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A DECOMPOSITION OF A SPACE OF MULTIPLE WIENER INTEGRALS BY THE DIFFERENCE OF TWO INDEPENDENT LÉVY PROCESSES IN TERMS OF THE LÉVY LAPLACIAN

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Abstract. In this paper, we consider the Lévy Laplacian acting on multiple Wiener integrals by the stochastic process given as a difference of two independent Lévy processes, and give a necessary and sufficient condition for eigenfunctions of the Lévy Laplacian. Moreover we give a decomposition of the L^2-space on Lévy noise probability space by eigenspaces consisting of multiple Wiener integrals by the above process in terms of the Lévy Laplacian. By this decomposition, we obtain an expression of the semigroup generated by the Lévy Laplacian related to the semigroup generated by the number operator.

1. Introduction

An infinite dimensional Laplacian was introduced by P. Lévy [7]. This Laplacian was introduced into the framework of white noise analysis by T. Hida [1] and has been studied by many authors from various aspects.

In the previous paper [2], we discussed the Lévy Laplacian

$$E[e^{irX(t)}] = \exp\{tf_X(r)\},\ r,t \in \mathbf{R},$$
$$f_X(r) = i\mu r - \frac{\sigma^2}{2}r^2 + \int_{|u|>0}\left(e^{iru} - 1 - \frac{iru}{1+u^2}\right)\frac{1+u^2}{u^2}d\beta(u),$$

where $\sigma \geq 0$,$\mu \in \mathbf{R}$ and β is a positive finite measure on $\mathbf{R}$ with $\beta(\{0\}) = \sigma^2$ and $\int_{\mathbf{R}} |u|^n d\beta(u) < +\infty$ for all $n \in \mathbf{N}$.

In this paper, we consider this Laplacian acting on multiple Wiener integrals by the stochastic process given as a difference $\Xi = \{\Xi(t)|t \in \mathbf{R}\}$ of two independent Lévy processes by the characteristic functions given as

$$E[e^{ir\Xi(t)}] = \exp\{tf_\Xi(r)\},\ r,t \in \mathbf{R},$$
$$f_\Xi(r) = -\sigma^2 r^2 + \int_{|u|>0}\left(e^{iru} + e^{-iru} - 2\right)\frac{1+u^2}{u^2}d\beta(u).$$

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and give a necessary and sufficient condition for $\mathcal{U}$-transforms $\mathcal{U}[I_n(f)], f \in L^2_{\mathbf{C}}(\mathbf{R}, dt)^{\hat{\otimes} n}$ as eigenfunctions of the Lévy Laplacian

$$(T1) \quad \beta = \sigma^2 \delta_0$$

$$(T2) \quad \sigma = 0, \beta = b\delta_a + d\delta_{-a} \text{ for some } a > 0, b \geq 0 \text{ and } d \geq 0,$$

where $I_n(f)$ is the multiple Wiener integral of order n. In this case, we have only one condition for eigenfunction of the Lévy Laplacian in Proposition 4.1. It is more clear than the result in my previous paper [2].

Moreover we give a decomposition by eigenspaces consisting of multiple Wiener integrals by the above process in terms of the Lévy Laplacian as follows:

1) A decomposition $(L^2) = \bigoplus_{n=0}^{\infty} W_n(0)$ holds if and only if (T1) holds.

2) A decomposition $(L^2) = \bigoplus_{n=0}^{\infty} W_n\Big(-na^2\Big)$ holds if and only if (T2) holds.

where $W_n(\lambda), n \in \mathbf{N} \cup \{0\}$ are eigenspaces of the Lévy Laplacian. The second part is more clear than the previous Theorem 4.4 in [2]. The decomposition implies an interesting expression of the semigroup generated by the Lévy Laplacian on (L^2) as the semigroup generated by some kind of the number operator. In the last section, we give the expression.

2. Preliminaries

Let $E = S(\mathbf{R})$ be the Schwartz space of rapidly decreasing $\mathbf{R}$-valued functions on $\mathbf{R}$ and let E^* be a dual space of E. The canonical bilinear form on $E^* \times E$ is denoted by $\langle \cdot, \cdot \rangle$.

