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THREE-DIMENSIONAL ELASTIC SOLUTION OF A POWER FORM FUNCTIONALLY GRADED RECTANGULAR PLATES

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ABSTRACT

The paper presents a three-dimensional analysis of a simply supported functionally graded plate subjected to normal and shear loadings on the lower and upper surfaces. The problem is formulated on the assumption that the elastic constants have the same dependence on the transverse coordinate *z*. The three-dimensional governing equations for an inhomogeneous isotropic plate are solved by expanding the corresponding physical quantities into Fourier series. An exact solution is obtained for an isotropic functionally graded rectangular plate based on the assumption that the Young's modulus has a general power form and the Poisson's ratio is constant. Numerical results for displacement and stress components, taking into account the variation of material nonhomogenous parameter and plate thickness, are shown graphically and discussed.

***Keywords*: Functionally graded material, Rectangular plate, Simply supported, Three-dimensional elasticity theory

1. INTRODUCTION

Functionally graded materials (FGMs) are heterogeneous composite materials which characterized by a gradual change in properties within the specimen as a function of the position coordinates. The property gradient in the material is typically caused by a position-dependent chemical composition, microstructure [1]. The advantage of FGMs is to eliminate interfacial stress jumps in composite materials and make the stress distribution smoother. Based on these characteristics, the application of FGMs has become more and more extensive in various industries [2-3].

With the increasing of the application of FGMs, lots of studies have been devoted to the response of FGM structures to different loadings. Some simplified theories and numerical techniques have been employed to study functionally graded beams, plates and shells. For instance, Tanigawa [4] used a laminated approximate model to analyze thermoelastic behaviour of functionally graded materials. Loy *et al.* [5] studied the vibration of functionally graded cylindrical shells using Love's shell theory. Cheng and Batra [6] used an asymptotic expansion method to analyze three-dimensional thermoelastic deformations of rigidly clamped functionally graded elliptic plates. Woo and Meguid [7] developed series solutions for large deflections of functionally graded plates under transverse loading and a temperature field using von Karman theory. Shen [8] and Yang and Shen [9] studied large deflections and postbuckling response of functionally graded plates with temperature-dependent material properties using a classical plate theory and a perturbation method. Tsukamoto [10] examined thermal stresses in a ceramic-metal plate subjected to through-thickness heat flow using the classical laminated plate theory.

Exact solutions for functionally grade materials based on elasticity theory are very few comparing to those for homogeneous materials, but they indeed play important roles in providing benchmark results for other simplified theories, numerical methodologies and experimental tests. In recent years, Jeon and Tanigawa [11] solved an axisymmetric problem of a FGM thick plate subjected to arbitrarily distributed load or a concentrated load on its surface. Morishita and Tanigawa [12] studied three-dimensional elastic solution of a FGM thick plate (layer) by

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introducing three kinds of displacement functions. Three-dimensional exact solutions were obtained for simply supported functionally graded plates [13, 14]. Mian and Spencer [15] established an exact three-dimensional solution for functionally graded plates with traction-free surface. Huang, Lü and Chen [16] provided an exact three-dimensional elasticity solution for the bending behavior of FGM thick plates on a Winkler-Pasternak foundation. Except for the researches on isotropic materials, Sankar [17] obtained an elasticity solution for an orthotropic simply supported functionally graded beam. Zhong and Shang [18, 19] developed three-dimensional analysis for orthotropic functionally graded piezoelectric and piezothermoelectric rectangular plates using state space approach. In the above works, elastic moduli were assumed to vary according to exponential law, reciprocal law or linear law through the thickness. To our knowledge, no exact solution has been found for functionally graded plates with elastic moduli given in a general power form.

Therefore, it is the purpose of the present paper to study the functionally graded plates with elastic moduli given in a general power form. Following Plevako's approach, two displacement functions are used to analyze the three-dimensional governing equations for an isotropic functionally graded plate and an exact solution are obtained by assuming that the Young's modulus has a general power form and the Poisson's ratio is constant. Numerical examples are provided to validate the proposed solution and examine the dependence of stress and displacement fields on the nonhomogenous parameter and plate thickness.

2. BASIC EQUATIONS OF A THREE-DIMENSIONAL ISOTROPIC FUNCTIONALLY GRADED PLATE

Consider an isotropic functionally graded rectangular plate of length a, width b and thickness h, as shown in Fig. 1. Introduce a Cartesian coordinate system xyz such that $0 \le x \le a$, $0 \le y \le b$, $0 \le z \le h$.

