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ON ELECTROMAGNETIC WAVE PROPAGATION IN A MEDIUM WITH SPATIAL AND TEMPORAL INHOMOGENEITIES

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ABSTRACT: An equation that models electromagnetic wave propagation in a medium with spatial and temporal inhomogeneities is considered. Asymptotic solutions and some Hamiltonian-related field properties of the equation in both non-caustic and caustic regions are developed.

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1. INTRODUCTION

Electromagnetic wave propagation in a medium with both spatial and temporal inhomogeneities is commonly modeled by an equation of the form

$$\nabla^2 \overline{E}(\overline{r}, t) - f(\overline{r}, t)\frac{\partial^2 \overline{E}(\overline{r}, t)}{\partial t^2} - g(\overline{r}, t)\overline{E}(\overline{r}, t) = 0. \tag{1}$$

In Equation 1, $\overline{E}(\overline{r}, t)$ is the wave function, $\overline{r}$ refers to the spatial coordinates and t is the time. When associated with a plasma, $f(\overline{r}, t)$ is related to the index of refraction and $g(\overline{r}, t)$ to the plasma oscillations. In modelling ionospheric propagation, often the ionosphere is regarded as stationary, hence time does not appear in the index of refraction and the medium oscillations are considered as negligible, hence $g(\overline{r}, t) = 0$. With regard to electromagnetic wave propagation in tokamaks, however, both the time variation of the medium and $g(\overline{r}, t)$ assume greater importance. Physically, the temporal inhomogeneity occurs when the characteristic frequencies of the medium, e.g., the resonant absorption frequency of the molecules, the cyclotron frequency of the plasma, lie within the frequency range of the source. In this case the component frequencies of the source are not uniformly absorbed and re-radiated. The re-radiated frequency components are propagated with different velocities, resulting in a distortion of the waveform[1, 2].

No general technique exists for obtaining exact solutions to Equation 1, or even the stationary model often considered in ionospheric propagation. Consequently, various approaches have been developed leading to approximate solutions, each approach valid under situation-specific assumptions. But the difficulty of obtaining even approximate solutions has long been noted, cf., [3]. One such approach is the asymptotic series, or geometrical optics algorithm, originally developed by Keller [4]; however, this formalism predicts infinite wave amplitudes at caustics, regions where the inhomogeneities of the medium effect a focusing of the wave trajectories. The geometrical optics algorithm has been modified to predict realistic wave amplitudes, associated with simple caustics for such as Equation 1, cf., [5, 6, 7]. More complicated caustics remain problematic, both in the analysis of propagation in tokamaks and in the time-independent propagation models associated with radio occultation, the remote analysis of planetary atmospheres [8, 9]. Further, such optical caustics have been produced in stationary media in a laboratory setting [10].

One approach to this difficulty is the Lagrange manifold formalism introduced by Arnol'd [11] and Maslov [12]. The finite wave amplitudes their approach predicts shows that while caustics are associated with regions which exhibit a focusing of energy, the wave amplitudes predicted by the geometrical optics approach arise as a consequence of the parametrization, rather than from a physical singularity. Our purpose here is to extend a formalism [13] developed to study more general caustics associated with scalar waves in spatially and temporally inhomogeneous media to asymptotic series solutions near caustics for electromagnetic wave propagation as modelled by Equation 1. In order to provide a better comparison of the geometrial optics and modified Lagrange formalisms, away from caustics we summarize Keller's asymptotic approach. Near caustics, we apply the modified Lagrange manifold approach, which allows inclusion of more complicated caustics than are ordinarily considered. In both regions, we develop transport equations that enable determination of the full asymptotic series solution and illustrate this approach with a simple example. Because both the geometrical optics and Lagrange manifold approaches are Hamiltonian formalisms, we illustrate that some field considerations associated with non-caustic regions may be extended to caustic regions.