Let $L^j = \{L^j(t) | t \in \mathbf{R}\}, j = 1, 2$ be independent Lévy processes on a probability space $(\Omega, \mathcal{F}, P)$, of which the characteristic functions are given by

$$E[e^{irL^j(t)}] = \exp\{th(r)\}, \ r, t \in \mathbf{R}, \ j = 1, 2$$
$$h(r) = i\mu r - \frac{\sigma^2}{2} r^2 + \int_{|u|>0} \left(e^{iru} - 1 - \frac{iru}{1+u^2} \right) \frac{1+u^2}{u^2} d\beta(u),$$

where $\sigma \geq 0, \mu \in \mathbf{R}$ and β is a positive finite measure on $\mathbf{R}$ with $\beta(\{0\}) = \sigma^2$ and $\int_{\mathbf{R}} |u|^n d\beta(u) < +\infty$ for all $n \in \mathbf{N}$. Define β_0 by the positive measure on $\mathbf{R}_* = \mathbf{R} - \{0\}$ given as

$$\beta_0(E) = \int_E \frac{1+u^2}{u^2} d\beta(u), \ E \in \mathcal{B}(\mathbf{R}_*).$$

Set $\Xi(t) = L^1(t) - L^2(t), t \in \mathbf{R}$. Then the characteristic function of Ξ is given by

$$E[e^{ir\Xi(t)}] = \exp\{tf_\Xi(r)\}, \ r, t \in \mathbf{R},$$
$$f_\Xi(r) = -\sigma^2 r^2 + \int_{|u|>0} \left(e^{iru} + e^{-iru} - 2 \right) \frac{1+u^2}{u^2} d\beta(u).$$

Set $C(\xi) = \exp\{\int_{\mathbf{R}} f_\Xi(\xi(t))dt\}$, $\xi \in E$. Then by the Bochner-Minlos Theorem, there exists a probability measure Λ on E^* such that

$$\int_{E^*} e^{i\langle x,\xi\rangle} d\Lambda(x) = C(\xi),\ \xi \in E.$$

The stochastic process Ξ is represented by

- $\Xi(t;x) = \langle x, 1_{[0,t]}\rangle$ if $t \geq 0$,
- $\Xi(t;x) = -\langle x, 1_{[t,0]}\rangle$ if $t < 0$.

Let $(L^2) \equiv L^2(E^*, \Lambda)$ be the Hilbert space of $\mathbf{C}$-valued square-integrable functions on (E^*, Λ). We denote the (L^2)-norm by $||\cdot||_0$. The $\mathcal{U}$-transform $\mathcal{U}[\varphi]$ of $\varphi \in (L^2)$ is defined by

$$\mathcal{U}[\varphi](\xi) = C(\xi)^{-1} \int_{E^*} e^{i\langle x,\xi\rangle}\varphi(x)d\Lambda(x),\ \xi \in E,$$

and the Wick product $\langle\cdot, f_1\rangle \diamond \cdots \diamond \langle\cdot, f_n\rangle$ of $\langle\cdot, f_j\rangle, j = 1, \cdots, n$, is given by

$$\mathcal{U}[\langle\cdot, f_1\rangle \diamond \cdots \diamond \langle\cdot, f_n\rangle] = \mathcal{U}[\langle\cdot, f_1\rangle] \cdots \mathcal{U}[\langle\cdot, f_n\rangle]\ ,\ f_1, \cdots, f_n \in E.$$

Fixing a finite interval T on $\mathbf{R}$, we take an orthonormal basis $\{\zeta_n\}_{n=0}^\infty \subset E$ for $L^2(T)$ which is equally dense and uniformly bounded (see [5]). Let $\mathcal{D}_L$ denote the set of all $\varphi \in (L^2)$ such that the limit

$$\widetilde{\Delta}_L \mathcal{U}[\varphi](\xi) \equiv \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} (\mathcal{U}\varphi)''(\xi)(\zeta_n, \zeta_n)$$

exists for each $\xi \in E$ and a functional $\widetilde{\Delta}_L \mathcal{U}[\varphi]$ is in $\mathcal{U}[(L^2)]$. The Lévy Laplacian Δ_L on $\mathcal{D}_L$ is defined by

$$\Delta_L \varphi = \mathcal{U}^{-1} \widetilde{\Delta}_L \mathcal{U}\varphi,\ \varphi \in \mathcal{D}_L.$$