In the absence of body forces, the equations of equilibrium in terms of displacements for an inhomogeneous isotropic elastic material can be written in the following form [14]:

$$\begin{aligned}
&E(z)\nabla^2 u + \frac{E(z)}{1-2\nu}\frac{\partial e}{\partial x} + \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)\frac{dE(z)}{dz} = 0 \\
&E(z)\nabla^2 v + \frac{E(z)}{1-2\nu}\frac{\partial e}{\partial y} + \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)\frac{dE(z)}{dz} = 0 \\
&E(z)\nabla^2 w + \frac{E(z)}{1-2\nu}\frac{\partial e}{\partial z} + e\frac{d}{dz}\left[\frac{2E(z)\nu}{1-2\nu}\right] + 2\frac{\partial w}{\partial x}\frac{dE(z)}{dz} = 0
\end{aligned}, \tag{1}$$

where x, y, z are Cartesian coordinates, u, v, w are displacement components, $e = \partial u / \partial x + \partial v / \partial y + \partial w / \partial z$ is the volumetric strain and $\nabla^2 = \partial^2 / \partial x^2 + \partial^2 / \partial y^2 + \partial^2 / \partial z^2$ is the Laplcian operator, E and ν are respectively the Young's modulus and the Poisson's ratio that depend on z coordinate only.

The plate is subjected to normal and shear loadings on its lower and upper surfaces, whose boundary conditions are given as:

$$\begin{aligned}
&\sigma_z = Z_0(x, y) \qquad \tau_{xz} = X_0(x, y) \qquad \tau_{yz} = Y_0(x, y) \quad \text{at } z = 0 \\
&\sigma_z = Z_1(x, y) \qquad \tau_{xz} = X_1(x, y) \qquad \tau_{yz} = Y_1(x, y) \quad \text{at } z = h,
\end{aligned} \tag{2}$$

where $Z_0(x, y)$ and $Z_1(x, y)$ are normal tractions, $X_0(x, y)$, $X_1(x, y)$, $Y_0(x, y)$, $Y_1(x, y)$ are shear tractions on the lower or upper surfaces of the plate.

If the plate is simply supported at its four edges, the edge boundary conditions are written as:

$$\begin{aligned}
&\sigma_x = v = w = 0 \quad (\text{at } x = 0, a) \\
&\sigma_y = u = w = 0 \quad (\text{at } y = 0, b)
\end{aligned} \tag{3}$$

Now the problem is reduced to solve the governing equations given in Eq. (1) under the boundary conditions (2) and (3).

3. SOLUTION

If the Poisson's ratio is constant, the displacement components are expressed as [20]:

$$
\begin{aligned}
u &= -\frac{1+v}{E}\left(\nu\nabla^2 - \frac{\partial^2}{\partial z^2}\right)\frac{\partial L}{\partial x} + \frac{\partial N}{\partial y} \\
v &= -\frac{1+v}{E}\left(\nu\nabla^2 - \frac{\partial^2}{\partial z^2}\right)\frac{\partial L}{\partial y} - \frac{\partial N}{\partial x} \\
w &= -2\frac{1+v}{E}\left(\nabla^2 - \frac{\partial^2}{\partial z^2}\right)\frac{\partial L}{\partial z} + \frac{\partial}{\partial z}\left[\frac{1+v}{E}\left(\nu\nabla^2 - \frac{\partial^2}{\partial z^2}\right)L\right]
\end{aligned} \tag{4}
$$

where $L = L(x, y, z)$ and $N = N(x, y, z)$ are two functions which satisfy the following partial differential equations:

$$\nabla^2\left(\frac{1}{E}\nabla^2 L\right) - \frac{1}{1-v}\left(\nabla^2 - \frac{\partial^2}{\partial z^2}\right)L\frac{d^2}{dz^2}\left(\frac{1}{E}\right) = 0 \tag{5}$$

$$\nabla^2 N + \frac{E'}{E}\frac{\partial N}{\partial z} = 0 \tag{6}$$

where $E' = dE/dz$.

