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# A PREEMPTIVE PRIORITY QUEUE WITH ACCESSIBLE BATCH SERVICE AND HETEROGENEOUS ARRIVALS AND WITH VACATION

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Abstract: In this paper a single server preemptive priority queuing system consisting of two types of customer units with Poisson arrival, exponential service time distribution and exponential vacation is analyzed. Two type of customer units of which higher priority units of unlimited Poisson input are served by bulk service rule and limited Poisson input of low priority units served singly. The bulk service rule admits in batches for service only if the batch size is not less than 'a' and not more than 'b' units. Such that the arriving units can enter service without affecting the service time. If the size of the batch being served is less than 'a' then the server turns to do service for the lower priority units with single service. Whenever the number of customers in the high priority queue is less than the quorum and there is no customers in low priority queue then the server takes an exponential vacation. Vacation is analyzed in two models as Model I and Model II. In Model I, the steady state probability vectors of the number of customers in the queue are obtained by the modified geometric method. The stability condition and mean queue length of customers are obtained. Numerical results are also presented.

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**Keywords:** Preemptive priority, Matrix-geometric algorithmic approach, Bulk service rule, Accessible batch, Vacation, Steady state solution.

## **1. INTRODUCTION**

Queue discipline generally means that FIFO (First In First Out) process. There are several other disciplines like LIFO (Last In First Out), queue with priority, etc. Priority discipline broadly classified as preemptive and non preemptive. In preemptive priority, the service to the customer depends upon the order of priority. If the customer who has the highest priority enter into queue then the server will suspend the service to the acting customer who will be in lower priority and the service is given to the customer who has highest priority.

In non preemptive priority, even though the customer has highest priority, the server allows the service only after the completion of service to the present customer even if this customer has low priority. The concept of accessibility into batches while receiving service has been considered by Gross and Harris [2], Kleinrock [4] and others. Heathcoat [3] has analyzed a preemptive priority queue model with two priorities.

Another model in which two independent units, type I (higher priority) and type II (lower priority) of Poisson classes arrive in separate queues to a single server was analyzed by Sivasamy [8]. The server serves to type II units one by one but one at a time, In type I units the server serves in groups (or) batches as per the bulk service rule introduced by Neuts [6]. According to this rule customers are served in batches of particular size  $x(a \le x \le b)$ . As soon as type I units arrive the server breaks the service to type II unit (if the service is going on) and remains ideal and wait until the queue size reaches a threshold value 'a', no matter how many type II are present.

Ayyappan [1] has discussed another model which consist of the single server facility .In which two independent Poisson classes of units viz type I and type II arrive and form separate queues. In this process, even though type I has preemptive priority over type II the server will respond to type II until type I units form a particular queue size 'a'. As soon as the minimum required size 'a' is attained, then immediately the server suspend the service to type II (if the service is going on) and serve to type I by bulk service rule.

The queuing process analyzed in this paper consists of a single server and two independent classes of unit's type I and type II arrive and form separate queue. If the number of customer in type I is less than 'a' and there is no customer in type II, then the server takes an exponential vacation. As soon as the server returns from the vacation, the server monitors whether queue size of type I has reached a threshold value 'a', if not then the server checks for customers arrivals in type II, if there is no customers, then the server will again avails the vacation. After vacation until the type I reaches a minimum value 'a' (or) atleast a single customer in type II, then the server again starts to avail the vacation and the process is continued.

This queuing model may be fitted in some real life situation. For example this model could be followed in practical usage for a public show (like circus, planetorium, etc.), where the server serves for entry-ticket. Here the higher priority may be considered as special attention to students (with concession) who are coming in batches, if the size of the student is sufficient then the server would give concession and allow all the students for the show in batches. If not then the server will issue tickets for public (which are consider as low priority) in one by one but one a time. If there is no customer in public line then the server goes for vacation.

#### **2. DESCRIPTION OF THE MODEL**

This model has single server facility in which two independent Poisson classes of customers with high priority and low priority arrive with different rates  $\lambda_1$  and  $\lambda_2$  and form type I and type II unit queues respectively. The waiting room capacity of type I unit is infinite and customer in type I unit are served in batches according to bulk service rule introduced by Neuts (1967). As per this rule the server serves the customers in batches of size  $x (a \le x \le b)$ . The waiting room capacity of type II customer is finite with maximum capacity N and they are served singly. The service time of the two units are exponentially distributed random variables with mean rate of  $1/\mu_1$  and  $1/\mu_2$  respectively. If the number of customer in the type I is less than 'a' and no customer in type II queue then the server goes for an exponential vacation with parameter  $\alpha$ .