3. Multiple Wiener Integral

Let $\mathcal{B}_b(\mathbf{R}^2_*)$ be the class of all bounded Borel subsets of $\mathbf{R}^2_* = \mathbf{R}^2 - \{(t,0)|t \in \mathbf{R}\}$. Define a measure ν on $\mathcal{B}_b(\mathbf{R}^2_*)$ by $d\nu(t,u) = d\beta_0(u)dt$. For each $A \in \mathcal{B}_b(\mathbf{R}^2_*)$, $N(A;\cdot)$ be a random variable on $(E^*, \mathcal{B}(E^*))$ defined by

$$N(A;x) = |\{(t,u) \in A : \Xi(t;x) - \Xi(t-;x) = u\}|$$

and set $N_0(A;x) = N(A;x) - \nu(A)$.

Let λ be a positive measure on $\mathcal{B}(\mathbf{R}^2)$ defined by

$$d\lambda(t,u) = (1+u^2)d\beta(u)dt.$$

Define a (L^2)-valued function M on $\{A \in \mathcal{B}(\mathbf{R}^2) : \lambda(A) < +\infty\}$ by

$$M(A) = \sigma \int_{\mathbf{R}} 1_A(t,0)dB(t) + \int_{\mathbf{R}^2_*} u 1_A(t,u)dN_0(t,u),$$

where $B = \{B(t) : t \in \mathbf{R}\}$ is a one dimensional Wiener process, independent of the system $\{N(A) : A \in \mathcal{B}_b(\mathbf{R}^2_*)\}$. Then we can define the multiple Wiener integral of order n with respect to M (see [6]). In notation, we write

$$I_n(f) = \int_{\mathbf{R}^2} \cdots \int_{\mathbf{R}^2} f(s_1, \cdots, s_n)dM(s_1)\cdots dM(s_n), f \in L^2_{\mathbf{C}}(\mathbf{R}^2, \lambda)^{\hat{\otimes} n},$$

where $L^2_{\mathbf{C}}(\mathbf{R}^2,\lambda)^{\hat{\otimes}n}$ is the complexication of n fold symmetric tensor product of $L^2(\mathbf{R}^2,\lambda)$.

Proposition 3.1. *(see [6]). Let $f \in L^2_{\mathbf{C}}(\mathbf{R}, dt)^{\hat{\otimes}n}$ be given, then*

$$I_n(f) = \int_{\mathbf{R}} \cdots \int_{\mathbf{R}} f(s_1, \cdots, s_n) d\Xi(s_1) \cdots d\Xi(s_n).$$

Let

$$\mathcal{D}_0 = \Big\{ \sum_{n=0}^{\infty} I_n(f_n) : f_n \in L^2_{\mathbf{C}}(\mathbf{R}, dt)^{\hat{\otimes}n} \text{ for all } n \in \mathbf{N} \cup \{0\} \\ \text{and } \sum_{n=0}^{\infty} n!\tau^n \int_{\mathbf{R}^n} |f_n(t)|^2 dt < +\infty \Big\}$$

where $\tau = 2\int_{\mathbf{R}}(1+u^2)d\beta(u)$, then the following theorem holds.

Theorem 3.2. *(see [6]). $\mathcal{D}_0 = (L^2)$ if and only if $\beta = c\delta_a$ for some $c > 0$ and $a \in \mathbf{R}$.*

4. Conditions for Eigenfunctions of the Lévy Laplacian.

Let $F_f(\xi) = \mathcal{U}[I_n(f)](\xi)$ with supp$f \subset T^n$. Then we have

$$F_f(\xi) = \int_{\mathbf{R}^n} f(t_1, \cdots, t_n) \prod_{j=1}^{n} G(\xi(t_j)) dt_1 \cdots dt_n \tag{4.1}$$

and we can calculate

$$\widetilde{\Delta}_L F_f(\xi) = -\frac{n}{|T|} \int_{\mathbf{R}^n} f(t_1, \cdots, t_n) H(\xi(t_1)) \prod_{j=2}^{n} G(\xi(t_j)) dt_1 \cdots dt_n, \tag{4.2}$$

where

$$\begin{aligned} G(r) &= 2i\sigma^2 r + \int_{|u|>0} (e^{iru} - e^{-iru}) \frac{1+u^2}{u} d\beta(u), \\ H(r) &= \int_{\mathbf{R}} u(1+u^2)(e^{iru} - e^{-iru}) d\beta(u). \end{aligned}$$