2 Non-Caustic Regions

The asymptotic procedures begin with a re-scaling of coordinates ($\overline{r} \to \overline{r}/\tau$, $t \to t/\tau$), where τ is a large parameter. Consequently, Equation 1 may be written as

$$\nabla^2 \overline{E}(\overline{r}, t) - f(\overline{r}, t)\frac{\partial^2 \overline{E}(\overline{r}, t)}{\partial t^2} - \tau^2 g(\overline{r}, t)\overline{E}(\overline{r}, t) = 0\,. \tag{2}$$

Physically, this implies the regime of long distances and observation times. Away from caustics, the geometrical optics approach assumes a solution of the form

$$\overline{E}(\overline{r}, t) = \overline{E}(\overline{r}, t, \tau)\, e^{i\tau\phi(\overline{r}, t)}\,, \tag{3}$$

where

$$\overline{E}(\overline{r}, t, \tau) = \sum_{n=0} \overline{E}_n(\overline{r}, t)\,(i\tau)^{-n}, \quad \overline{E}_{-n} = 0\,. \tag{4}$$

In Equation 3, $\overline{E}\,(\overline{r}, t, \tau)$ may be regarded as the wave amplitude and $\phi\,(\overline{r}, t)$ as a phase. Then substituting Equation 3 into Equation 2, followed by a re-grouping by powers of $i\tau$, obtains

$$\begin{aligned}\sum_{n=0} \Bigg\{ &(i\tau)^2 \left[(\nabla\phi)^2 - f(\overline{r}, t)\left(\frac{\partial\phi}{\partial t}\right)^2 + g(\overline{r}, t) \right] \overline{E} \\ &+ (i\tau)\left[2\left(\nabla\phi(\nabla\cdot\overline{E}) + f(\overline{r}, t)\frac{\partial\phi}{\partial t}\frac{\partial\overline{E}}{\partial t} \right) + (\nabla^2\phi)\,\overline{E} - f(\overline{r}, t)\frac{\partial^2\phi}{\partial t^2}\overline{E} \right] \\ &+ (i\tau)^0 \left[\nabla^2\overline{E} - f(\overline{r}, t)\frac{\partial^2\overline{E}}{\partial t^2} \right] \Bigg\} e^{i\tau\phi(\overline{r}, t)} \sim 0. \end{aligned} \tag{5}$$

Next, introducing the wavevector and frequency

$$\overline{k} = \nabla\phi, \quad \omega = -\frac{\partial\phi}{\partial t}, \tag{6}$$

respectively, into the coefficient of the $(i\tau)^2$ term determines the eikonal equation

$$\overline{k}\cdot\overline{k} - f(\overline{r}, t)\,\omega^2 + g(\overline{r}, t) = 0\,, \tag{7}$$

which may be regarded as a Hamiltonian

$$H = \overline{k}\cdot\overline{k} - f(\overline{r}, t)\,\omega^2 + g(\overline{r}, t)\,. \tag{8}$$

Then Hamilton's equations

$$\frac{d\overline{r}}{dt} = \nabla_k H, \quad \frac{d\overline{k}}{dt} = -\nabla_r H \tag{9}$$

$$\frac{dt}{d\gamma} = -\frac{\partial H}{\partial\omega}, \quad \frac{d\omega}{d\gamma} = \frac{\partial H}{\partial t}, \tag{10}$$

where γ is a raypath parameter, e.g., the arclength, may be used to determine the phase. The solution of Equations 9 and 10

$$\overline{r} = \overline{r}(\gamma, \overline{\sigma}), \quad \overline{k} = \overline{k}(\gamma, \overline{\sigma}) \tag{11}$$

$$t = t(\gamma, \overline{\sigma}), \quad \omega = \omega(\gamma, \overline{\sigma}), \tag{12}$$

where $\overline{\sigma}$ is a parameterized initial condition, e.g., direction cosines, are the space-time ray trajectories. The phase is determined by an integration along the trajectories

$$\phi(\overline{r}, t) = \int_{(r_0, t_0)}^{(r,t)} \overline{k} \cdot d\overline{r} - \omega d\gamma + \phi(\overline{r}_0, t_0). \tag{13}$$

With the phase thus determined, a transport equation for the wave amplitudes proceeds from the $(i\tau)$ and $(i\tau)^0$ terms in Equation 5, again using Hamilton's Equations, specifically from Equation 8,

$$\frac{d\overline{r}}{dt} = \nabla_k H, \quad \frac{dt}{d\gamma} = -\frac{\partial H}{\partial \omega} = 2\omega f(\overline{r}, t). \tag{14}$$

Then substitution of Equations 14 into Equation 5 determines the transport equation

$$\frac{d\overline{E}_n}{d\gamma} + \left(\nabla^2 \phi - f(\overline{r}, t)\frac{\partial^2 \phi}{\partial t^2}\right)\overline{E}_n + \nabla^2 \overline{E}_{n-1} - f(\overline{r}, t)\frac{\partial^2 \overline{E}_{n-1}}{\partial t^2} = 0, \quad n \geq 1. \tag{15}$$

