$$

$$d\tilde{X}_t=\tilde{X}_t\left(-\frac{e^{-\Lambda_t}}{1-Z_{t\wedge\tau_n}}\right)dN_t+\sum_{i=1}^d\sum_{j=1}^r \tilde{X}_tF_j^i\left(\tilde{X}_t-(1-Z_t)\right)dY_t^j.$$

We shall apply Itô's formula to the function $f(z)=(1+|z|^2)^p$. We have

$$\begin{aligned}
&f(\tilde{X}_t^u(x))-f(x)\\
&=\sum_{i,j}\int_u^t\frac{\partial f}{\partial z_i}\left(\tilde{X}_s^u(x)\right)\\
&\qquad\times\left[\tilde{X}_s(x)\left(-\frac{e^{-\Lambda_s}}{1-Z_{s\wedge\tau_n}}\right)dN_s+\tilde{X}_s(x)F_j^i\left(\tilde{X}_s(x)-(1-Z_s)\right)dY_s^j\right]\\
&\quad+\frac{1}{2}\sum_{i,j,k,l}\int_u^t\frac{\partial^2 f}{\partial z_i\partial z_j}\left(\tilde{X}_s^u(x)\right)\\
&\qquad\times\left[\tilde{X}_s(x)\left(-\frac{e^{-\Lambda_s}}{1-Z_{s\wedge\tau_n}}\right)dN_s+\tilde{X}_s(x)F_k^i\left(\tilde{X}_s(x)-(1-Z_s)\right)dY_s^k\right]\\
&\qquad\times\left[\tilde{X}_s(x)\left(-\frac{e^{-\Lambda_s}}{1-Z_{s\wedge\tau_n}}\right)dN_s+\tilde{X}_s(x)F_l^j\left(\tilde{X}_s(x)-(1-Z_s)\right)dY_s^l\right],
\end{aligned}$$

$f(\tilde{X}_t^u(x))-f(x)=\tilde{I}_t+\tilde{J}_t$ such that

$$\begin{aligned}
\tilde{I}_t=&\sum_{i,j}\int_u^t\frac{\partial f}{\partial z_i}\left(\tilde{X}_s^u(x)\right)\\
&\times\left[\tilde{X}_s(x)\left(-\frac{e^{-\Lambda_s}}{1-Z_{s\wedge\tau_n}}\right)dN_s+\tilde{X}_s(x)F_j^i\left(\tilde{X}_s(x)-(1-Z_s)\right)dY_s^j\right],\\
\tilde{J}_t=&\frac{1}{2}\sum_{i,j,k,l}\int_u^t\frac{\partial^2 f}{\partial z_i\partial z_j}\left(\tilde{X}_s^u(x)\right)\\
&\times\left[\tilde{X}_s(x)\left(-\frac{e^{-\Lambda_s}}{1-Z_{s\wedge\tau_n}}\right)dN_s+\tilde{X}_s(x)F_k^i\left(\tilde{X}_s(x)-(1-Z_s)\right)dY_s^k\right]\\
&\times\left[\tilde{X}_s(x)\left(-\frac{e^{-\Lambda_s}}{1-Z_{s\wedge\tau_n}}\right)dN_s+\tilde{X}_s(x)F_l^j\left(\tilde{X}_s(x)-(1-Z_s)\right)dY_s^l\right].
\end{aligned}$$

For $\tilde{I}_t$, we have

$$\begin{aligned}
\tilde{I}_t=&\sum_{i,j}\int_u^t\frac{\partial f}{\partial z_i}\left(\tilde{X}_s^u(x)\right)\\
&\times\left[\tilde{X}_s(x)\left(-\frac{e^{-\Lambda_s}}{1-Z_{s\wedge\tau_n}}\right)dN_s+\tilde{X}_s(x)F_j^i\left(\tilde{X}_s(x)-(1-Z_s)\right)dY_s^j\right],
\end{aligned}$$

$$\tilde{I}_t = \sum_{i,j} \int_u^t \frac{\partial f}{\partial z_i} \left(\tilde{X}_s^u(x)\right) \times \tilde{X}_s(x) \left(-\frac{e^{-\Lambda_s}}{1 - Z_{s\wedge\tau_n}}\right) dN_s$$
$$+ \sum_{i,j} \int_u^t \frac{\partial f}{\partial z_i} \left(\tilde{X}_s^u(x)\right) \times \tilde{X}_s(x) F_j^i \left(\tilde{X}_s(x) - (1 - Z_s)\right) dY_s^j.$$

