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WEIGHTED DIFFERENTIATION COMPOSITION OPERATORS FROM BERGMAN SPACES WITH B´EKOLL´E WEIGHTS TO WEIGHTED TYPE SPACES

ZAHEER ABBAS, SONALI MAGOTRA AND S.D.SHARMA

Abstract : In this paper, we characterize boundedness and compactness of weighted differentiation composition operators acting from Bergman spaces with B´ekoll´e weights to weighted type spaces.

INTRODUCTION AND PRELIMINARIES

Let $\mathbb{D}$ denote the open unit disk in the complex plane $\mathbb{C}$ and $\mathcal{H}(\mathbb{D})$ denotes the space of all complex valued holomorphic functions on $\mathbb{D}$. We shall call the function $\Psi: \mathbb{D} \longrightarrow (0,\infty)$ defined as $\Psi(z) = \psi(|z|)$, where $\psi : [0,1) \longrightarrow (0,\infty)$ is a continuous function, a weight or a weight function. For $0 < p < \infty$ and Ψ a weight on $\mathbb{D}$, we denote by $\mathcal{A}^p(\Psi)$ the weighted Bergman space consisting of all holomorphic functions f on $\mathbb{D}$ satisfying:

$$\| f \|_{\mathcal{A}^p(\Psi)}^p = \int_{\mathbb{D}} \left| f(z) \right|^p \Psi(z) dA(z) < \infty,$$

where $dA(z) = \frac{1}{\pi} dxdy = \frac{1}{\pi} rdrd\theta$ is the normalized area measure on $\mathbb{D}$. The weights considered in this paper are so called Be'kolle' weights. For $\alpha > -1$, let

$$dA_\alpha(z) = (\alpha+1)\left(1-|z|^2\right)^\alpha dA(z).$$

For $p_0 > 1$ and $\alpha > -1$, the class $B_{p_0}(\alpha)$ consists of weights Ψ with the property that there exists a constant C > 0 such that

$$\left(\int_{s(\theta,h)} \Psi dA_\alpha \right) \left(\int_{s(\theta,h)} \Psi^{\frac{-p_0'}{p_0}} dA_\alpha \right)^{\frac{-p_0}{p_0'}} \leq c \left[A_\alpha\left(s(\theta,h)\right) \right]^{p_0}, \qquad \textbf{(1)}$$

for any Carleson square

$$s(\theta,h) = \left\{ z = re^{\iota\phi} : 1-h < r < 1, |\theta-\phi| < \frac{h}{2} \right\}, \theta \in [0,2\pi], h \in (0,1),$$

where $\frac{1}{p_0} + \frac{1}{p_0'} = 1$. It can be seen that all normal weights are Be'kolle' weights [12]. Recall that a weight ν is normal if there exist positive numbers

η *and* $\tau, 0 < \eta < \tau$, and $\delta \in [0,1)$ such that

$$\frac{\nu(z)}{(1-r)^{\eta}} \text{ is decreasing on } [\delta,1) \text{ and } \lim_{r\to 1}\frac{\nu(z)}{(1-r)^{\eta}}=0,$$

$$\frac{\nu(z)}{(1-r)^{\tau}} \text{ is decreasing on } [\delta,1) \text{ and } \lim_{r\to 1}\frac{\nu(z)}{(1-r)^{\tau}}=\infty.$$

It is well known that classical weights Ψ_α (z) = $(1-|z|^2)^\alpha$, $\alpha > -1$ are normal weights. A weight function Ψ is in the space C_{p_0} , $p_0 > 1$, if there exists a positive constant C such that

$$\left(\int_{D_\lambda(r)} \Psi dA(z)\right)\left(\int_{D_\lambda(r)} \Psi^{\frac{-p_0'}{-p_0}} dA(z)\right)^{\frac{-p_0}{p_0'}} \leq C\left[A\left(D_\lambda(r)\right)\right]^{p_0},$$

(2)

for every disk $D_\lambda(r) = \{z \in \mathbb{D} : |z-\lambda| < r(1-|\lambda|)\}$. Here, $r \in (0,1)$ is fixed, but the class C_{p_0} is actually independent of r $\in (0,1)$. Moreover $B_{p_0(\alpha)} \subset C_{p_0}$ for every $\alpha > -1$ and the inclusion is strict. For more about the classes $B_{p_0(\alpha)}$ and C_{p_0} and properties satisfied by the weights in these classes, we refer [2], [3] and [4] and the references therein. For a normal weight ν, the weighted type space $\mathcal{A}_\nu$ on$\mathbb{D}$ is the space of all holomorphic functions f on $\mathbb{D}$ such that