Consequently, if the wave amplitude $\overline{E}_0$ is specified at the source, $\overline{r}_0$ at some initial time, t_0, with initial wave vectors, $\overline{k}_0$, and frequency, ω_0, satisfying Equation 7, then the wave amplitudes at any space-time field point $(\overline{r}, t)$ may be determined from

$$\overline{E}_0(\overline{r}, t) = \overline{E}_0(\overline{r}_0, t_0)\left[\frac{f(\overline{r}_0, t_0) J_{t_0}(\overline{r}_0, t_0)}{f(\overline{r}, t) J_t(\overline{r}, t)}\right]^{\frac{1}{2}}, \tag{16}$$

where J_t is the Jacobian of the ray transformation from parameter space to coordinate space, $\overline{\mu} \rightarrow \overline{r}$, at each time t, i.e.,

$$J_t(\overline{r}, \overline{\mu}) = \frac{\partial(\overline{r})}{\partial(\overline{\mu})}, \tag{17}$$

where $\overline{\mu} = (\overline{\sigma}, \gamma)$. With $\overline{E}_0(\overline{r}, t)$ determined, the other $\overline{E}_n$,'s may be obtained recursively.

This algorithm suffices to determine the asymptotic solution at most field points. At caustic points, however, the spatial and temporal inhomogeneities of the medium effect a focusing of trajectories and the ray transformation from parameter space to coordinate space becomes singular, i.e., $J_t(\overline{r}, \overline{\mu}) = 0$. Consequently, at such points, this algorithm predicts unrealistic wave amplitudes and a modified approach is necessary.

3. CAUSTIC REGIONS

One approach that applies near caustics is a variation of a formalism developed by Arnol'd [11] and Maslov [12] that has been adapted to study caustics associated with scalar waves in media with an inhomogeneous non-stationary flow [13]. Analogous to Equation 3, near caustics we assume a solution of the form

$$\overline{E}(\overline{r},t) = \int \overline{E}(\overline{r},\overline{k},t,\tau)\, e^{i\tau\phi}\, d\overline{k}\ . \tag{18}$$

In Equation 18, the wave amplitude

$$\overline{E}(\overline{r},\overline{k},t,\tau) = \sum_{n=0} \overline{E}_n(\overline{r},\overline{k},t,\tau)\,(i\tau)^{-n}; \quad \overline{E}_{-n} = 0\,, \tag{19}$$

and its derivatives are assumed bounded, and

$$\phi(\overline{r},\overline{k},t) = \overline{r}\cdot\overline{k} - S(\overline{k},t)\,, \tag{20}$$

is analytic. Then carrying the differentiation in Equation 2 across the integral in Equation 18, re-grouping by powers of $i\tau$ leads to

$$\begin{aligned}\int \Bigg\{ &(i\tau)^2 \left[(\nabla\phi)^2 - f(\overline{r},t)\left(\frac{\partial\phi}{\partial t}\right)^2 + g(\overline{r},t) \right] \overline{E} \\ &+ (i\tau)\left[2\left(\nabla\phi(\nabla\cdot\overline{E}) + f(\overline{r},t)\,\frac{\partial\phi}{\partial t}\,\frac{\partial\overline{E}}{\partial t} \right) - f(\overline{r},t)\,\frac{\partial^2\phi}{\partial t^2}\,\overline{E} \right] \\ &+ (i\tau)^0 \left[\nabla^2\overline{E} - f(\overline{r},t)\,\frac{\partial^2\overline{E}}{\partial t^2} \right]\Bigg\} e^{i\tau\phi}\, d\overline{k} \sim 0. \end{aligned} \tag{21}$$

By introducing the wavevector and frequency from Equation 6, the coefficient of the $(i\tau)^2$ term is seen to be the Hamiltonian, Equation 8, from which we obtain the phase.