The stress components can also be expressed in terms of functions L and N as:

$$
\begin{aligned}
\sigma_x &= \left(\nu\frac{\partial^2}{\partial y^2}\nabla^2 + \frac{\partial^4}{\partial x^2\partial z^2}\right)L + \frac{E}{1+\nu}\frac{\partial^2 N}{\partial x\partial y} \\
\sigma_y &= \left(\nu\frac{\partial^2}{\partial x^2}\nabla^2 + \frac{\partial^4}{\partial y^2\partial z^2}\right)L - \frac{E}{1+\nu}\frac{\partial^2 N}{\partial x\partial y} \\
\sigma_z &= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)^2 L \\
\tau_{yz} &= -\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\frac{\partial^2 L}{\partial y\partial z} - \frac{E}{2(1+\nu)}\frac{\partial^2 N}{\partial x\partial z} \\
\tau_{xz} &= -\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\frac{\partial^2 L}{\partial x\partial z} + \frac{E}{2(1+\nu)}\frac{\partial^2 N}{\partial y\partial z} \\
\tau_{xy} &= -\left[\nu\nabla^2 - \frac{\partial^2}{\partial z^2}\right]\frac{\partial^2 L}{\partial x\partial y} - \frac{E}{2(1+\nu)}\left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}\right)N
\end{aligned} \tag{7}
$$

Assuming that functions L and N have the following forms:

$$
\begin{aligned}
L(x,y,z) &= \sum_{m,n=1}^{\infty}\psi_{mn}(z)\sin(\alpha_m x)\sin(\beta_n y) \\
N(x,y,z) &= \sum_{m,n=1}^{\infty}\phi_{mn}(z)\cos(\alpha_m x)\cos(\beta_n y)
\end{aligned} \tag{8}
$$

with

$$\alpha_m = \frac{m\pi}{a} \quad \beta_n = \frac{n\pi}{b} \tag{9}$$

where m and n are positive integers. It can be shown that Eq. (8) gives solutions of Eqs. (5) and (6) that satisfies the simply supported edge boundary conditions given by Eq. (3) [14].

Substituting Eq. (8) into Eqs. (5) and (6) leads to the following governing equations for unknown functions ψ_{mn} and ϕ_{mn}:

$$\begin{aligned}&\frac{d^4\psi_{mn}}{dz^4}-2\frac{E'}{E}\frac{d^3\psi_{mn}}{dz^3}+\left[2\left(\frac{E'}{E}\right)^2-\frac{E''}{E}-2\lambda\right]\frac{d^2\psi_{mn}}{dz^2}+2\frac{E'}{E}\lambda\frac{d\psi_{mn}}{dz}\\&+\left\{\lambda+\frac{\nu}{1-\nu}\left[2\left(\frac{E'}{E}\right)^2-\frac{E''}{E}\right]\right\}\lambda\psi_{mn}=0\end{aligned} \tag{10}$$

$$\frac{d^2\phi_{mn}}{dz^2}+\frac{E'}{E}\frac{d\phi_{mn}}{dz}-\lambda\phi_{mn}=0 \tag{11}$$

where

$$\lambda=\lambda_{mn}=\alpha_m^2+\beta_n^2 \tag{12}$$

If the normal tractions, $Z_0(x, y)$ and $Z_1(x, y)$, and the shear tractions, $X_0(x, y)$, $X_1(x, y)$, $Y_0(x, y)$ and $Y_1(x, y)$, on the upper and lower surfaces of the plate, can be expanded into double Fourier series:

$$\begin{aligned}Z_0(x,y)&=\sum_{m,n=1}^{\infty}\bar{Z}_{0mn}\sin(\alpha_m x)\sin(\beta_n y)\\Z_1(x,y)&=\sum_{m,n=1}^{\infty}\bar{Z}_{1mn}\sin(\alpha_m x)\sin(\beta_n y)\\X_0(x,y)&=\sum_{m,n=1}^{\infty}\bar{X}_{0mn}\cos(\alpha_m x)\sin(\beta_n y)\\X_1(x,y)&=\sum_{m,n=1}^{\infty}\bar{X}_{1mn}\cos(\alpha_m x)\sin(\beta_n y)\\Y_0(x,y)&=\sum_{m,n=1}^{\infty}\bar{Y}_{0mn}\sin(\alpha_m x)\cos(\beta_n y)\\Y_1(x,y)&=\sum_{m,n=1}^{\infty}\bar{Y}_{1mn}\sin(\alpha_m x)\cos(\beta_n y)\end{aligned} \tag{13}$$