## Model I

This model relates to the queue with accessible service system with vacation. When the server complete service then looks number of customers in both type I and type II. If the size of the batch in type I is less than the batch size 'a' and there is no customers in type II, then the server avails for vacation. After completion of the vacation the server returns to the system and looks again batch size of type I, if it reaches 'a' and at least one customer waiting in type II queue then the server starts servicing. Otherwise if the batch size of type I is less than 'a' and there is no customers in type II then the server again takes an vacation. In this manner server avails vacation until the batch size of type I reaches 'a' and there is at least one customer waiting in the type II queue.

### **3. STEADY STATE PROBABILITY VECTOR**

The process can be formulated as a continues time Marko chain with state space

$$S = \{(i, j, k), 0 \le i \le a - 1, 0 \le j \le N, K = 0, 1\}$$
$$U\{(i, j, k), i \ge a, 0 \le j \le N, K = 0\}$$
$$U\{(i, j, m, k), i = 0, 0 \le j \le N, a \le m \le b - 1, K = 1\}$$
$$U\{(i, j, m, k), i \ge 1, 0 \le j \le N, m = b, K = 1\}$$

where *i* denote number of type I customers waiting in the queue, *j* denotes number of type II customers waiting in the queue, m denotes number of customers in accessible service batch, *K* denotes about the server state, if k = 0 then the server is in vacation otherwise if k = 1 then the server is in busy state. The corresponding generator *Q* of the Marko process is given by



Where  $D_0$ ,  $A_0$ ,  $A_1$ ,  $A_2$  are square matrix of order 2(N + 1).

The matrix  $D_0$  is given by

Where  $\lambda = \lambda_1 + \lambda_2$ . In all the matrices the unmarked entries are treated as zeros.  $A_0 = \lambda_1 I$ , *I* is unit matrix of order 2(N + 1). Then the matrix  $A_1$  is of order 2(N + 1) is given by

 $A_2$  is the matrix of order 2(N + 1). In which the second (N + 1) matrix has main diagonal elements as  $\mu_1$  and all other elements in the matrix are zero.

Let us denote by X the vector of steady state probabilities associated with Q such that

$$\underline{X} Q = \underline{0}, \text{ and } \underline{X} \underline{e} = 1,$$
 (1)

where  $\underline{e} = (1, 1, 1, ..., 1)^T$ . Let us partition  $\underline{X}$  as  $\underline{X} = (\underline{X}_0, \underline{X}_1, ..., \underline{X}_{a-1}, \underline{X}_a, ..., \text{ etc.})$ , where  $\underline{X}_i$  for  $i \ge 0$ .

$$\underline{X}_{i} = [X_{i00}, X_{i10}, \dots, X_{iN0}, X_{i01}, X_{i11}, \dots, X_{iN1}] \quad \text{where } 0 \le i \le a - 1,$$
  
$$\underline{X}_{i} = [X_{i00}, X_{i10}, \dots, X_{iN0}, X_{00i1}, X_{01i1}, \dots, X_{0Ni1}] \quad \text{where } a \le i \le b$$
  
$$\underline{X}_{i} = [X_{i00}, X_{i10}, \dots, X_{iN0}, X_{i001}, X_{i1b1}, \dots, X_{iNb1}] \quad \text{where } i \ge 1.$$

Following Neuts (1978) we examine the existence of a solution of the form,

$$\underline{X}_{i} = \underline{X}_{a-1} R^{i-a+1}; \quad i \ge a$$

$$\tag{2}$$

where the matrix R is unique non-negative solution, with spectral radius less than one of the matrix equation

$$A_0 + RA_1 + R^{(b+1)}A_2 = 0 (3)$$

the matrix R is given by  $\lim_{j\to\infty} R_j$ , where the sequence of the matrices  $\{R_j\}$  is defined as:

$$R_0 = 0,$$
  

$$R_{j+1} = -A_0 A_1^{-1} - R_j^{b-1} A_2 A_1^{-1}, \qquad j = 0, 1, 2, \dots$$
(4)