The determination of $S(\overline{k}, t)$, and hence the phase, proceeds from the stationary phase ($\nabla_k \phi = 0$) evaluation of the integral at any caustic point ($\overline{r}$, t), leading to the time-parameterized Lagrange manifold

$$\overline{r} = \nabla_k S(\overline{k}, t), \tag{22}$$

a coordinate transformation between configuration space and momentum space. To determine $S(\overline{k}, t)$, we again apply Hamilton's Equations 9 and 10 to find the trajectories (map) in Equations 11 and 12. Next the wavevector and time transformations are inverted to obtain

$$\overline{k} = \overline{k}(\gamma, \overline{\varpi}), \qquad t = t(\gamma, \overline{\varpi}). \tag{23}$$

Then substitution of Equation 23 into the coordinate space map, determines the Lagrange Manifold explicitly:

$$\bar{r} = \bar{r}\,(\gamma\,(\bar{k},\, t),\, \bar{\sigma}\,(\bar{k},\, t)) = \nabla_k S\,(\bar{k},\, t), \tag{24}$$

where time appears as a parameter. Finally, an integration along the trajectories obtains

$$S(\bar{k}, t) = \int_{\bar{k}_0}^{\bar{k}} \bar{r} \cdot d\bar{k} \tag{25}$$

and, hence, the phase

$$\phi(\bar{r}, \bar{k}, t) = \bar{r} \cdot \bar{k} - S(\bar{k}, t)\,. \tag{26}$$

In the parametrization specified by the Lagrange Manifold, caustic points are those space-time points at which

$$\det\left\{\frac{\partial^2 \phi}{\partial k_i \partial k_j}\right\} = \det\left\{\frac{\partial^2 S}{\partial k_i \partial k_j}\right\} = 0\,. \tag{27}$$

Each triplet $(\bar{k})$ that satisfies Equation 27 at a particular time t corresponds to a caustic point in configuration space obtained by substitution into the Lagrange manifold, Equation 22. The locus of these points specifies the caustic curve in configuration space at that particular time. Solving Equation 27 for various values of time, with repeated substitution into the Lagrange manifold, obtains the time-evolution of the caustic in configuration space.

We determine a transport equation for the wave amplitudes by Taylor-expanding the Hamiltonian near the Lagrange manifold

$$\begin{aligned} \bar{k} \cdot \bar{k} - f(\bar{r}, t)\, \omega^2 + g(\bar{r}, t) &= \bar{k} \cdot \bar{k} - f(\nabla_k S, t)\, \omega^2 \\ &+ g(\nabla_k S, t) + (\bar{r} - \nabla_k S) \cdot \bar{D} = (\bar{r} - \nabla_k S) \cdot \bar{D}\,, \end{aligned} \tag{28}$$

where

$$\bar{D} = \int_0^1 \nabla_r H\,(\zeta(\bar{r} - \nabla_k S) + \nabla_k S)\, d\zeta\,. \tag{29}$$

Substitution of Equation 29 into Equation 21, followed by a partial integration, leads to

$$\begin{aligned} \int \Bigg\{ (i\tau) \Bigg[(\nabla_k \cdot \bar{D})\bar{E} - \bar{D}(\nabla_k \cdot \bar{E}) + 2\left(\nabla\phi(\nabla \cdot \bar{E}) + f(\bar{r}, t)\, \frac{\partial \phi}{\partial t}\, \frac{\partial \bar{E}}{\partial t} \right) \\ - f(\bar{r}, t)\, \frac{\partial^2 \phi}{\partial t^2}\, \bar{E} \Bigg] + (i\tau)^0 \left[\nabla^2 \bar{E} - f(\bar{r}, t)\, \frac{\partial^2 \bar{E}}{\partial t^2} \right] \Bigg\} e^{i\tau\phi}\, d\bar{k} \sim 0. \end{aligned} \tag{30}$$

Finally, introducing the non-Hamiltonian flow

$$\frac{d\bar{r}}{d\gamma} = 2\bar{k}, \quad \frac{dt}{d\gamma} = 2\,\omega f(\bar{r}, t), \quad \frac{d\bar{k}}{d\gamma} = -\bar{D}, \tag{31}$$

into the integral in Equation 30 determines the transport equation

$$\frac{d\bar{E}_n}{d\gamma} - \left(\bar{\nabla}_k \cdot \bar{D} + f(\bar{r}, t)\frac{\partial^2 \phi}{\partial t^2}\right)\bar{E}_n + \nabla^2 \bar{E}_{n-1} - f(\bar{r}, t)\frac{\partial^2 \bar{E}_{n-1}}{\partial t^2} = 0, \quad n \geq 1, \tag{32}$$

for the evolution of the amplitude in Equation 21.