$$\sup_{z\in\mathbb{D}} \nu(z)|f(z)| < \infty.$$

The little weighted type space $\mathcal{A}_{\nu,0}$ consists of all $f \in \mathcal{A}_\nu$ such that

$$\lim_{z\to 1} \nu(z)|f(z)| = 0.$$

Both the space are Banach spaces with the norm

$$\|f\|_{\mathcal{A}_\nu} = \sup_{z\in\mathbb{D}} \nu(z)|f(z)|.$$

Also, $\mathcal{A}_{\nu,0}$ is a closed subspace of $\mathcal{A}_\nu$. Let $\psi \in H(\mathbb{D})$, φ a holomorphic self-map of $\mathbb{D}$ and n a non-negative integer. Then the weighted differentiation composition operator $D^n_{\psi,\varphi}$ is a linear operator on $H(\mathbb{D})$ defined by $D^n_{\psi,\varphi} f = \psi \,.\, f^{(n)}\, o\, \varphi$ for $f \in H(\mathbb{D})$. It is of interest to provide function theoretic characterizations involving ψ *and* φ of boundedness and compactness of $D^n_{\psi,\varphi}$ acting between different function spaces. Recently, several authors have studied these type of operators, composition operators, product of composition operators with differentiation, etc, on different spaces of analytic functions, see for example, [1], [6]-[38] and the related references therein. In this paper, we shall study boundedness and compactness of $D^n_{\psi,\varphi}$ between $\mathcal{A}^p(\Psi)$ and $\mathcal{A}_\nu(\mathcal{A}_{\nu,0})$.

Throughout this paper, we denote by $\mathbb{N}_0$ the set of non-negative integers, constants are denoted by C, they are positive and not necessarily the same at each occurrence. The notations $A \lesssim B$ means that A is less than equal to constant times B and $D \gtrsim E$, means that D is greater than equal to constant times E. When $A \lesssim B$ as well as $A \gtrsim B$, then we write A ≍ B.

SOME AUXILARY RESULTS

In this section we collect some auxiliary results which we use to prove the main results of this paper. The first among these estimates the growth of the derivatives of the functions in $\mathcal{A}^p(\Psi)$. When the weight Ψ is in C_{p_0}, then we have the following growth property for derivatives of functions in $\mathcal{A}^p(\Psi)$ (see [3]).

Lemma 1. Let $p > 0$, $p_0 > 1$, Ψ a weight in C_{p_o} and $n \in \mathbb{N}_0$. Then for each $f \in \mathcal{A}^p(\Psi)$, there is a constant $C > 0$ such that

$$\left|f^{(n)}(z)\right| \leq C \frac{\left(\int_{B(z,r)} \Psi dA\right)^{-\frac{1}{p}}}{\left(1-|z|^2\right)^n} \|f\|_{\mathcal{A}^p(\Psi)}.$$

The following criterion for compactness follows by standard arguments similar to those outlined in the proposition 3.11 in [7].

Lemma 2. Let Ψ be a weight such that $\frac{\Psi}{\left(1-|z|^2\right)^\alpha}$ belongs to $B_{p_0}(\alpha)$, ν a normal weight and $D^n_{\psi,\varphi}$: $\mathcal{A}^p(\Psi) \longrightarrow \mathcal{A}_\nu$ is bounded. Then $D^n_{\psi,\varphi}$: $\mathcal{A}^p(\Psi) \longrightarrow \mathcal{A}_\nu$ is compact if for any bounded sequence $\{f_n\}_{n \, \varepsilon \, \mathbb{N}}$ in $\mathcal{A}^p(\Psi)$ which converges to zero uniformly on compact subset of $\mathbb{D}$, we have $\| D^n_{\psi,\varphi} f_n \|_{\mathcal{A}_\nu} \longrightarrow 0$ as $n \longrightarrow \infty$. The next two lemmas provide test functions in $\mathcal{A}^p(\Psi)$, which are in conjunction with Lemma 2, see [34].