$\tilde{I}_t = \tilde{I}_t^1 + \tilde{I}_t^2$ such that

$$\tilde{I}_t^1 = \sum_{i,j} \int_u^t \frac{\partial f}{\partial z_i} \left(\tilde{X}_s^u(x)\right) \times \tilde{X}_s(x) \left(-\frac{e^{-\Lambda_s}}{1 - Z_{s\wedge\tau_n}}\right) dN_s,$$

$$\tilde{I}_t^2 = \sum_{i,j} \int_u^t \frac{\partial f}{\partial z_i} \left(\tilde{X}_s^u(x)\right) \times \tilde{X}_s(x) F_j^i \left(\tilde{X}_s(x) - (1 - Z_s)\right) dY_s^j.$$

For $\tilde{I}_t^1$, note $\dfrac{\partial f}{\partial z_i} = 2\, p\, z_i\, (1+|z|^2)^{p-1}$ and the hypothesis $H_Y(C)$ is always assumed, so

$$\tilde{I}_t^1 = \sum_{i,j} \int_u^t \frac{\partial f}{\partial z_i} \left(\tilde{X}_s^u(x)\right) \times \tilde{X}_s(x) \left(-\frac{e^{-\Lambda_s}}{1 - Z_{s\wedge\tau_n}}\right) dN_s,$$

$$\sum_i \left| \frac{\partial f}{\partial z_i} \left(\tilde{X}_s^u(x)\right) \times \tilde{X}_s(x) \right| \le 2|p||z_i|(1 + |z|^2)^{p-1} \sqrt{d} |\tilde{X}_s(x)|$$
$$\le 2|p|\sqrt{d} \left(1 + |\tilde{X}_s(x)|^2\right)^p.$$

Therefore,

$$\tilde{I}_t^1 \le 2|p|\sqrt{d}\, r\, Q_t \int_u^t \left(1 + |\tilde{X}_s(x)|^2\right)^p ds.$$

For $\tilde{I}_t^2$, we have

$$\tilde{I}_t^2 = \sum_{i,j} \int_u^t \frac{\partial f}{\partial z_i} \left(\tilde{X}_s^u(x)\right) \times \tilde{X}_s(x) F_j^i \left(\tilde{X}_s(x) - (1 - Z_s)\right) dY_s^j.$$

Note

$$\tilde{V}_j^i(\tilde{X}_s^x) = \tilde{X}_s(x) F_j^i \left(\tilde{X}_s(x) - (1 - Z_s)\right).$$

Let $\tilde{K}$ be a positive constant such that

$$\tilde{V}_j^i(\tilde{X}_s^x) \le \tilde{K} \left(1 + |\tilde{X}_s(x)|^2\right)^{\frac{1}{2}},$$

$$\sum_i \left| \frac{\partial f}{\partial z_i} \left(\tilde{X}_s^u(x)\right) \times \tilde{V}_j^i(\tilde{X}_s^x) \right| \le 2\, |p|\, |z_i|\, (1 + |z|^2)^{p-1} \sqrt{d}\, \tilde{K} \left(1 + |\tilde{X}_s(x)|^2\right)^{\frac{1}{2}}$$
$$\le 2\sqrt{d}\, |p|\, \tilde{K} \left(1 + |\tilde{X}_s(x)|^2\right)^p.$$

So

$$\tilde{I}_t^2 \le 2\sqrt{d}\, |p|\, r\, \tilde{K} \int_u^t \left(1 + |\tilde{X}_s(x)|^2\right)^p ds.$$

Therefore,

$$\begin{aligned}\tilde{I}_t \leq & 2|p|\sqrt{d}\, r\, Q_t \int_u^t \left(1+|\tilde{X}_s(x)|^2\right)^p ds \\ & + 2\sqrt{d}\,|p|\, r\, \tilde{K} \int_u^t \left(1+|\tilde{X}_s(x)|^2\right)^p ds \\ \leq & 2|p|\sqrt{d}\, r\, (Q_t+\tilde{K}) \int_u^t \left(1+|\tilde{X}_s(x)|^2\right)^p ds.\end{aligned}$$