4. Example

As an example, we consider propagation from a point source at the origin $\bar{r} = (0, 0, 0)$ at time t_0 with initial frequency Ω, initial wave vector $\bar{k}_0$ and direction cosines $(\cos(\theta), (\sin(\theta), \cos(\phi))$, where θ and ϕ are the usual horizontal and vertical direction angles, respectively. For expositional clarity, let the medium have an index of refraction $f(\bar{r}, t) = 1$ and plasma oscillation index $g(\bar{r}, t) = c^2 - at - bz$, where a, b and c are positive constants. Then assuming a solution of the form in Equation 3 leads to the Hamiltonian

$$H = \bar{k} \cdot \bar{k} - \omega^2 + at + bz - c^2,$$

and from Hamilton's Equations, the configuration space and wavevector maps

$$\begin{aligned} x &= 2\gamma k_0 \cos(\theta)\sin(\phi) & k_x &= k_0 \cos(\theta)\sin(\phi) \\ y &= 2\gamma k_0 \sin(\theta)\sin(\phi) & k_y &= k_0 \sin(\theta)\sin(\phi) \\ z &= -b\gamma^2 + 2\gamma k_0 \cos(\phi) & k_z &= -b\gamma + k_0 \cos(\phi) \end{aligned}$$

We note that the map from parameter space to coordinate space $(\bar{\mu} \to \bar{r})$ becomes singular on the curve $\gamma b \cos(\phi) + k_0 = 0$ or if $\phi = 0$. For such caustic points, we invert the map from parameter space to wavevector space $(\bar{\mu} \to \bar{k})$ and substitute into Equations 24, to explicitly obtain the Lagrange Manifold

$$x = \frac{2k_x}{(b^2 - a^2)}\left(F^{\frac{1}{2}}(k_x, k_y, k_z, t) + a\Omega - bk_z\right)$$

$$y = \frac{2k_y}{(b^2 - a^2)}\left(F^{\frac{1}{2}}(k_x, k_y, k_z, t) + a\Omega - bk_z\right)$$

$$z = \frac{2k_z}{(b^2 - a^2)}\left(F^{\frac{1}{2}}(k_x, k_y, k_z, t) + a\Omega - bk_z\right) + \frac{1}{(b^2 - a^2)}\left(2b(a\Omega - bk_z)^2\right.$$
$$\left. + 2b(a\Omega - bk_z)\left(F^{1/2}(k_x, k_y, k_z, t) + (b^2 - a^2)F(k_x, k_y, k_z, t)\right)\right),$$

where

$$F(k_x, k_y, k_z, t) = (a\Omega - bk_z)^2 - (b^2 - a^2)(k_x^2 + k_y^2 + k_z^2 - \Omega^2 - c^2 + at).$$

Then, from Equations 25 and 26, we determine the phase

$$\phi(\bar{r}, \bar{k}, t) = \bar{r} \cdot \bar{k} - S(\bar{r}, \bar{k}, t),$$

where

$$S(\bar{k}, t) = bk_z^3 + a\Omega k_z^2 + b(k_x^2 + k_y^2 - \Omega^2 - c^2 + at) - \frac{2(a\Omega - bk_z)^3}{3(a^2 - b^2)^2} - \frac{2F^{\frac{3}{2}}(k_x, k_y, k_z, t)}{(a^2 - b^2)}.$$

Finally, from Equation 32, we determine the transport equation

$$\frac{d\bar{E}_n}{d\gamma} - \left(\bar{\nabla}_k \cdot \bar{D} + f(\bar{r}, t)\frac{\partial^2 \phi}{\partial t^2} \right) \bar{E}_n + \left(\nabla^2 \bar{E}_{n-1} - f(\bar{r}, t)\frac{\partial^2 \bar{E}_{n-1}}{\partial t^2} \right) = 0, \quad n \geq 1$$

where $\bar{D} = (0, 0, b)$.