Lemma 3. Let $r \in (0,1)$ be fixed, p > 0, $p_0 > 1$ and $\eta > -1$. Assume that $p_0 \geq p$, Ψ is a weight function such that $\frac{\Psi}{\left(1-|z|^2\right)^\alpha}$ belongs to $B_{p_0}(\alpha)$ and $\gamma \geq \frac{(n+2)p_0}{p-2}$. Let $K^\gamma_\lambda = \frac{1}{\left(1-\bar{\lambda} z\right)^{\gamma+2}}$ be the reproducing kernel of the Bergman space $\mathcal{A}^p(\Psi_\alpha)$. Then

$$\| K_\lambda^\gamma \|_{\mathcal{A}^p(\Psi)} \asymp \frac{\left(\int_{B(z,r)} \Psi dA\right)^{-\frac{1}{p}}}{\left(1-|\lambda|^2\right)^{\gamma+2}}.$$

Lemma 4. Let $r \in (0, 1)$ be fixed, $p > 0$, $p_0 > 1$ and $\eta > -1$. Assume that $p_0 \geq p$, Ψ is a weight function such that $\Psi/(1-|z|^2)^\alpha$ belongs to to $B_{p_0}(\alpha)$. Then

$$f_\lambda(z) = \frac{\left(1-|\lambda|^2\right)^{\frac{(n+2)p_0}{p+1}}}{\left(\int_{B(\lambda,r)} \Psi dA\right)^{\frac{1}{p}} \left(1-\bar{\lambda} z\right)^{\frac{(n+2)p_0}{p+1}}}. \qquad \textbf{(3)}$$

is in $\mathcal{A}^p(\Psi)$. Moreover, $\| K_\lambda^\gamma \|_{\mathcal{A}^p(\Psi)} \asymp 1$ and f_λ converges to zero, uniformly on compact subset of $\mathbb{D}$.

The following criterion for closed subset $L \subset \mathcal{A}_{\nu,0}$ follows by standard arguments which appear for the first time in [11]. We omit the details.

Lemma 5. A closed set L in $\mathcal{A}_{\nu,0}$ is compact if and only if it is bounded with respect to the norm $\|.\|_{\mathcal{A}_\nu}$ and satisfies

$$\lim_{|z|\to 1} \sup_{f\in L} \nu(z)|f(z)| = 0.$$

BOUNDEDNESS AND COMPACTNESS OF $D_{\psi,\varphi}^n : \mathcal{A}^p(\Psi) \longrightarrow \mathcal{A}_\nu(\mathcal{A}_{\nu,0})$

In this section, we characterize boundedness and compactness of weighted composition operators acting from weighted Bergman spaces $\mathcal{A}^p(\Psi)$ to Bloch type spaces $\mathcal{A}_\nu$ and $\mathcal{A}_{\nu,0}$.

Theorem 1. Let $r \in (0, 1)$ be fixed, $p > 0$, $p_0 > 1$ and $\alpha > -1$, ν a normal weight, $\psi \in H(\mathbb{D})$, φ a holomorphic self-map of $\mathbb{D}$ and $n \in \mathbb{N}_0$. Assume that that $p_0 \geq p$, and Ψ is a weight function such that $\frac{\Psi}{\left(1-|z|^2\right)^\alpha}$ belongs to to $B_{p_0}(\alpha)$. Then $D_{\psi,\varphi}^n : \mathcal{A}^p(\Psi) \longrightarrow \mathcal{A}_\nu$ is bounded if and only if

$$M = \sup_{z\in\mathbb{D}} \frac{\nu(z)|\psi(z)|}{\left(1-\varphi(z)^2\right)^n} \left(\int_{B(\varphi(z),r)} \Psi dA\right)^{\frac{-1}{p}} < \infty. \qquad \textbf{(4)}$$

Moreover, if $D^n_{\psi,\varphi}:\mathcal{A}^p(\Psi)\longrightarrow\mathcal{A}_\nu$ is bounded, then

$$\| D^n_{\psi,\varphi}\|_{\mathcal{A}^p(\Psi)\to\mathcal{A}_\nu}\asymp \text{M}. \quad \textbf{(5)}$$

Proof. First suppose that condition (4) hold. Then by Lemma 1, we have

$$\nu(z)\left|\left(D^n_{\psi,\varphi}\text{f}\right)(z)\right|=\nu(z)\left|\psi(z)\right|\left|f^{(n)}\left(\varphi(z)\right)\right|$$

$$\lesssim \frac{\nu(z)\left|\psi(z)\right|}{\left(1-\left|\varphi(z)\right|^2\right)^n}\left(\int_{B(\varphi(z),r)}\Psi dA\right)^{\frac{-1}{p}}\| f\|_{\mathcal{A}^p(\Psi)}. \quad \textbf{(6)}$$

Using (6), we see that

$$\| D^n_{\psi,\varphi}f\|_{\mathcal{A}_\nu}=\sup_{z\in\mathbb{D}}\nu(z)\left|\psi(z)\right|\left|f^{(n)}\left(\varphi(z)\right)\right|\le M\| f\|_{\mathcal{A}^p(\Psi)}.$$