where

$$\begin{aligned}\bar{Z}_{0mn}&=\frac{1}{4ab}\int_0^a\int_0^b Z_0(x,y)\sin(\alpha_m x)\sin(\beta_n y)dxdy\\\bar{Z}_{1mn}&=\frac{1}{4ab}\int_0^a\int_0^b Z_1(x,y)\sin(\alpha_m x)\sin(\beta_n y)dxdy\\\bar{X}_{0mn}&=\frac{1}{4ab}\int_0^a\int_0^b X_0(x,y)\cos(\alpha_m x)\sin(\beta_n y)dxdy\\\bar{X}_{1mn}&=\frac{1}{4ab}\int_0^a\int_0^b X_1(x,y)\cos(\alpha_m x)\sin(\beta_n y)dxdy\\\bar{Y}_{0mn}&=\frac{1}{4ab}\int_0^a\int_0^b Y_0(x,y)\sin(\alpha_m x)\cos(\beta_n y)dxdy\end{aligned} \tag{14}$$

$$\bar{Y}_{1mn} = \frac{1}{4ab}\int_0^a\int_0^b Y_1(x,y)\sin(\alpha_m x)\cos(\beta_n y)dxdy$$

we get from Eq. (2) the boundary conditions for ψ_{mn} and ϕ_{mn} as:

$$\psi_{mn}(0) = \frac{\bar{Z}_{0mn}}{\lambda_{mn}^2} \quad \psi_{mn}(h) = \frac{\bar{Z}_{1mn}}{\lambda_{mn}^2}$$

$$\left.\frac{d\psi_{mn}}{dz}\right|_{z=0} = \frac{1}{\lambda_{mn}^2}\left(\bar{X}_{0mn}\alpha_m + \bar{Y}_{0mn}\beta_n\right)$$

$$\left.\frac{d\psi_{mn}}{dz}\right|_{z=h} = \frac{1}{\lambda_{mn}^2}\left(\bar{X}_{1mn}\alpha_m + \bar{Y}_{1mn}\beta_n\right) \tag{15}$$

and

$$\left.\frac{d\phi_{mn}}{dz}\right|_{z=0} = -\frac{2(1+\nu)}{E\lambda_{mn}}\left(\bar{X}_{0mn}\beta_n - \bar{Y}_{0mn}\alpha_m\right)$$

$$\left.\frac{d\phi_{mn}}{dz}\right|_{z=h} = -\frac{2(1+\nu)}{E\lambda_{mn}}\left(\bar{X}_{1mn}\beta_n - \bar{Y}_{1mn}\alpha_m\right) \tag{16}$$

Therefore, the next task is to find the solutions of the ordinary differential equations with variable coefficients, Eqs. (10), (11), under boundary conditions (15) and (16).

4. POWER FUNCTIONS

When we take a power-law dependence of the elastic modulus on the coordinate z in the form

$$E(z) = E_0(1 + cz)^p \tag{17}$$

where c and p are two material parameters describing the nonhomogeneity of $E(z)$. In this case, Eqs. (10) and (11) are reduced to

$$\frac{d^4\psi_{mn}}{dz^4} - \frac{2cp}{1+cz}\frac{d^3\psi_{mn}}{dz^3} + \left[\frac{pc^2(p+1)}{(1+cz)^2} - 2\lambda\right]\frac{d^2\psi_{mn}}{dz^2} + \frac{2pc\lambda}{1+cz}\frac{d\psi_{mn}}{dz} + \left[\lambda + \frac{v(p+1)c^2p}{(1-v)(1+cz)^2}\right]\lambda\psi_{mn} = 0 \tag{18}$$

$$\frac{d^2\phi_{mn}}{dz^2} + \frac{pc}{1+cz}\frac{d\phi_{mn}}{dz} - \lambda\phi_{mn} = 0 \tag{19}$$

whose solution can be expressed, as follows:

for $v \neq \frac{1}{p+1}$ and $p \neq -1$

$$\psi_{mn} = z_1^{g-1/2}\left\{A_1W_{\chi,g}(2\eta z_1) + A_2W_{-\chi,g}(2\eta z_1) + A_3W_{\chi,g}(-2\eta z_1) + A_4W_{-\chi,g}(-2\eta z_1)\right\} \tag{20}$$