For stable systems, the sequence  $\{R_j\}$  is monotonically increasing and converges to R. Hence R can be evaluated by successive substitutions using equation (4), until a desired level of convergence is achieved.

$$R_{1,1} = \frac{\lambda_{1}}{\lambda + \alpha}$$

$$R_{2,2} = \frac{\lambda_{1}}{\lambda + \alpha}$$

$$\vdots$$

$$R_{N,N} = \frac{\lambda_{1}}{\lambda_{1} + \alpha}$$

$$R_{(N+1),(N+1)} = \frac{\lambda_{1}}{\lambda + \mu_{1}} + \frac{\mu_{1}}{\lambda + \mu_{1}} R_{N,N}^{b-1}$$

$$\vdots$$

$$R_{2N,2N} = \frac{\lambda_{1}}{\lambda_{1} + \mu_{1}} + \frac{\mu_{1}}{\lambda_{1} + \mu_{1}} R_{2N-1,2N-1}^{b-1}$$
(5)

All the elements of *R* below the leading diagonal are zeros. It is clear from (5) that the matrix *R* is reducible and its Eigen values of the matrix *R* are its diagonal elements. Since  $X_k \rightarrow 0$  as  $k \rightarrow \infty$  the spectral radius of the matrix *R* must be less than one.

Equation (5) can be written as

$$R_{ij} = f_i(R_{ij}), \qquad N < j < 2N \tag{6}$$

For each *j*,  $R_{jj}$  is the smallest non– negative solution of the corresponding equation. Now, for  $N \le j \le 2N - 1$ ,  $f_j(0) > 0$  and  $f_j(1) < 1$ , which implies that the convex non-decreasing graph of  $y = f_j(z)$ , has a unique point of intersection with the line y = z in (0, 1) Since  $f_{2N}(0) > 0$  and  $f_{2N}(1) = 1$ , the equation  $z = f_{2N}(z)$  has a unique solution  $R_{jj}$  in (0, 1), if and only if  $f'_{2n+1}(1-\varepsilon) > 1$  In the case  $R_{(2N+1), (2N+1)} < 1$  if and only if

$$\lambda_1 < b\mu_1 \tag{7}$$

This is the equilibrium condition. The chain Q is positive recurrent if and only if this condition is satisfied. We are now left with finding  $(\underline{X}_0, \underline{X}_1, ..., \underline{X}_{a-1})$ 

$$Q^{*} = \begin{bmatrix} 0 & 1 & 2 & \cdots & a-1 \\ D_{0} & A_{0} & & & \\ 0 & D_{0} & A_{0} & & & \\ 0 & D_{0} & A_{0} & & & \\ 0 & D_{0} & A_{0} & & & \\ 0 & 0 & A_{0} & & &$$

**Lemma**:  $\mathbf{Q}^*$  is an infinitesimal generator

**Proof:** To prove  $\mathbf{Q}^* \underline{e} = 0$ , it is enough to consider the last row of  $\mathbf{Q}^*$  since the other rows are identical to the rows of  $\mathbf{Q}$ .

(Last row of 
$$\mathbf{Q}^*$$
)  $e = (I - R)^{-1} (I - R^{b+1}) A_2 \underline{e} - A_0 \underline{e} - A_2 \underline{e}$   

$$= (I - R)^{-1} (A_2 \underline{e} - R^{b+1} A_2 \underline{e} - (I - R) A_0 \underline{e} - (I - R) A_2 \underline{e})$$

$$= (I - R)^{-1} (-R^{b+1} A_2 \underline{e} + R (A_0 + A_2) \underline{e} - A_0 \underline{e})$$

$$= (I - R)^{-1} (-R^{b+1} A_2 \underline{e} - RA_1 \underline{e} - A_0 \underline{e})$$

$$= 0, \text{ by equation (3).}$$

The matrix  $\mathbf{Q}^*$  is irreducible .Now from  $\underline{\mathbf{X}}^* \mathbf{Q}^* = \underline{\mathbf{0}}$ , Where  $X^* = (\underline{X}_0, \underline{X}_1, \underline{X}_2, ..., \underline{X}_{a-1})$ , the vectors  $\underline{X}_{i_0} \mathbf{0} \le i \le a-2$ ,