Proposition 4.1. *The functionals F_f for all $f \in L^2_{\mathbf{C}}(\mathbf{R}, dt)^{\hat{\otimes}n}$ are eigenfunctions of $\widetilde{\Delta}_L$ if and only if there exists $C \in \mathbf{R}$ such that the following equalities hold:*

$$(P1) \qquad -\frac{n}{|T|} \int_{\mathbf{R}} u^{2k}(1+u^2) d\beta(u) = C \int_{\mathbf{R}} u^{2(k-1)}(1+u^2) d\beta(u), \; k \in \mathbf{N},$$

Proof. Let F_f be an eigenfunction of $\widetilde{\Delta}_L$. Then there exists $C \in \mathbf{R}$ such that $\widetilde{\Delta}_L F_f = C F_f$. By (4.1) and (4.2), we have

$$H(r) - CG(r) = 0 \text{ for all } r \in \mathbf{R}.$$

We note that

$$\begin{aligned} G(r) &= 2ir\Big\{\sigma^2 + \int_{|u|>0}(1+u^2)d\beta(u)\Big\} \\ &\quad +2\sum_{k=1}^{\infty}\frac{i^{2k+1}}{(2k+1)!}r^{2k+1}\int_{\mathbf{R}}u^{2k}(1+u^2)d\beta(u) \\ &= 2ir\int_{\mathbf{R}}(1+u^2)d\beta(u) \\ &\quad +2\sum_{k=1}^{\infty}\frac{i^{2k+1}}{(2k+1)!}r^{2k+1}\int_{\mathbf{R}}u^{2k}(1+u^2)d\beta(u) \end{aligned} \tag{4.3}$$

and

$$\begin{aligned} H(r) &= 2ir\int_{\mathbf{R}}u^2(1+u^2)d\beta(u) \\ &\quad +2\sum_{k=1}^{\infty}\frac{i^{2k+1}}{(2k+1)!}r^{2k+1}\int_{\mathbf{R}}u^{2(k+1)}(1+u^2)d\beta(u). \end{aligned} \tag{4.4}$$

By (4.2),(4.3) and (4.4), we have (P1). Conversely, if there exists $C \in \mathbf{R}$ such that (P1) holds, then we can check $\widetilde{\Delta}_L F_f = CF_f$ by the above calculations (4.1), (4.2), (4.3) and (4.4). □

Theorem 4.2. *The functionals F_f for all $f \in L^2_{\mathbf{C}}(\mathbf{R}, dt)^{\hat{\otimes} n}$ are eigenfunctions of $\widetilde{\Delta}_L$ if and only if either of the following equality holds:*

$(T1)$ $\quad \beta = \sigma^2\delta_0$

$(T2)$ $\quad \sigma = 0, \beta = b\delta_a + d\delta_{-a}$ *for some* $a > 0, b \geq 0$ *and* $d \geq 0$

Proof. If the functionals F_f for all $f \in L^2_{\mathbf{C}}(\mathbf{R}, dt)^{\hat{\otimes} n}$ are eigenfunctions of $\widetilde{\Delta}_L$, then (P1) holds. Since

$$\int_{\mathbf{R}} u^{2k}\left(u^2 + \frac{|T|}{n}C\right)\left(1+u^2\right)d\beta(u) = 0\ ,\ \ k \in \mathbf{N}$$

from (P1), we have

$$\begin{aligned} &\int_{\mathbf{R}} u^2\left(u^2 + \frac{|T|}{n}C\right)^2\left(1+u^2\right)d\beta(u) \\ &= \int_{\mathbf{R}} u^4\left(u^2 + \frac{|T|}{n}C\right)\left(1+u^2\right)d\beta(u) \\ &\quad + \frac{|T|}{n}C\int_{\mathbf{R}} u^2\left(u^2 + \frac{|T|}{n}C\right)\left(1+u^2\right)d\beta(u) \\ &= 0. \end{aligned} \tag{4.5}$$