for $v = \frac{1}{p+1}$ or $p = -1$

$$\psi_{mn} = z_1^{\,g}\{A_1 I_g(\eta z_1) + A_2 K_g(\eta z_1)$$
$$+A_3[I_g(\eta z_1)\int K_g^2(\eta z_1)dz_1 - K_g(\eta z_1)\int I_g(\eta z_1)K_g(\eta z_1)dz_1]$$
$$+A_4[I_g(\eta z_1)\int I_g(\eta z_1)K_g(\eta z_1)dz_1 - K_g(\eta z_1)\int I_g^2(\eta z_1)dz_1]\} \tag{21}$$

$$\phi_{mn} = z_1^{-g+\frac{3}{2}}\left[A_5 I_{-g+3/2}(\eta z_1) + A_6 K_{-g+3/2}(\eta z_1)\right] \tag{22}$$

with

$$z_1 = 1 + cz, \eta = \frac{\sqrt{\lambda}}{c}, g = 1 + p/2, \chi = \frac{1}{2}\sqrt{(p+1)\left(1 - \frac{vp}{1-v}\right)}$$

where $W_{\pm\chi,g}(\pm 2\eta z_1)$ are Whittaker functions, $I_g(\eta z_1)$ and $K_g(\eta z_1)$ are the gth order modified Bessel functions of the first and second kind respectively, $I_{-g+3/2}(\eta z_1)$ and $K_{-g+3/2}(\eta z_1)$ are the $-g + 3 / 2$ order modified Bessel functions of the first and second kind respectively [21]. The coefficients A_1, A_2, A_3, A_4, A_5, A_6 need to be determined from boundary conditions (15) and (16).

5. NUMERICAL RESULTS AND DISCUSSION

In this section we will make numerical study of an isotropic FGM square plate ($a = b = 1m$, $h = 0.1m$) based on the above exact solutions. The plate is simply supported on its four lateral edges and subjected to a sinusoidal normal traction at its upper surface, i.e.,

$$Z_1(x, y) = -q\sin\left(\frac{\pi}{a}x\right)\sin\left(\frac{\pi}{b}y\right),\ q = 1Pa$$
$$X_0(x, y) = Y_0(x, y) = Z_0(x, y) = X_1(x, y) = Y_1(x, y) = 0$$

from which only one term solution is used ($m = 1$, $n = 1$) and attention is focused on the influence of different nonhomogenous parameter p on the displacement and stress fields in the plate. The Young's moduli at the upper and lower surfaces of the plate are given as $E(0) = 1GPa$, $E(h)$, $E(h) = 10GPa$. Accordingly, material model for FGM used in the present study is given as in Eq. (17) with $p = -2, -1, 0, 1, 2$, $E_0 = E(0)$ and :

$$c = \frac{1}{h}\left\{\left[\frac{E(h)}{E_0}\right]^{\frac{1}{p}} - 1\right\}$$

Fig. 2 shows the variations of the Young's modulus along the thickness direction for different nonhomogenous parameter p when $E(h) / E(0) = 10$.

Based on the above models of Young's modulus, the variation of displacements u and w, stresses σ_x, τ_{xy}, σ_z and τ_{xz} at a chosen position ($x / a = 0.25$, $y / b = 0.25$), along the thickness direction, are shown in Fig. 3 for a thin plate ($h / a = 0.1$) and Fig. 4 for a thick plate ($h / a = 0.4$). The displacement ν, stresses σ_y and τ_{yz} are not depicted because their distributions along the plate thickness direction are similar to those of u, σ_x and τ_{xz} respectively, because of the symmetry of the problem.

From Fig. 3 and Fig. 4, the following observations can be made:

For a thin FGM plate, the vertical displacement w (Fig. 3(b)) demonstrates essentially uniform distribution along the thickness direction and the horizontal displacement u (Fig. 3(a)) shows a linear variation across the thickness of the plate. For a thick FGM plate, the vertical displacement w (Fig. 4(b)) is no longer uniform and the horizontal displacement u (Fig. 4(a)) shows a deviation from linear distribution across the thickness.