We have

$$\left|\mathbb{E}\, \tilde{I}_t\right| \leq \; 2|p|\sqrt{d}\, r\, (Q_t+\tilde{K}) \int_u^t \mathbb{E}\left(1+|\tilde{X}_s(x)|^2\right)^p ds. \tag{3.3}$$

Next, for $\tilde{J}_t$ we have

$$\begin{aligned}\tilde{J}_t =& \frac{1}{2}\sum_{i,j,k,l}\int_u^t \frac{\partial^2 f}{\partial z_i \partial z_j}\left(\tilde{X}_s^u(x)\right) \\ & \times \left[\tilde{X}_s(x)\left(-\frac{e^{-\Lambda_s}}{1-Z_{s\wedge\tau_n}}\right) dN_s + \tilde{X}_s(x) F_k^i\left(\tilde{X}_s(x)-(1-Z_s)\right) dY_s^k\right] \\ & \times \left[\tilde{X}_s(x)\left(-\frac{e^{-\Lambda_s}}{1-Z_{s\wedge\tau_n}}\right) dN_s + \tilde{X}_s(x) F_l^j\left(\tilde{X}_s(x)-(1-Z_s)\right) dY_s^l\right].\end{aligned}$$

Note

$$\begin{aligned}\tilde{V}_k^i(\tilde{X}_s^x) &= \tilde{X}_s(x) F_k^i\left(\tilde{X}_s(x)-(1-Z_s)\right), \\ \tilde{V}_l^j(\tilde{X}_s^x) &= \tilde{X}_s(x) F_l^j\left(\tilde{X}_s(x)-(1-Z_s)\right).\end{aligned}$$

So

$$\begin{aligned}\tilde{J}_t =& \frac{1}{2}\sum_{i,j,k,l}\int_u^t \frac{\partial^2 f}{\partial z_i \partial z_j}\left(\tilde{X}_s^u(x)\right) \\ & \times \left[\tilde{X}_s(x)^2\left(-\frac{e^{-\Lambda_s}}{1-Z_{s\wedge\tau_n}}\right)^2 dN_s dN_s + \tilde{V}_k^i(\tilde{X}_s^x)\,\tilde{V}_l^j(\tilde{X}_s^x)\, dY_s^k\, dY_s^l \right. \\ & + \tilde{X}_s(x)\left(-\frac{e^{-\Lambda_s}}{1-Z_{s\wedge\tau_n}}\right)\tilde{V}_l^j(\tilde{X}_s^x)\, dN_s\, dY_s^l \\ & \left. + \tilde{X}_s(x)\left(-\frac{e^{-\Lambda_s}}{1-Z_{s\wedge\tau_n}}\right)\tilde{V}_k^i(\tilde{X}_s^x) dN_s dY_s^k\right].\end{aligned}$$

Note $\tilde{J}_t = \frac{1}{2}\left[\tilde{J}_t^1 + \tilde{J}_t^2 + \tilde{J}_t^3 + \tilde{J}_t^4\right]$, such that

$$\tilde{J}_t^1 = \sum_{i,j,k,l}\int_u^t \frac{\partial^2 f}{\partial z_i \partial z_j}\left(\tilde{X}_s^u(x)\right)\tilde{X}_s(x)^2\left(-\frac{e^{-\Lambda_s}}{1-Z_{s\wedge\tau_n}}\right)^2 dN_s dN_s,$$

$$\tilde{J}_t^2 = \sum_{i,j,k,l}\int_u^t \frac{\partial^2 f}{\partial z_i \partial z_j}\left(\tilde{X}_s^u(x)\right)\tilde{V}_k^i(\tilde{X}_s^x)\,\tilde{V}_l^j(\tilde{X}_s^x)\, dY_s^k\, dY_s^l,$$

$$\tilde{J}_t^3 = \sum_{i,j,k,l}\int_u^t \frac{\partial^2 f}{\partial z_i \partial z_j}\left(\tilde{X}_s^u(x)\right)\tilde{X}_s(x)\left(-\frac{e^{-\Lambda_s}}{1-Z_{s\wedge\tau_n}}\right)\tilde{V}_l^j(\tilde{X}_s^x)\, dN_s\, dY_s^l,$$