In the case of $C \geq 0$, by (4.5) we have

$$\beta(\mathbf{R} - \{0\}) = 0.$$

Therefore β is expressed by

$$\beta = c\delta_0 \ , \ c \geq 0.$$

Since $\beta(\{0\}) = \sigma^2$, we get

$$c = \int_{\mathbf{R}} 1_{\{0\}}(u) d\beta(u) = \sigma^2.$$

Hence we have (T1). In the case of $C < 0$, by (4.5) we have

$$\beta(\mathbf{R} - \{0, a, -a\}) = 0$$

where $a = \sqrt{-\frac{|T|C}{n}}$. This means that β is expressed by

$$\beta = c\delta_0 + b\delta_a + d\delta_{-a} \ , \ b, c, d \geq 0.$$

Since $\beta(\{0\}) = \sigma^2$, we get

$$c = \int_{\mathbf{R}} 1_{\{0\}}(u) d\beta(u) = \sigma^2.$$

By (P1), we have

$$(b+d)a^2(1+a^2) = a^2\{\sigma^4 + (b+d)(1+a^2)\}$$

and hence

$$\sigma = 0,$$

thus we have (T2). If β is given by the form (T1), we have

$$-\frac{n}{|T|} \int_{\mathbf{R}} u^k (1+u^2) d\beta(u) = 0, \ k \in \mathbf{N}.$$

Therefore (P1) holds for arbitrary constant C in case of $\sigma = 0$, and holds by setting $C = 0$ if otherwise. By Proposition 4.1, the function F_f is an eigenfunction of $\widetilde{\Delta}_L$. If β is given by the form (T2), setting $C = -\frac{n}{|T|}a^2$, we have

$$\begin{aligned} C \int_{\mathbf{R}} u^{2(k-1)} (1+u^2) d\beta(u) &= -\frac{n}{|T|} a^2 (b+d) a^{2(k-1)} (1+a^2) \\ &= -\frac{n}{|T|} \int_{\mathbf{R}} u^{2k} (1+u^2) d\beta(u), \end{aligned}$$

where $k \in \mathbf{N}$. Hence (P1) holds. By Proposition 4.1, the function F_f is an eigenfunction of $\widetilde{\Delta}_L$. □

Example 4.3. Standard Gaussian white noise measure

$$\mu = 0 \ , \ \beta = \sigma^2 \delta_0.$$

By Theorem 4.2, the function F_f is an eigenfunction of $\widetilde{\Delta}_L$ with $C = 0$.

Example 4.4. Poisson white noise measure

$$\mu = \frac{1}{2} \ , \ \beta = \frac{1}{2}\delta_1.$$

By Theorem 4.2, the function F_f is an eigenfunction of $\widetilde{\Delta}_L$ with $C = -\frac{n}{|T|}$.

Example 4.5. Gamma white noise measure

$$\mu = \int_0^\infty \frac{e^{-u}}{1+u^2} du \ , \ \beta(E) = \int_{E\cap[0,+\infty)} \frac{ue^{-u}}{1+u^2} du.$$

Since

$$\int_{\mathbf{R}} u^k (1+u^2) d\beta(u) = (2k)! \qquad (k \in \mathbf{N}),$$

by Proposition 4.1, there exist $f \in L^2_{\mathbf{C}}(\mathbf{R}, dt)^{\hat{\otimes} n}$ such that F_f is not an eigenfunction of $\widetilde{\Delta}_L$.