Either for a thin plate or a thick plate, the magnitude of the vertical displacement for $p = 1$ is smallest while that for $p = 0$ is biggest (Fig. 3(b) and Fig. 4(b)). This can be explained by the fact that the bending rigidities of the plate

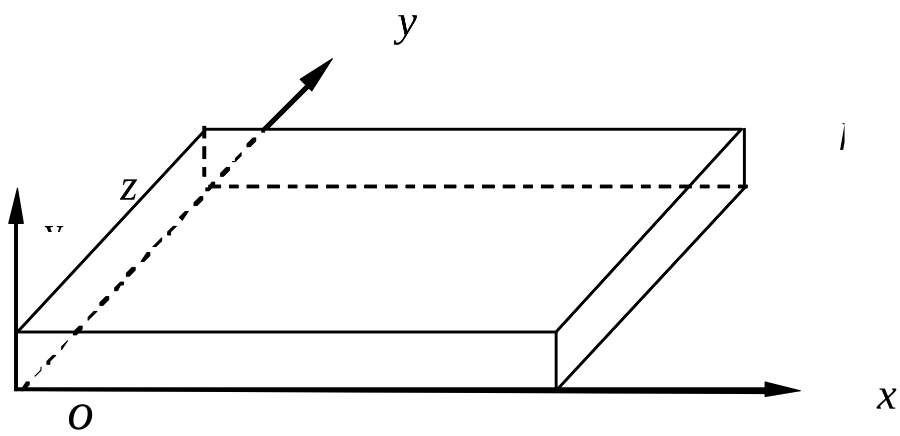

Figure 1: A Schematic of a Functionally Graded Plate

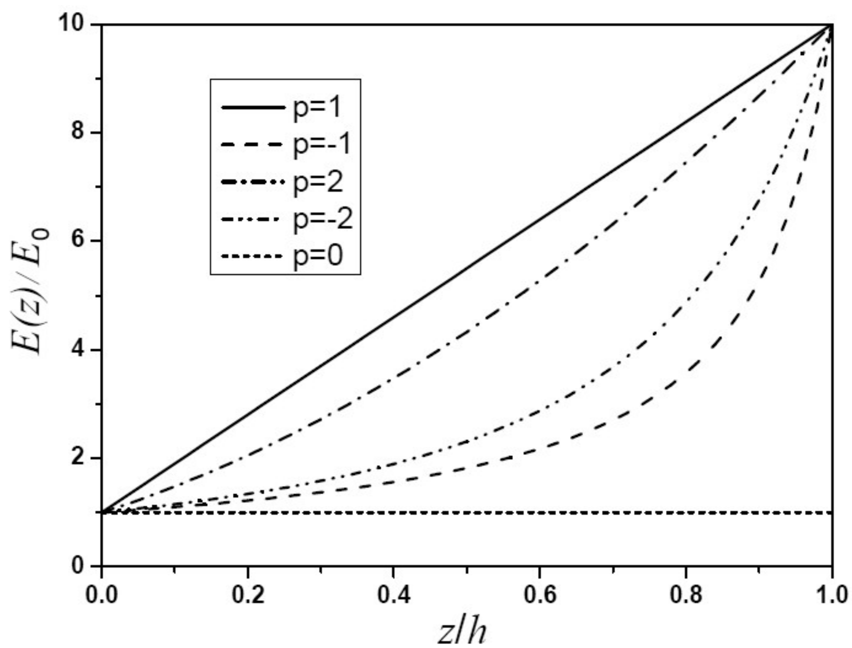

Figure 2: Variation of Young's Modulus along the Thickness of the Plate

are different for different nonhomogenous models. It can also be found that for the same graded model of elastic modulus, a thin plate has a much larger deflection than that of a thick plate since the bending rigidity of a thin plate is less than that of a thick plate.

The in-plane stress concentrations (Fig. 3(c) and Fig. 4(c)) in the plate are quite different for different grade models of Young's modulus. This enables an optimal design of the plate by selecting the appropriate graded parameter of a functionally graded material.

The out-of-plane stresses, σ_z and τ_{zx}, are negligible compared to the in-plane stress σ_x for a thin FGM plate. But for a thick plate this is not true since σ_z and τ_{zx} are comparable with σ_x. This observation should be considered in establishing a simplified FGM thin plate theory.

6. CONCLUDING REMARKS

An exact solution is obtained for a simply supported isotropic FGM plate subjected to normal and shear loadings on its lower and upper surfaces by assuming that the Young's modulus is of power form and the Poisson's ratio is a constant. The obtained solution can be used to assess the validity and accuracy of various approximate theoretical and numerical models of functionally graded plates.