$$\tilde{J}_t^4 = \sum_{i,j,k,l} \int_u^t \frac{\partial^2 f}{\partial z_i \partial z_j} \left(\tilde{X}_s^u(x)\right) \tilde{X}_s(x) \left(-\frac{e^{-\Lambda_s}}{1 - Z_{s\wedge\tau_n}}\right) \tilde{V}_k^i(\tilde{X}_s^x) dN_s dY_s^k,$$

and note that

$$\frac{\partial^2 f}{\partial z_i \partial z_j} = 2p\,(1+|z|^2)^{p-1}\delta_{ij} + 4p\,(p-1)\,z_i\,z_j\,(1+|z|^2)^{p-2},$$

where δ_{ij} is the Kronecker delta, then for $\tilde{J}_t^1$ we have

$$\tilde{J}_t^1 = \sum_{i,j,k,l} \int_u^t \frac{\partial^2 f}{\partial z_i \partial z_j} \left(\tilde{X}_s^u(x)\right) \tilde{X}_s(x)^2 \left(-\frac{e^{-\Lambda_s}}{1 - Z_{s\wedge\tau_n}}\right)^2 dN_s dN_s,$$

$$\begin{aligned}
&\sum_{i,j} \left| \frac{\partial^2 f}{\partial z_i \partial z_j} \left(\tilde{X}_s^u(x)\right) \tilde{X}_s(x)^2 \right| \\
&\le \left| \left(2p\,(1+|z|^2)^{p-1}\delta_{ij} + 4p\,(p-1)\,z_i\,z_j\,(1+|z|^2)^{p-2}\right) \tilde{X}_s(x)^2 \right| \\
&\le 2|p|\,(2(p-1)+d) \left(1 + |\tilde{X}_s(x)|^2\right)^p.
\end{aligned}$$

Therefore,

$$\tilde{J}_t^1 \le 2|p|\,(2(p-1)+d) \int_u^t \left(1 + |\tilde{X}_s(x)|^2\right)^p ds \int_u^t \left(-\frac{e^{-\Lambda_s}}{1 - Z_{s\wedge\tau_n}}\right)^2 dN_s dN_s.$$

By hypothesis $H_Y(C)$, we have

$$\tilde{J}_t^1 \le 2|p|\,(2(p-1)+d)\,Q_t^2 \int_u^t \left(1 + |\tilde{X}_s(x)|^2\right)^p ds.$$

For $\tilde{J}_t^2$, we have

$$\tilde{J}_t^2 = \sum_{i,j,k,l} \int_u^t \frac{\partial^2 f}{\partial z_i \partial z_j} \left(\tilde{X}_s^u(x)\right) \tilde{V}_k^i(\tilde{X}_s^x)\,\tilde{V}_l^j(\tilde{X}_s^x)\,dY_s^k\,dY_s^l,$$

$$\begin{aligned}
&\sum_{i,j} \left| \frac{\partial^2 f}{\partial z_i \partial z_j} \left(\tilde{X}_s^u(x)\right) \tilde{V}_k^i(\tilde{X}_s^x)\,\tilde{V}_l^j(\tilde{X}_s^x) \right| \\
&\le | \left(2p\,(1+|z|^2)^{p-1}\delta_{ij} + 4p\,(p-1)\,z_i\,z_j\,(1+|z|^2)^{p-2}\right) \times \tilde{K}^2(1 + |\tilde{X}_s(x)|^2)| \\
&\le 2|p|\,(2(p-1)+d)\,\tilde{K}^2(1 + |\tilde{X}_s(x)|^2)^p.
\end{aligned}$$

Therefore,

$$\tilde{J}_t^2 \le 2|p|\,(2(p-1)+d)\,\tilde{K}^2\,r^2 \int_u^t \left(1 + |\tilde{X}_s(x)|^2\right)^p ds.$$

For $\tilde{J}_t^3$, we have

$$\tilde{J}_t^3 = \sum_{i,j,k,l} \int_u^t \frac{\partial^2 f}{\partial z_i \partial z_j} \left(\tilde{X}_s^u(x)\right) \tilde{X}_s(x) \left(-\frac{e^{-\Lambda_s}}{1 - Z_{s\wedge\tau_n}}\right) \tilde{V}_l^j(\tilde{X}_s^x)\,dN_s\,dY_s^l,$$

$$\sum_{i,j}\left|\frac{\partial^2 f}{\partial z_i \partial z_j}\left(\tilde{X}_s^u(x)\right)\tilde{V}_l^j(\tilde{X}_s^x)\right|$$
$$\leq |\left(2p\,(1+|z|^2)^{p-1}\delta_{ij}+4p(p-1)\,z_i\,z_j\,(1+|z|^2)^{p-2}\right)\times\tilde{K}(1+|\tilde{X}_s(x)|^2)^{\frac{1}{2}}\,\tilde{X}_s(x)|$$
$$\leq 2|p|\,(2(p-1)+d)\,\tilde{K}(1+|\tilde{X}_s(x)|^2)^p.$$