Thus $D^n_{\psi,\varphi}:\mathcal{A}^p(\Psi)\longrightarrow\mathcal{A}_\nu$ is bounded and

$$\left\| D^n_{\psi,\varphi}\right\|_{\mathcal{A}^p(\Psi)\to\mathcal{A}_\nu}\lesssim \text{M}. \quad \textbf{(7)}$$

Conversely, suppose that $D^n_{\psi,\varphi}:\mathcal{A}^p(\Psi)\longrightarrow\mathcal{A}_\nu$ is bounded. Then by taking $f_0(z)=\dfrac{z^n}{n!}$ in $\mathcal{A}^p(\Psi)$, we have that

$$\sup_{z\in\mathbb{D}}\nu(z)\left|\psi(z)\right|=\| D^n_{\psi,\varphi}\text{f}_0\|_{\mathcal{A}_\nu}\lesssim\| D^n_{\psi,\varphi}\|_{\mathcal{A}^p(\Psi)\to\mathcal{A}_\nu}. \quad \textbf{(8)}$$

Let $\lambda=\varphi(\varsigma)$ and consider the function f_λ defined in (3). Then

$$f^{(n)}_\lambda(z)=\left(\bar{\lambda}\right)^n\prod_{j=1}^{n}\left((\eta+2)\frac{p_0}{p}+j\right)\frac{\left(1-|\lambda|^2\right)^{\frac{(n+2)p_0}{p+1}}}{\left(\int_{B(\lambda,r)}\Psi dA\right)^{\frac{1}{p}}\left(1-\bar{\lambda}z\right)^{\frac{n+1+(n+2)p_0}{p}}}$$

and
$$f^{(n)}_\lambda(z)=\left(\bar{\lambda}\right)^n\prod_{j=1}^{n}\left((\eta+2)\frac{p_0}{p}+j\right)\frac{\left(\int_{B(\lambda,r)}\Psi dA\right)^{\frac{-1}{p}}}{\left(1-|\lambda|^2\right)^n}. \quad \textbf{(9)}$$

Thus,

$$\| D^n_{\psi,\varphi}\|_{\mathcal{A}^p(\Psi)\to\mathcal{A}_\nu}\gtrsim\| D^n_{\psi,\varphi}\text{f}_\lambda\|_{\mathcal{A}_\nu}\ge\nu(\varsigma)\left|\psi(\varsigma)f^{(n)}_\lambda\left(\varphi(\varsigma)\right)\right|$$

$$\geq \nu(\varsigma)|\varphi(\varsigma)|^n \prod_{j=1}^{n}\left((\eta+2)\frac{p_0}{p}+j\right)|\psi(\varsigma)| \frac{\left(\int_{B(\varphi(\varsigma),r)} \Psi dA\right)^{\frac{-1}{p}}}{\left(1-|\varphi(\varsigma)|^2\right)^n}, \quad \textbf{(10)}$$

that is, we have that

$$M = \sup_{\varsigma\in\mathbb{D}} \nu(\varsigma)|\varphi(\varsigma)|^n |\psi(\varsigma)| \frac{\left(\int_{B(\varphi(\varsigma),r)} \Psi dA\right)^{\frac{-1}{p}}}{\left(1-|\varphi(\varsigma)|^2\right)^n} \leq \| D^n_{\psi,\varphi} \|_{\mathcal{A}^p(\Psi)\to\mathcal{A}_\nu}. \quad \textbf{(11)}$$

Thus for fixed $\delta \in (0,1)$, we have

$$\sup_{|\varphi(\varsigma)|>\delta} \nu(\varsigma)|\varphi(\varsigma)|^n |\psi(\varsigma)| \frac{\left(\int_{B(\varphi(\varsigma),r)} \Psi dA\right)^{\frac{-1}{p}}}{\left(1-|\varphi(\varsigma)|^2\right)^n} \leq \| D^n_{\psi,\varphi} \|_{\mathcal{A}^p(\Psi)\to\mathcal{A}_\nu}. \quad \textbf{(12)}$$

Now,

$$1 = \left| \mathrm{K}_\lambda^{\frac{-2+(\eta+2)p_0}{p}}(0) \right|$$

$$\leq C\left(\int_{B\left(0,\frac{1}{2}\right)} \Psi dA\right)^{\frac{-1}{p}} \| \mathrm{K}_\lambda^{\frac{-2+(\eta+2)p_0}{p}} \|_{\mathcal{A}^p(\Psi)}$$

$$\asymp \frac{\left(\int_{B(\lambda,r)} \Psi dA\right)^{\frac{1}{p}}}{\left(1-|\lambda|^2\right)^{\frac{(\eta+2)p_0}{p}}}.$$