Then we get,

$$\begin{split} \underline{X}_0 D_0 + \underline{X}_{a-1} R \left( I - R \right)^{-1} & (I - R^{b-a+1}) A_2 = 0 \\ \underline{X}_0 &= - \underline{X}_{a-1} \left( R \left( I - R \right)^{-1} & (I - R^{b-a+1}) A_2 \right) D^{-1} \\ \underline{X}_0 A_0 + \underline{X}_1 D_0 + \underline{X}_{a-1} R^{b-a+1} A_2 \cdot R^{b-a+1} A_2 = 0 \\ \underline{X}_1 &= \underline{X}_{a-1} \left[ R \left( I - R \right)^{-1} & (I - R^{b-a+1}) A_2 D_0^{-1} A_0 - R^{b-a+2} A_2 \right] D_0^{-1} \end{split}$$

In this way we can calculate all the  $X_i$ 's, for  $0 \le a - 2$  in terms of  $X_{a-1}$ . Finally we get,

$$\underline{X}_{a-1}[D_0 + EA_0 + R^b A_2] = \underline{0}$$

where the matrix *E* is readily computed,  $\underline{X}_{a-1}$  is the left Eigen vector of matrix  $D_1 + EA_2 + R^bA_0$  of order 2(N + 1) corresponding to Eigen value zero. It is normalized so that

$$\sum_{i=0}^{a-2} \underline{X}_i \underline{e} + \underline{X}_{a-1} (I-R)^{-1} \underline{e} = 1$$

Thus the vector  $(\underline{X}_0, \underline{X}_1, \underline{X}_2, ..., \underline{X}_{a-1})$  are uniquely determined.

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**Theorem:** Provided  $\lambda_1 < b\mu_1$  the queue is stable. The steady state probability vector of the system is given by  $\underline{X} = (\underline{X}_0, \underline{X}_1, ..., \underline{X}_{a-1}, \underline{X}_a, ..., \text{etc.})$ , Where  $\underline{X}_i, i \ge 0$  are  $1 \times 2(N+1)$  vectors . The vectors  $\underline{X}_i, i \ge a - 1$  such that  $\underline{X}_i = \underline{X}_{a-1}R^{i-(a-1)}$ , Where *R* is of order 2(N+1) and is the unique solution in the set of non negative matrices of spectral radius less than one of the equation  $A_0 + RA_1 + R^{(b+1)}A_2 = 0$ . The matrix *R* is reducible. The vector  $\underline{X}_{a-1}$  is the left Eigen vector of the matrix  $EA_2 + D_1 + R^bA_0$  of order 2(N+1) corresponding to the Eigen value zero.  $\underline{X}_{a-1}$  may be normalized by

$$\sum_{i=0}^{a-2} \underline{X}_i \underline{e} + \underline{X}_{a-1} (I-R)^{-1} \underline{e} = 1.$$

#### 4. NUMERICAL RESULTS

By theorem 1 of Latouche and Neuts (1980) *R* is the limit of the sequence  $\{R(n)\}, n \ge 0$  of matrices defined by the relation in equation (4).

For numerical illustration, let us take  $\lambda_1 = 1$ ,  $\lambda_2 = 3$ ,  $\mu_1 = 1$ ,  $\mu_2 = 2$ ,  $\alpha = 2$ , a = 3, b = 5, N = 3. Then  $A_2 = 1I$ , where *I* is unit matrix of order (8 × 8). The matrix *R* is given by

	0.1667	0.0833	0.0417	0.0417	0.0667	0.0734	0.0609	0.1483
<b>R</b> =	0	0.1667	0.833	0.833	0	0.0667	0.0734	0.2091
	0	0	0.1667	0.1667	0	0	0.0667	0.2825
	0	0	0	0.3333	0	0	0	0.3492
	0	0	0	0	2000	0.1201	0.0721	0.1165
	0	0	0	0	0	0.2000	0.1201	0.1886
	0	0	0	0	0	0	0.2000	0.3086
	0	0	0	0	0	0	0	0.5087

The vectors,  $\underline{X}_i$ ,  $i \ge 0$  are row vectors of order 8. Now  $\underline{X}^* = (\underline{X}_0, \underline{X}_1, \underline{X}_2)$ .