The hypothesis $H_Y(C)$ is always assumed, so

$$\tilde{J}_t^3 \leq 2|p|\,(2(p-1)+d)\;\tilde{K}\,r\,Q_t\int_u^t\left(1+|\tilde{X}_s(x)|^2\right)^p ds.$$

For $\tilde{J}_t^4$, we have also

$$\tilde{J}_t^4 \leq 2|p|\,(2(p-1)+d)\;\tilde{K}\,r\,Q_t\int_u^t\left(1+|\tilde{X}_s(x)|^2\right)^p ds.$$

Therefore,

$$\begin{aligned}\tilde{J}_t =&\frac{1}{2}\left[\tilde{J}_t^1+\tilde{J}_t^2+\tilde{J}_t^3+\tilde{J}_t^4\right]\\ \leq&\frac{1}{2}\Bigg[2|p|\,(2(p-1)+d)\,Q_t^2\int_u^t\left(1+|\tilde{X}_s(x)|^2\right)^p ds\\ &+2|p|\,(2(p-1)+d)\,\tilde{K}^2\,r^2\int_u^t\left(1+|\tilde{X}_s(x)|^2\right)^p ds\\ &+4|p|\,(2(p-1)+d)\;\tilde{K}\,r\,Q_t\int_u^t\left(1+|\tilde{X}_s(x)|^2\right)^p ds\Bigg]\\ \leq&|p|\,(2(p-1)+d)\int_u^t\left(1+|\tilde{X}_s(x)|^2\right)^p ds\left(Q_t^2+\tilde{K}^2\,r^2\;+\;2\,\tilde{K}\,r\,Q_t\right).\end{aligned}$$

So

$$\left|\mathbb{E}\,\tilde{J}_t\right| \leq\; |p|\,(2(p-1)+d)\left(Q_t+r\,\tilde{K}\right)^2\int_u^t\mathbb{E}\left(1+|\tilde{X}_s(x)|^2\right)^p ds. \tag{3.4}$$

Summing up these two inequalities (3.3) and (3.4), we obtain

$$\mathbb{E}\left(1+|\tilde{X}_s(x)|^2\right)^p \leq \left(1+|x|^2\right)^p+\;\text{const}\times\int_u^t\mathbb{E}\left(1+|\tilde{X}_s(x)|^2\right)^p ds.$$

By Grönwall's inequality, we have

$$\mathbb{E}\left(1+|\tilde{X}_s(x)|^2\right)^p \leq \left(1+|x|^2\right)^p\times\exp\left(\tilde{C}_{p,u}\right),$$

such that

$$\tilde{C}_{p,u} =\;\text{const}\times\int_u^t\mathbb{E}\left(1+|\tilde{X}_s(x)|^2\right)^p ds,$$

and

$$\tilde{K}_{p,u}^3=\exp\left(\tilde{C}_{p,u}\right).$$

So, we have the inequality of the lemma 2.8

$$\mathbb{E}\left(1+|\tilde{X}_s(x)|^2\right)^p \leq \tilde{K}_{p,u}^3\left(1+|x|^2\right)^p.$$

Now, taking negative p in the above calculus, we see that $\left|\tilde{X}_t(x)\right|$ tends to infinity in probability as x tends sequentially to infinity. We shall prove a stronger convergence.