That is,

$$\frac{\left(1-|\lambda|^2\right)^{\frac{(\eta+2)p_0}{p}}}{\left(\int_{B(\lambda,r)} \Psi dA\right)^{\frac{1}{p}}}. \quad \textbf{(13)}$$

Also by using (10) and (13), we have that

$$\sup_{|\varphi(\varsigma)|\le\delta} \nu(\varsigma)|\psi(\varsigma)||\varphi(\varsigma)|^n \frac{\left(\int_{B(\varphi(\varsigma),r)}\Psi dA\right)^{\frac{-1}{p}}}{\left(1-|\varphi(\varsigma)|^2\right)^n}$$

$$= \sup_{|\varphi(\varsigma)|\le\delta} \frac{\nu(\varsigma)|\psi(\varsigma)||\varphi(\varsigma)|^n}{\left(1-|\varphi(\varsigma)|^2\right)^{\frac{(\eta+2)p_0}{p+1}}} \frac{\left(1-|\varphi(\varsigma)|^2\right)^{\frac{(\eta+2)p_0}{p}}}{\left(\int_{B(\varphi(\varsigma),r)}\Psi dA\right)^{\frac{1}{p}}}$$

$$\lesssim \frac{1}{\left(1-|\delta|^2\right)^{\frac{(\eta+2)p_0}{p+n}}} \| D^n_{\psi,\varphi} \|_{\mathcal{A}^p(\Psi)\to\mathcal{A}_{\nu.}} . \qquad \textbf{(14)}$$

Hence from (12) and (14) we have

$$\sup_{\varsigma\in\mathbb{D}} |\psi(\varsigma)|\nu(\varsigma) \frac{\left(\int_{B(\varphi(\varsigma),r)}\Psi dA\right)^{\frac{-1}{p}}}{\left(1-|\varphi(\varsigma)|^2\right)^n} \lesssim \| D^n_{\psi,\varphi} \|_{\mathcal{A}^p(\Psi)\to\mathcal{A}_{\nu.}} . \qquad \textbf{(15)}$$

Combing (7) and (15), (5) holds.

Theorem 2. Let $r \in (0,1)$ be fixed, $p > 0$, $p_0 > 1$, $\alpha > -1, \nu$ a normal weight, $\psi \in H(\mathbb{D}), \varphi$ a holomorphic self-map of $\mathbb{D}$ and n $\in \mathbb{N}_0$. Assume that $p_0 \ge p, \Psi$ is a weight function such that $\frac{\Psi}{\left(1-|z|^2\right)^\alpha}$ belongs to $\mathrm{B}_{p_0}(\alpha)$ and $D^n_{\psi,\varphi} : \mathcal{A}^p(\Psi) \longrightarrow \mathcal{A}_\nu$ is bounded. Then $D^n_{\psi,\varphi} : \mathcal{A}^p(\Psi) \longrightarrow \mathcal{A}_\nu$ is compact if and only if

$$\lim_{|\varphi(z)|\to 1} \frac{\nu(z)|\psi(z)|}{\left(1-|\varphi(z)|^2\right)^n} \left(\int_{B(\varphi(\varsigma),r)}\Psi dA\right)^{\frac{-1}{p}} = 0 . \qquad \textbf{(16)}$$

Proof. First suppose that condition (16) hold. Then by Lemma 2, it is sufficient to show that if $\{f_k\}_{k\in\mathbb{N}}$ is a bounded sequence in $\mathcal{A}^p(\Psi)$ that converges to zero uniformly on compact subsets of $\mathbb{D}$, then $\| D^n_{\psi,\varphi}\mathrm{f} \|_{\mathcal{A}_\nu} \longrightarrow 0 \; as \; k \longrightarrow \infty$.

Let $\{f_k\}_{k\in\mathbb{N}}$ be a sequence in $\mathcal{A}^p(\Psi)$ such that $\sup_{n\in\mathbb{N}} \| f_k \|_{\mathcal{A}^p(\Psi)} \leq L$ and $\{f_k\}$ converges to zero uniformly on compact subsets of $\mathbb{D}$ as k $\longrightarrow \infty$. By the condition (16), we have that for any $\varepsilon > 0$, there is a $\delta > 0, 0 < \delta < 1$ such that

$$\frac{\nu(z)|\psi(z)|}{\left(1-|\varphi(z)|^2\right)^n}\left(\int_{B(\varphi(\varsigma),r)} \Psi dA\right)^{\frac{-1}{p}} < \in, \tag{17}$$