The determination of the electromagnetic field $\bar{E} = (\bar{r}, t)$ then follows from the asymptotic evaluation of the field integrals, from Equations 18 and 19,

$$\bar{E}(\bar{r}, t) = \sum_{n=0} \int \bar{E}_n(\bar{r}, \bar{k}, t)\, e^{i\tau\phi}\, d\bar{k}. \tag{33}$$

5. Calculational Considerations

The asymptotic evaluation of the field integrals

$$\int \bar{E}_n(\bar{r}, \bar{k}, t)\, e^{i\tau\phi}\, d\bar{k}$$

at any field point, taken for definiteness at $(\bar{r}_f, \bar{k}_f, t_f)$, proceeds by transforming the phase to its canonical form. If the Hessian determinant, i.e.,

$$\det \left(\frac{\partial(\bar{r})}{\partial(\bar{k})} \right)$$

has one zero eigenvalue, the canonical form for the phase is

$$\hat{\phi}(\bar{r}_f, \bar{\beta}, t_f) = \phi(\bar{r}_f, \bar{k}_f, t_f) \pm \beta_1^2 \pm \beta_2^2 \pm \beta_3^j, \quad 3 \leq j.$$

(The case $j = 2$ corresponds to points in the non-caustic region and are most expeditiously handled using the geometrical optics approach.) The value of the exponent 'j' may be found by forming

$$G(\lambda) = \phi(\overline{r}_f, k_{fx} + e_1\lambda, k_{fy} + e_2\lambda, k_{fz} + e_3\lambda, t_0),$$

where e_1, e_2 and e_3 are the components of the zero eigenvector and $\overline{k}_f = (k_{fx}, k_{fy}, f_{fz})$. The first non-vanishing term in the Taylor series of $G(\lambda)$ determines the value of j. The sign β_3 corresponds to the sign of this Taylor coefficient, and the signs of β_1 and β_2 correspond to the signs of the non-zero eigenvalues. The values $j = 3, 4$ correspond to the Airy and Pearcy functions, or the fold and cusp forms of Thom [14, 15], and the coordinate transformations carrying the eikonal phase to the canonical form may be determined algebraically by completing the cubic or quartic, as appropriate. If $j \leq 5$, where $j = 5, 6$ correspond to the swallowtail and butterfly forms of Thom, the coordinate transformations require an application of the implicit function theorem.

The resulting integral

$$\int \overline{E}_n(\overline{r}_f, \overline{k}(\overline{\beta}), t_f) \left(\frac{\partial(\overline{k})}{\partial(\overline{\beta})} \right) e^{i\tau(\pm\beta_1^2 \pm \beta_2^2 \pm \beta_3^j)} \, d\overline{\beta}$$

is evaluated by factoring the triple integral into an integral over (β_1, β_2) followed by an integral over β_3. The evaluation proceeds using the classical stationary phase technique for the integral over (β_1, β_2), followed by a modified stationary phase approach for the integral over β_3.

The case where the Hessian determinant has two zero eigenvalues corresponds to more complicated caustic structures. The canonical form for the phase is

$$\hat{\phi}(\overline{r}_f, \overline{\beta}, t_f) = \phi(\overline{r}_f, \overline{k}_f, t_f) \pm \beta_1^2 + \theta(\beta_2, \beta_3),$$

where $\theta(\beta_2, \beta_3)$ is the appropriate Thom umbilic and the sign of β_1 corresponds to the sign of the non-zero eigenvalue. To transform the eikonal phase to this canonical form we first apply the splitting lemma and complete the square in the quadratic. The umbilic corresponding to the eikonal phase is then determined from the cubic terms in β_2 and β_3, i.e.,

$$t_{30}\beta_2^3 + t_{21}\beta_2^2\beta_3 + t_{12}\beta_2\beta_3^2 + t_{03}\beta_3^3,$$

where the t_{ij} are constants. Setting the cubic equal to zero leads to one of the four possible root combinations, each corresponding to a specific canonical form:

three real equal roots	β_3^3	fold
three real unequal roots	$\beta_2^3 - \beta_2\beta_3^2$	elliptic umbilic
three real roots, two equal	$\beta_2^2\beta_3 + \beta_2^4$	parabolic umbilic
one real root one complex conjugate pair	$\beta_2^2 + \beta_2\beta_3^2$ or $\beta_2^3 + \beta_3^3$	hyperbolic umbilic

Two special cases are noteworthy. If $t_{30} = t_{03} = 0$ and both t_{12} and $t_{21} \neq 0$, the corresponding form is the parabolic umbilic. If $t_{30} = t_{03} = 0$ and either t_{12} or $t_{21} = 0$, there is no corresponding form. Additionally, we note that the case corresponding to three equal roots represents a symmetry where propagation in three dimensions may be modelled as two dimensional. The canonical forms corresponding to the Hessian matrix having three zero eigenvalues have been given by Arnol'd [16]. This case is not considered here.