Using  $\underline{X}^* Q^* = 0$ , the system governing equation's are obtained whose solution is given by

- $\underline{X}_0 = [0.000289416, 0.00153168, 0.0102857, 0.0721648, 0.000578831, 0.0041609, 0.00285597, 0.0928185]$
- $\underline{X}_1 = [0.000570615, 0.0026984, 0.0158109, 0.0906672, 0.000996523, 0.000646663, 0.0382423, 0.0762021]$
- $\underline{X}_2 = \begin{bmatrix} 0.000808879, 0.00349873, 0.0185659, 0.0965926, 0.00133245, \\ 0.00793367, 0.0425442, 0.0717064 \end{bmatrix}$

Since the matrix *R* and the vector  $\underline{x}_2$  are known, the remaining vectors  $\underline{x}_i$ ,  $i \ge 3$  can be evaluated using the relation  $\underline{X}_i = \underline{X}_2 R^{i-2}$ :  $i \ge 3$  obtained from (8)

- $\underline{X}_3 = [0.000002246, 0.000119656, 0.000629758, 0.0125021, 0.000007308, 0.000499671, 0.00277251, 0.0637405]$
- $\underline{X}_4 = \begin{bmatrix} 0.000003745, 0.000002181, 0.000115867, 0.00428324, 0.000001611, \\ 0.000118343, 0.000671942, 0.037953 \end{bmatrix}$
- $\underline{X}_5 = [0.00000063, 0.00000395, 0.000002128, 0.00144903, 0.00000347, 0.000002733, 0.000159323, 0.0210704]$
- $\underline{X}_6 = \begin{bmatrix} 0.000000104, 0.00000071, 0.000000390, 0.000486914, 0.000000073, \\ 0.000000619, 0.000003714, 0.0112854 \end{bmatrix}$
- $\underline{X}_7 = [0.000000001, 0.000000012, 0.000000071, 0.000163018, 0.000000015, 0.0000000138, 0.000000854, 0.00592445]$
- $\underline{X}_8 = [0.000000000, 0.000000002, 0.000000013, 0.000005446, 0.000000003, 0.0000000194, 0.00307361]$

It may be noted that  $\underline{X}_k \to 0$  as  $k \to \infty$  for the numerical parameters chosen above,  $\underline{X}_{46} \to 0$  and the sum of the steady state probabilities becomes 0.999999998.

#### **5. MEAN QUEUE LENGTH**

The expected number of type I customer is

$$\sum_{i=0}^{a-1} i \left\{ \sum_{k=0,1} X_{ijk} \right\} + \sum_{i=a}^{\infty} i \left\{ \sum_{k=0,1} X_{ijk} \right\}, \quad j = 0, 1, ..., N$$

The expected number of type I I customer is

$$\sum_{j=0}^{N} j \left\{ \sum_{i=0}^{\infty} X_{ijk} + \sum_{n=a}^{b} X_{0jn1} + \sum_{m=1}^{\infty} X_{mjb1} \right\}, \quad k = 0, 1.$$

#### Model II

Model II is same as model I with slight variation. Here we consider the vacation as single vacation. When the server complete service then looks number of customers in both type I and type II. If the size of the batch in type I is less than the batch size 'a' and there is no customers in type II, then the server avails for vacation. After completion of the vacation the server returns to the system and again looks the batch size of type I, if it reaches 'a' and if at least any one customer waiting in the type II queue then the server start servicing. Otherwise the server will remain ideal until the batch size becomes 'a' and there is at least one customer waiting in type II queue.

Steady State Probability Vector

The process can be formulated as a continues time Marko chain with state space

$$S = \{(i, j, 0), 0 \le i \le a - 1, j = 0, 0\} U\{(i, j, I), 0 \le i \le a - 1, j = 0, I\}$$
$$U\{(i, j, B), 0 \le i \le a - 1, 0 \le j \le N, B\} U\{(i, j, m, B), i = 0, 0 \le j \le N, a \le m \le b, B\}$$
$$U\{(i, j, m, B), i \ge 0, 0 \le j \le N, m = b, B\}$$

where *i* denote number of type I customers waiting in the queue, j denotes number of type II customers waiting in the queue, '0' denotes the server is in vacation. B denotes the server is in busy state. I denote server is in ideal state. m denotes the customers in accessible batch.

The first process denote that 'The server is in vacation' and the second one is 'The server stay ideal', third is 'The server busy with type II customers', fourth one is 'The server busy with type I customers and this is same as last one.

The technique used for the analysis of Model I is successfully applied for the above described Model II and the results are verified with suitable numerical example. The details are not presented here as they are similar to that of Model I.

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