whenever $\delta < |\varphi(z)| < 1$. Let $K = \{z \in \mathbb{D} : |z| \leq \delta\}$ be a compact subset of $\mathbb{D}$. Then $\| D^n_{\psi,\varphi} f \|_{\mathcal{A}_\nu} = \sup_{\varsigma\in\mathbb{D}} \nu(\varsigma)|\psi(\varsigma)|\left|f_k^{(n)}(\varphi(\varsigma))\right|$

$$\leq \sup_{\{\varsigma\in\mathbb{D}:\,\varphi(\varsigma)\in K\}} \nu(\varsigma)|\psi(\varsigma)|\left|f_k^{(n)}(\varphi(\varsigma))\right| + \sup_{\{\varsigma\in\mathbb{D}:\delta<|\varphi(\varsigma)|<1\}} \nu(\varsigma)|\psi(\varsigma)|\left|f_k^{(n)}(\varphi(\varsigma))\right|$$

$$\leq \| \psi \|_{\mathcal{A}_\nu} \sup_{z\in K}\left|f_k^{(n)}(z)\right| + \sup_{\{\varsigma\in\mathbb{D}:\delta<|\varphi(\varsigma)|<1\}} \frac{\nu(\varsigma)|\psi(\varsigma)|}{\left(1-|\varphi(\varsigma)|^2\right)^n} \times \left(\int_{B(\varphi(\varsigma),r)} \Psi dA\right)^{\frac{-1}{p}} \| f_k \|_{\mathcal{A}^p(\Psi)}, \tag{18}$$

where we have used the fact that $\psi \in \mathcal{A}_\nu$. Using (17) along with facts that $\sup_{z\in K}\left|f_k^{(n)}(z)\right| < \in$, for all k $\geq N_0$ in (18), we have that $\| D^n_{\psi,\varphi} f \|_{\mathcal{A}_\nu} < C \in$ for $k \geq N_0$. Since $\in > 0$ is arbitrary, it follows that $\| D^n_{\psi,\varphi} f \|_{\mathcal{A}_\nu} \longrightarrow 0$ *as* $n \longrightarrow \infty$. Hence $D^n_{\psi,\varphi} : \mathcal{A}^p(\Psi) \longrightarrow \mathcal{A}_\nu$ is compact. Conversely suppose that $D^n_{\psi,\varphi} : \mathcal{A}^p(\Psi) \longrightarrow \mathcal{A}_\nu$ is compact. Let ς_k be a sequence in $\mathbb{D}$ such that $|\varphi(\varsigma_k)| \longrightarrow 1$ *as* $k \longrightarrow \infty$. Choose $f_{\varphi(\varsigma_k)}$ as in (3). Then $\| f_{\varphi(\varsigma_k)} \|_{\mathcal{A}^p(\Psi)} \lesssim 1$ and $\{f_{\varphi(\varsigma_k)}\}$ converges to zero uniformly on compact subset of $\mathbb{D}$ as k $\longrightarrow \infty$. Since $D^n_{\psi,\varphi} : \mathcal{A}^p(\Psi) \longrightarrow \mathcal{A}_\nu$ is compact, we have that $\| D^n_{\psi,\varphi} g_n \|_{\mathcal{A}_\nu} \longrightarrow 0$ *as* $n \longrightarrow \infty$. Moreover, as in Theorem 1, we have

$$\| D^n_{\psi,\varphi} f \|_{\mathcal{A}_\nu} \geq \frac{\nu(\varsigma)|\psi(\varsigma)||\psi(\varsigma_k)|}{\left(1-|\varphi(\varsigma_k)|^2\right)^n}\left(\int_{B(\varphi(\varsigma_k),r)} \Psi dA\right)^{\frac{-1}{p}}.$$

Thus,

$$\lim_{|\varphi(\varsigma_k)|\to 1} \frac{\nu(z)|\psi(\varsigma_k)||\varphi'(\varsigma_k)\|\varphi(\varsigma_k)\|}{\left(1-|\varphi(\varsigma_k)|^2\right)^n}\left(\int_{B(\varphi(\varsigma_k),r)}\Psi dA\right)^{\frac{-1}{p}}=0,$$

as desired.

Next we characterize boundedness and compactness of $D^n_{\psi,\varphi} : \mathcal{A}^p(\Psi) \longrightarrow \mathcal{A}_{\nu,0}$.