5. Decomposition by Eigenspaces of (L^2)

Let $\mathcal{D}_{\overline{L}}$ be the set of white noise distribution $\Phi \in (E)^*$ such that $\lim_{|T|\to\infty} |T|\Delta_L \Phi$ exists in $(E)^*$. For any $\Phi \in \mathcal{D}_{\overline{L}}$, we define an operator $\overline{\Delta_L}$ by

$$\overline{\Delta_L}\Phi = \lim_{|T|\to\infty} |T|\Delta_L \Phi.$$

We use the same notaition $\overline{\Delta_L}$ for the $\mathcal{U}$-transform $\mathcal{U}\Phi$ of $\Phi \in \mathcal{D}_{\overline{L}}$ by $\overline{\Delta_L}\mathcal{U}\Phi = \mathcal{U}\overline{\Delta_L}\Phi$. If (T2) holds, we can calculate

$$\overline{\Delta_L}\Phi = \lim_{|T|\to\infty} |T|\Delta_L \Phi = -na^2, \ \Phi \in \mathcal{D}_{\overline{L}}.$$

Let $K_0 = \mathbf{C}$, $K_n = \{I_n(f) | f \in L^2_{\mathbf{C}}(\mathbf{R}, dt)^{\hat{\otimes} n}\}$ for each $n \in \mathbf{N}$. Set $\mathbf{K} = \bigoplus_{n=0}^\infty K_n$. By Proposition 3.1, we can obtain the following Proposition.

Proposition 5.1. *If (T1) or (T2) holds, then* $\mathbf{K} = (L^2)$.

Proposition 5.2. *Let $n \in \mathbf{N}$. Then any φ in K_n is an eigenfunction of $\overline{\Delta_L}$ if and only if either of (T1) and (T2) holds.*

Proof. By Theorem 4.2, all F in $\mathcal{U}[K_n]$ are eigenfunctions of $\overline{\Delta_L}$ with same eigenvalue if and only if (T1) or (T2) holds. Then, all φ in K_n are eigenfunctions of $\overline{\Delta_L}$ with same eigenvalue if and only if (T1) or (T2) holds. □

Let $W_n(\lambda) = \{\varphi \in K_n \,|\, \overline{\Delta_L}\varphi = \lambda\varphi\}$ for any $n \in \mathbf{N} \cup \{0\}$ and $\lambda \in \mathbf{R}$. Then Proposition 5.1 and Proposition 5.2 imply the following Theorem.

Theorem 5.3. *We have the following assertions:*

1) *The decomposition $(L^2) = \bigoplus_{n=0}^\infty W_n(0)$ holds if and only if (T1) holds.*
2) *The decomposition $(L^2) = \bigoplus_{n=0}^\infty W_n\Big(-na^2\Big)$ holds if and only if (T2) holds.*

6. A Semi-group Generated by the Lévy Laplacian

For each $t \geq 0$, we consider an operator G_t on $\mathbf{K}$ is defined by

$$G_t\varphi = \sum_{n=0}^\infty e^{C_n t}\varphi_n$$

for $\varphi = \sum_{n=0}^\infty \varphi_n \in \mathbf{K}$, where C_n is defined by

- $C_n = 0$ if (T1) holds,

- $C_n = -na^2$ if (T2) holds.

Then we have the following Theorem.

Theorem 6.1. *If (T1) or (T2) holds, then the family* $\{G_t; t \geq 0\}$ *is an equi-continuous semigroup of class* (C_0) *generated by* $\overline{\Delta_L}$ *as a continuous linear operator densely defined on* (L^2).

Proof. By Proposition 5.1, $(L^2) = \mathbf{K}$ holds. Let $\varphi = \sum_{n=0}^{\infty} \varphi_n \in \mathbf{K}$, for any $t \geq 0$ the norm $\|G_t \varphi\|_0$ is estimated as follows:

$$\begin{aligned} \|G_t\varphi\|_0^2 &= \sum_{n=0}^{\infty} \|e^{C_n t}\varphi_n\|_0^2 \\ &\leq \sum_{n=0}^{\infty} \|\varphi_n\|_0^2 \\ &= \|\varphi\|_0^2. \end{aligned}$$