The coordinate transformations carrying the eikonal phase to the appropriate umbilic are detailed elsewhere [14, 15] and after a factoring result in the integral

$$e^{i\tau\phi(\bar{r}_f,\bar{k}_f,t_f)}\int\int e^{i\tau\phi(\beta_2,\beta_3)}\,d\beta_2\,d\beta_3\int \bar{E}_n(\bar{r}_f,\bar{k}(\bar{\beta}),t_f)\left(\frac{\partial(\bar{k})}{\partial(\bar{\beta})}\right)e^{\pm\tau\beta_1^2}\,d\beta_1\,,$$

where $\theta(\beta_2, \beta_3)$ is the umbilic and the sign of β_1 follows from the sign of the non-zero eigenvalue. The evaluation of this triple integral begins with a straightforward stationary phase integration over β_1, followed by a substantially modified stationary phase approach for the integration over β_2 and β_3 [17].

6. Field Considerations

Because the Lagrange Manifold formalism is essentially an integral representation of the geometrical optics approach, various Hamiltonian and field properties away from the caustic carry over into the caustic region. Since these algorithms develop identical eikonal equations and Hamiltonians on and off the caustic, i.e.,

$$\bar{k}\cdot\bar{k} - f(\bar{r},t)\omega^2 + g(\bar{r},t) = 0\,, \tag{7}$$

and

$$H = \bar{k}\cdot\bar{k} - f(\bar{r},t)\omega^2 + g(\bar{r},t)\,, \tag{8}$$

the dispersion relation both on and off the caustic becomes

$$\omega^2 = \frac{k_x^2 + k_y^2 + k_z^2 + g(\bar{r},t)}{f(\bar{r},t)}\,. \tag{34}$$

Consequently, both on and off the caustic the phase and group velocities, respectively, are

$$v_{p_x} = \frac{\omega}{k_x} = \sqrt{\frac{k_x^2 + k_y^2 + k_z^2 + g(\bar{r},t)}{f(\bar{r},t)\,k_x^2}}$$

$$v_{p_y} = \frac{\omega}{k_y} = \sqrt{\frac{k_x^2 + k_y^2 + k_z^2 + g(\bar{r},t)}{f(\bar{r},t)\,k_y^2}}$$

$$v_{p_z} = \frac{\omega}{k_z} = \sqrt{\frac{k_x^2 + k_y^2 + k_z^2 + g(\overline{r}, t)}{f(\overline{r}, t)\, k_z^2}} \tag{35}$$

and

$$c_{g_x} = \frac{\partial\omega}{\partial k_x} = \frac{k_x}{f(\overline{r}, t)\,\omega}$$

$$c_{g_y} = \frac{\partial\omega}{\partial k_y} = \frac{k_y}{f(\overline{r}, t)\,\omega}$$

$$c_{g_z} = \frac{\partial\omega}{\partial k_z} = \frac{k_z}{f(\overline{r}, t)\,\omega}\,. \tag{36}$$

Also, Hamilton's equations will be the same in both regions, specifically,

$$\frac{d\overline{r}}{d\gamma} = 2\overline{k}$$

$$\frac{d\overline{r}}{d\gamma} = -\nabla_{\overline{r}} H = \nabla f(\overline{r}, t) - \nabla g(\overline{r}, t)$$

$$\frac{d\omega}{d\gamma} = \frac{\partial H}{\partial t} = -\frac{\partial f(\overline{r}, t)}{\partial t}\,\omega^2 + \frac{\partial g(\overline{r}, t)}{\partial t}$$

$$\frac{dt}{d\gamma} = -\frac{\partial H}{\partial \omega} = 2\omega f(\overline{r}, t)\,. \tag{37}$$

Hamilton's equations may be recast as derivatives with respect to time by merely dividing each equation by $\frac{dt}{d\gamma}$ to obtain

$$\frac{d\overline{r}}{d\gamma} = \frac{\overline{k}}{\omega f(\overline{r}, t)} = \overline{c}_{v_g}$$

$$\frac{d\overline{k}}{dt} = \frac{\nabla f(\overline{r}, t) - \nabla g(\overline{r}, t)}{2\omega f(\overline{r}, t)}$$

$$\frac{d\omega}{dt} = \frac{-\frac{\partial f(\overline{r}, t)}{\partial t}\,\omega^2 + \frac{\partial g(\overline{r}, t)}{\partial t}}{2\omega f(\overline{r}, t)} \tag{38}$$

From these equations, we observe the following phenomenological predictions apply both away from and on the caustic:

(1) the direction of propagation coincides with the group velocity;

(2) any change in the local wave vector in a given direction must be the result of a spatial inhomogeneity acting in that direction. If the medium is cyclic in that direction, the wavevector is conserved in that direction;

(3) any change in the frequency must be the result of a temporal inhomogeneity in the medium. If the medium is autonomous, then the frequency is conserved.