Theorem 3. Let $r \in (0,1)$ be fixed, p > 0, p_0 > 1, $\alpha > -1, \nu$ a normal weight, $\psi \in H(\mathbb{D})$, φ a holomorphic self-map of $\mathbb{D}$ and n $\in \mathbb{N}_0$. Assume that $p_0 \geq p, \Psi$ is a weight function such that $\frac{\Psi}{\left(1-|z|^2\right)^\alpha}$ belongs to $\mathrm{B}_{p_0}(\alpha)$. Then $D^n_{\psi,\varphi} : \mathcal{A}^p(\Psi) \longrightarrow \mathcal{A}_{\nu,0}$ is bounded if and only if $D^n_{\psi,\varphi} : \mathcal{A}^p(\Psi) \longrightarrow \mathcal{A}_\nu$ is bounded, $\psi \in \mathcal{A}_{\nu,0}$.

Proof: First suppose that $D^n_{\psi,\varphi} : \mathcal{A}^p(\Psi) \longrightarrow \mathcal{A}_{\nu,0}$ is bounded. Then it is obvious that $D^n_{\psi,\varphi} : \mathcal{A}^p(\Psi) \longrightarrow \mathcal{A}_\nu$ is also bounded. By taking $f(z)=\frac{z^n}{n!}$ in $\mathcal{A}^p(\Psi)$, we have that $\psi \in \mathcal{A}_{\nu,0}$ Conversely, assume that $D^n_{\psi,\varphi} : \mathcal{A}^p(\Psi) \longrightarrow \mathcal{A}_\nu$ is bounded, $\psi \in \mathcal{A}_{\nu,0}$. Then for each polynomial $p(z)$, we have that

$$\nu(z)\left|\left(D^n_{\psi,\varphi}p\right)(z)\right| \leq \nu(z)|\psi(z)|\,\|\,p^{(n)}\varphi(z)\,\|. \qquad \textbf{(19)}$$

Using the fact that $\psi \in \mathcal{A}_{\nu,0}$ in (19), it follows that $D^n_{\psi,\varphi}\, p \in \mathcal{A}_{\nu,0}$. Also the set of all polynomials is dense in $\mathcal{A}^p(\Psi)$, so we have that for every f $\in \mathcal{A}^p(\Psi)$, there is a sequence of polynomial $\{p_n\}_{n\in\mathbb{N}}$ such that $\| f - p_n \|_{\mathcal{A}^p(\Psi)} \longrightarrow 0$ $as\ n \longrightarrow \infty$. Hence by the boundedness of $D^n_{\psi,\varphi} : \mathcal{A}^p(\Psi) \longrightarrow \mathcal{A}_\nu$, we have,

$$\| D^n_{\psi,\varphi} f - D^n_{\psi,\varphi} p_n \|_{\mathcal{A}_\nu} \leq \| D^n_{\psi,\varphi} \|_{\mathcal{A}^p(\Psi)\to\mathcal{A}_\nu} \| f - p_n \|_{\mathcal{A}^p(\Psi)} \longrightarrow 0 \text{ as } n \longrightarrow \infty.$$

Since $\mathcal{A}_{\nu,0}$ is a closed subspace of $\mathcal{A}_\nu$, we have that $D^n_{\psi,\varphi}(\mathcal{A}^p(\Psi)) \subset \mathcal{A}_{\nu,0}$ and so $D^n_{\psi,\varphi} : \mathcal{A}^p(\Psi) \longrightarrow \mathcal{A}_{\nu,0}$ is bounded.

Theorem 4. Let $r \in (0,1)$ be fixed, p > 0, p_0 > 1, $\alpha > -1, \nu$ a normal weight, $\psi \in H(\mathbb{D}), \varphi$ a holomorphic self-map of $\mathbb{D}$ and n $\in \mathbb{N}_0$. Assume that $p_0 \geq$

p, Ψ is a weight function such that $\frac{\Psi}{\left(1-|z|^2\right)^{\alpha}}$ belongs to $\mathrm{B}_{p_0}(\alpha)$ and $D^n_{\psi,\varphi}$: $\mathcal{A}^p(\Psi) \longrightarrow \mathcal{A}_{\nu,0}$ is bounded. Then the following conditions are equivalent:

(i) $D^n_{\psi,\varphi}$: $\mathcal{A}^p(\Psi) \longrightarrow \mathcal{A}_{\nu,0}$ is compact.