Hence the family $\{G_t; t \geq 0\}$ is an equi-continuous in t. It is easily checked that $G_0 = I, G_t G_s = G_{t+s}$ for each $t, s \geq 0$. We can also estimate that

$$\begin{aligned} \|G_t\varphi - G_{t_0}\varphi\|_0^2 &= \sum_{n=0}^{\infty} |e^{C_n t} - e^{C_n t_0}|^2 \|\varphi_n\|_0^2 \\ &\leq 4\sum_{n=0}^{\infty} \|\varphi_n\|_0^2 \\ &= 4\|\varphi\|_0^2 \end{aligned}$$

for each $t, t_0 \geq 0$. Therefore, by the Lebesgue convergence theorem, we get that

$$\lim_{t \to t_0} G_t \varphi = G_{t_0}\varphi \text{ in } (L^2)$$

for each $t_0 \geq 0$ and $\varphi \in \mathbf{K}$. Thus the family $\{G_t; t \geq 0\}$ is an equi-continuous semigroup of class (C_0). We next prove that the infinitensional generator of the semigroup is given by $\overline{\Delta_L}$. Let

$$\mathcal{D} = \{\varphi \in \mathbf{K} \,|\, \overline{\Delta_L}\varphi \text{ exists in } \mathbf{K}\}.$$

Since polynomial functionals of white noise are included in $\mathcal{D}$, we can see that $\mathcal{D}$ is dense in (L^2). For $\varphi = \sum_{n=0}^{\infty} \varphi_n \in \mathcal{D}$, we have

$$\left\| \frac{G_t\varphi - \varphi}{t} - \overline{\Delta_L}\varphi \right\|_0^2 = \sum_{n=0}^{\infty} \left\| \frac{e^{C_n t} - 1}{t}\varphi_n - C_n\varphi_n \right\|_0^2.$$

By the mean value theorem, for any $t > 0$ there exists a constant $\theta \in (0, 1)$ such that

$$\left| \frac{e^{C_n t} - 1}{t} \right| = C_n e^{C_n t\theta} \leq C_n.$$

Therefore we can estimate the following term:

$$\begin{aligned} \left\| \frac{e^{C_n t} - 1}{t}\varphi_n - C_n\varphi_n \right\|_0^2 &= \left| \frac{e^{C_n t} - 1}{t} - C_n \right|^2 \|\varphi_n\|_0^2 \\ &\leq 4C_n \|\varphi_n\|_0^2. \end{aligned}$$

By

$$\lim_{t\to 0}\left|\frac{e^{C_n t}-1}{t}-C_n\right|=0$$

and the Lebesgue convergence theorem, we obtain

$$\lim_{t\to 0}\left\|\frac{G_t\varphi-\varphi}{t}-\overline{\Delta_L}\varphi\right\|_0^2=0.$$

Thus the proof is completed. □

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References

1. Hida, T.: Analysis of Brownian Functionals, *Carleton Math. Lecture Notes, No.13.* (1975), Carleton University, Ottawa.
2. Ishikawa, A.: A Decomposition of multiple wiener integrals by the Lévy process and Lévy Laplacian, *Communication on Stochastic Analysis Vol.2, No.3.* (2008), 459–467.
3. Ishikawa, A. and Saitô, K: The Lévy Laplacian and the Lévy process, *RIMS Kokyuroku 1507* (2006) 14–25.
4. Ishikawa, A., Saitô, K., and Tsoi, A. H.: Poisson Noise Analysis based on the Lévy Laplacian, *Quantum Information IV* (2001) 103–114.
5. Kuo, H.-H.: White Noise Distribution Theory, *CRC Press*, Boca Raton, 1996.
6. Lee, Y.-J. and Shih, H.-H.: Analysis of generalized Lévy white noise functionals, *Journal of Functional Analysis 211* (2004) 1–70.
7. Lévy, P.: Lecons d'analyse fonctionnelle, *Gauthier-Villars* Paris, 1922.
8. Saitô, K.: An infinite dimensional Laplacian acting on multiple Wiener integrals by some Lévy process, *Infinite Dimensional Harmonic Analysis III* Paris, (2005) 265–276.
9. Saitô, K. and Tsoi, A. H.: The Lévy Laplacian as a self-adjoint operator, *Quantum Information* (1999) 159–171.
10. Saitô, K. and Tsoi, A. H.: The Lévy Laplacian acting on poisson noise functionals, *Infinite Dimensional Analysis, Quantum Probability and Related Topics 2* (1999) 503–510.

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