Two vector field phenomenologies are also worth noting. The first-order approximation to the time-averaged Poynting vector (power density or, equivalently the magnitude of the field vector, E_0^2) $\overline{S}$ both on and off the caustic becomes

$$\overline{S} = \mathrm{Re}(\overline{E} \times \overline{H}^*) \sim \sqrt{\varepsilon_0/\mu_0}\ \mathrm{E}_0^2 \overline{k}\,, \tag{39}$$

where ε_0 and μ_0 are the permittivity and permeability, respectively, of vacuum, proceeds from the $n = 0$ term of the appropriate transport equation. For example, in the off-caustic region, from $i\tau$ term Equation 5, we have

$$2\left(\nabla\phi(\nabla\cdot\overline{E}_0) + f(\overline{r},t)\frac{\partial\phi}{\partial t}\frac{\partial\overline{E}_0}{\partial t}\right) + (\nabla^2\phi)\,\overline{E}_0 - f(\overline{r},t)\frac{\partial^2\phi}{\partial t^2}\,\overline{E}_0 = 0\,. \tag{40}$$

Then multiplying Equation 40 scalarly by $\overline{E}_0^*$ and the complex conjugate of Equation 40 by $\overline{E}_0$, adding and introducing ω from Equations 6 and the Hamiltonian flow from Equations 14, leads to

$$\frac{dE_0^2}{d\gamma} - 2\left(f(\overline{r},t)\frac{\partial^2\phi}{\partial t^2} - \nabla^2\phi\right)E_0^2 = 0\,, \tag{41}$$

and thus

$$E_0^2(\gamma) = E_0^2(0)\exp\left\{2\int\left(f(\overline{r},t)\frac{\partial^2\phi}{\partial t^2} - \nabla^2\phi\right)d\gamma\right\}. \tag{42}$$

Analogously on the caustic, we consider the τ\tau term in the integral in Equation 30

$$-(\nabla_k\cdot\overline{D})\overline{E}_0 - \overline{D}(\nabla_k\cdot\overline{E}_0) + 2\left(\nabla\phi\cdot\nabla\overline{E}_0 + f(\overline{r},t)\frac{\partial\phi}{\partial t}\frac{\partial\overline{E}_0}{\partial t}\right) - f(\overline{r},t)\frac{\partial^2\phi}{\partial t^2}\overline{E}_0 = 0\,. \tag{43}$$

Applying the same procedure but with the flow from Equations 31 leads to

$$\frac{dE_0^2}{d\gamma} - 2\left(f(\bar{r}, t)\frac{\partial^2\phi}{\partial t^2} + \nabla_{\bar{k}} \cdot \bar{D} \right) E_0^2 = 0\,, \tag{44}$$

and

$$E_0^2(\gamma) = E_0^2(0)\exp\left\{ 2\int \left(f(\bar{r}, t)\frac{\partial^2\phi}{\partial t^2} + \nabla_{\bar{k}} \cdot \bar{D} \right) d\gamma \right\} = 0\,. \tag{45}$$

A similar consideration leads to an interesting result for the zeroth-order approximation for the polarization

$$\bar{P} = \frac{\bar{E}_0}{(\bar{E}_0 \cdot \bar{E}_0^*)}\,. \tag{46}$$

In the off-caustic region, substituting for $\nabla^2\phi$ from Equation 40 into Equation 41 and re-grouping leads to the result that on the Hamiltonian flow

$$\frac{d\bar{P}}{d\gamma} = 0\,. \tag{47}$$

Analogously, substituting for $\nabla \cdot \bar{D}$ from Equation 43 into Equation 44 and introducing the non-Hamiltonian flow from Equation 31, also yields Equation 47 on the caustic. Consequently, this model predicts that away from the caustic on the Hamiltonian flow and that on the caustic on the non-Hamiltonian flow in Equation 31, the first-order approximation to the polarization is a constant. Equations 41, 44 and 47 correspond to similar results obtained for spatially inhomogeneous stationary media [18, 19].

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