(ii) $\psi \in \mathcal{A}_{\nu,0}$ and $\lim_{|\varphi(z)|\to 1} \frac{\nu(z)|\psi(z)|}{\left(1-|\varphi(z)|^2\right)^n}\left(\int_{B(\varphi(z),r)} \Psi dA\right)^{\frac{-1}{p}} = 0.$

(20)

(iii) $\lim_{|z|\to 1} \frac{\nu(z)|\psi(z)|}{\left(1-|\varphi(z)|^2\right)^n}\left(\int_{B(\varphi(z),r)} \Psi dA\right)^{\frac{-1}{p}} = 0.$ **(21)**

Proof: Proof.(ii) ⇒ (iii): Suppose that $\psi \in \mathcal{A}_{\nu,0}$ and (20) hold. By (20), for every $\varepsilon > 0$, there exist $\delta \in (0,1)$ such that

$$\frac{\nu(z)|\psi(z)|}{\left(1-|\varphi(z)|^2\right)^n}\left(\int_{B(\varphi(z),r)} \Psi dA\right)^{\frac{-1}{p}} < \epsilon.$$

when $\delta < |\varphi(z)| < 1$. Since $\psi \in \mathcal{A}_{\nu,0}$, there exist $\gamma \in (0,1)$ such that

$$\nu(z)|\psi(z)| \le \epsilon\left(\int_{B(\varphi(z),r)} \Psi dA\right)^{\frac{1}{p}},$$

whenever $\gamma < |z| < 1$. Thus if $\gamma < |z| < 1$ and $\delta < |\varphi(z)| < 1$, we have that

$$\nu(z)|\psi(z)|\left(\int_{B(\varphi(z),r)} \Psi dA\right)^{\frac{-1}{p}} < \epsilon. \quad \textbf{(22)}$$

Again $\gamma < |z| < 1$ and $|\varphi(z)| \le \delta$, then from (13), we have

$$\frac{\nu(z)|\psi(z)|}{\left(1-|\varphi(z)|^2\right)^n}\left(\int_{B(\varphi(z),r)}\Psi dA\right)^{\frac{-1}{p}}=\frac{\nu(z)|\psi(z)|}{\left(1-|\lambda|^2\right)^{n+(\eta+2)\frac{p_0}{p}}}\frac{\left(1-|\lambda|^2\right)^{(\eta+2)\frac{p_0}{p}}}{\left(\int_{B(\varphi(z),r)}\Psi dA\right)^{\frac{1}{p}}}$$

$$\leq C\frac{\nu(z)|\psi'(z)|}{\left(1-|\delta|^2\right)^{n+(\eta+2)\frac{p_0}{p}}}<\in. \qquad \mathbf{(23)}$$

Combining (22) and (23), we obtain that(ii) ⇒ (iii).

(i) ⇒(ii). Suppose that $D^{\eta}_{\psi,\Phi}:\mathcal{A}^p(\Psi)\to\mathcal{A}_{\mathcal{V},0}$ is compact. By taking $f(z)=z^{\eta}/n!$ in $\mathcal{A}^p(\Psi)$, we have that $\psi\in\mathcal{A}_{\mathcal{V},0}$. Proceeding as in the proof of Theorem 2, we have

$$\lim_{|\varphi(z)|\to 1}\frac{\nu(z)|\psi(z)\varphi'(z)|}{\left(1-|\varphi(z)|^2\right)^n}\left(\int_{B(\varphi(z),r)}\Psi dA\right)^{\frac{-1}{p}}=0. \qquad \mathbf{(24)}$$

(iii) ⇒(i). Suppose that (iii) holds. Then from (6), we have that

$$\nu(z)\left|\left(D^n_{\psi,\varphi}f\right)(z)\right|\leq\frac{\nu(z)|\psi(z)\varphi'(z)|}{1-|\varphi(z)|^2}\left(\int_{B(\varphi(z),r)}\Psi dA\right)^{\frac{-1}{p}}\|f\|_{\mathcal{A}^p(\Psi)}. \qquad \mathbf{(25)}$$

Taking suprimum in the inequality over all $f\in\mathcal{A}^p(\Psi)$ such that $\| f \|_{\mathcal{A}^p(\Psi)}\leq 1$, and then letting $|z|\to 1$, we obtain that

$$\lim_{|z|\to 1}\sup_{\|f\|_{\mathcal{A}^p(\Psi)}\leq 1}\mathcal{V}(z)|(D^n_{\psi,\varphi}f)'(z)|=0.$$

Thus by lemma 5, we obtain we obtain that $D^n_{\psi,\varphi}:\mathcal{A}^p(\Psi)\to\mathcal{A}_{\mathcal{V},0}$ is compact.

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Zaheer Abbas and Sonali Magotra,
Department of Mathematical Sciences,
BGSB University, Rajouri-185234, J&K, India.
Email address: az11292000@yahoo.co.in,
sonalimagotra@rocketmail.com.

S.D.Sharma
Department of Mathematics,
Central University of Jammu, (Bagla) Raya-Suchani,
Samba-181143, J&K, India.
Email address: somdattjammu@gmail.com