Let $\bar{\mathbb{R}}^d = \mathbb{R}^d \cup \{\infty\}$ be the one point compactification of $\mathbb{R}$. Set

$$\tilde{X}_t^u(x) = x + \int_u^t \tilde{X}_s \left(-\frac{e^{-\Lambda_s}}{1 - Z_{s\wedge\tau_n}}\right) dN_s + \int_u^t \sum_{i=1}^d \sum_{j=1}^r \tilde{X}_s F_j^i \left(\tilde{X}_s - (1 - Z_s)\right) dY_s^j,$$

$$\tilde{\eta}_t(x) = \begin{cases} \dfrac{1}{1 + \left|\tilde{X}_t(x)\right|}, & \text{if } x \in \mathbb{R}^d, \\ 0, & \text{if } x = \infty. \end{cases}$$

Evidently $\tilde{\eta}_t(x)$ is continuous in $[0,\infty) \times \mathbb{R}^d$. Thus just to prove the continuity in the vicinity of infinity. Suppose $p > 2(2d+1)$. We have

$$|\tilde{\eta}_t(x) - \tilde{\eta}_s(y)|^p \le \tilde{\eta}_t(x)^p \tilde{\eta}_s(y)^p \left|\tilde{X}_t(x) - \tilde{X}_s(y)\right|^p.$$

By Hölder's inequality, proposition 2.1 and lemma 2.8, we have

$$\begin{aligned} \mathbb{E}\,|\tilde{\eta}_t(x) - \tilde{\eta}_s(y)|^p \le& \left(\mathbb{E}\,\tilde{\eta}_t(x)^{4p}\right)^{\frac{1}{4}} \left(\mathbb{E}\,\tilde{\eta}_s(y)^{4p}\right)^{\frac{1}{4}} (\mathbb{E}|\tilde{X}_t(x) - \tilde{X}_s(y)|^{2p})^{\frac{1}{2}} \\ \le& \tilde{C}_{p,T} \left(1 + |x|\right)^{-p} \left(1 + |y|\right)^{-p} \left(|x-y|^p + |t-s|^{\frac{p}{2}}\right), \end{aligned}$$

if $t, s \in [0,T]$ and $x, y \in \mathbb{R}^d$, where $\tilde{C}_{p,T}$ is a positive constant. Set

$$\frac{1}{x} = (x_1^{-1}, x_2^{-1}, ..., x_d^{-1}).$$

Since

$$\frac{|x-y|}{(1+|x|)(1+|y|)} \le \left|\frac{1}{x} - \frac{1}{y}\right|,$$

we get the inequality

$$\mathbb{E}\,|\tilde{\eta}_t(x) - \tilde{\eta}_s(y)|^p \le \tilde{C}_{p,T} \left(\left|\frac{1}{x} - \frac{1}{y}\right|^p + |t-s|^{\frac{p}{2}}\right).$$

Define

$$\bar{\eta}_t(x) = \begin{cases} \tilde{\eta}_t(\frac{1}{x}), & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}$$

Then the above inequality implies

$$\mathbb{E}|\bar{\eta}_t(x) - \bar{\eta}_s(y)|^p \le \tilde{C}_{p,T} \left(|x-y|^p + |t-s|^{\frac{p}{2}}\right),\ x \neq 0,\ y \neq 0.$$

In case $y = 0$, we have

$$\mathbb{E}|\bar{\eta}_t(x)|^p \le \tilde{C}_{p,T}\,|x|^p.$$

Therefore, $\bar{\eta}_t(x)$ is continuous in $[0,\infty) \times \mathbb{R}^d$ by Kolmogorov's theorem. This proves that $\tilde{\eta}_t(x)$ is continuous in $[0,\infty) \times$ neighborhood of infinity.

So, define a stochastic process $\bar{X}_t$ on $\bar{\mathbb{R}}^d = \mathbb{R}^d \cup \{\infty\}$ by

$$\bar{X}_t(x) = \begin{cases} \tilde{X}_t(x), & \text{if } x \in \mathbb{R}^d, \\ \infty, & \text{if } x = \infty. \end{cases}$$

Then $\bar{X}_t(x)$ is continuous on $[0,\infty)\times\mathbb{R}^d$ by the previous lemma. Thus for each $t > 0$, the map $\bar{X}_t(\cdot,\omega)$ is homotopic to the identity map on $\bar{\mathbb{R}}^d$, which is homeomorphic to d-dimensional sphere S^d. Then $\bar{X}_t(\cdot,\omega)$ is an onto map of $\bar{\mathbb{R}}^d$ by the well known homotopic theory. Now, the map $\bar{X}_t$ is a homeomorphism of $\bar{\mathbb{R}}^d$, since it is one to one, onto and continuous. Since ∞ is the invariant point of the map $\bar{X}_t$, we see that $\tilde{X}_t$ is a homeomorphism of $\mathbb{R}^d$. This completes the proof of Theorem 2.2.

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