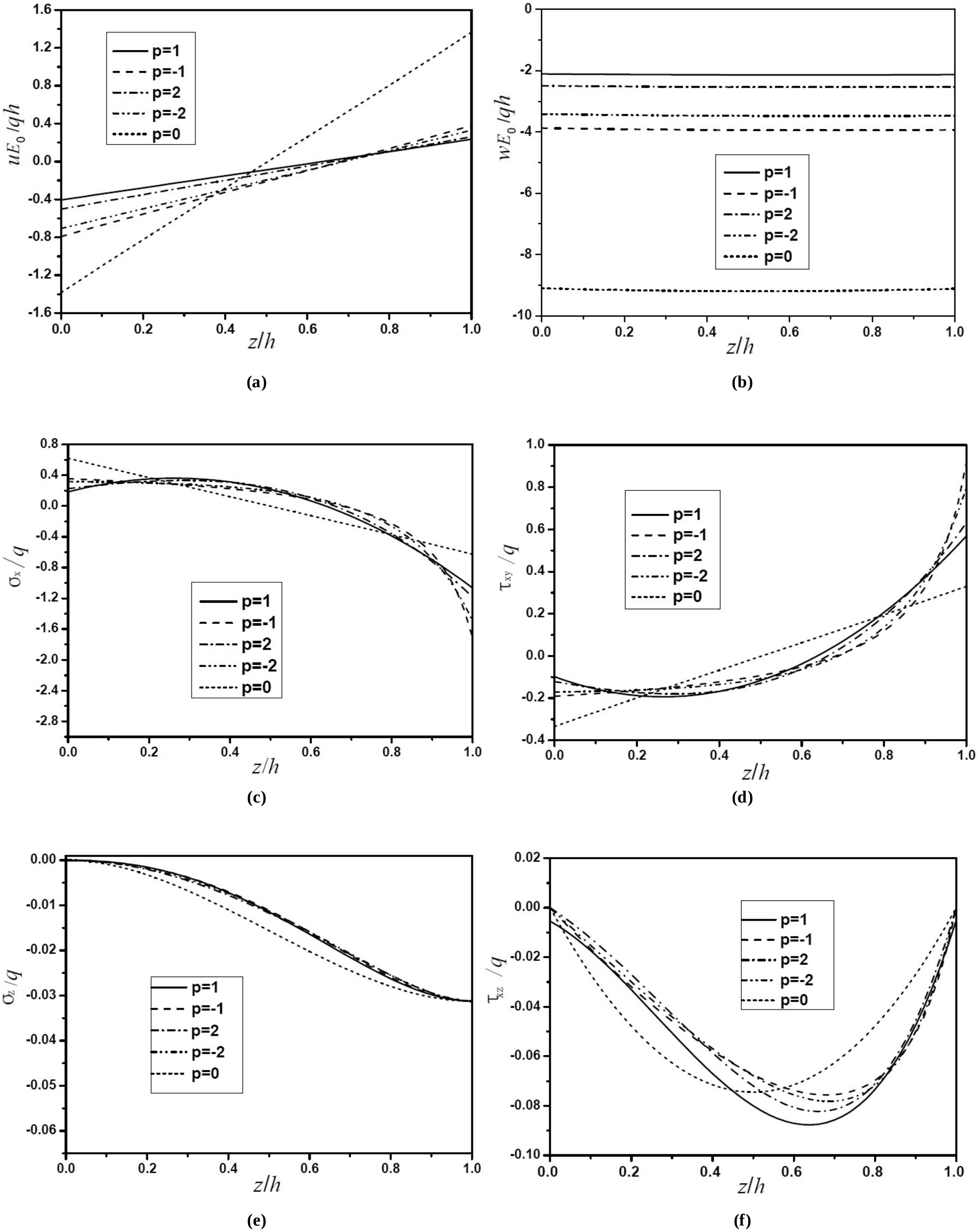

Figure 3: Variation of Physical Quantities with *z*-coordinate at a Chosen Location (*x/a* = 0.25, *y/b* = 0.25) for a thin Plate (*h/a* = 0.1): (a) displacement *u*, (b) displacement *w*, (c) stress σ_x, (d) stress τ_{xy}, (e) stress σ_z, (f) stress τ_{zx}

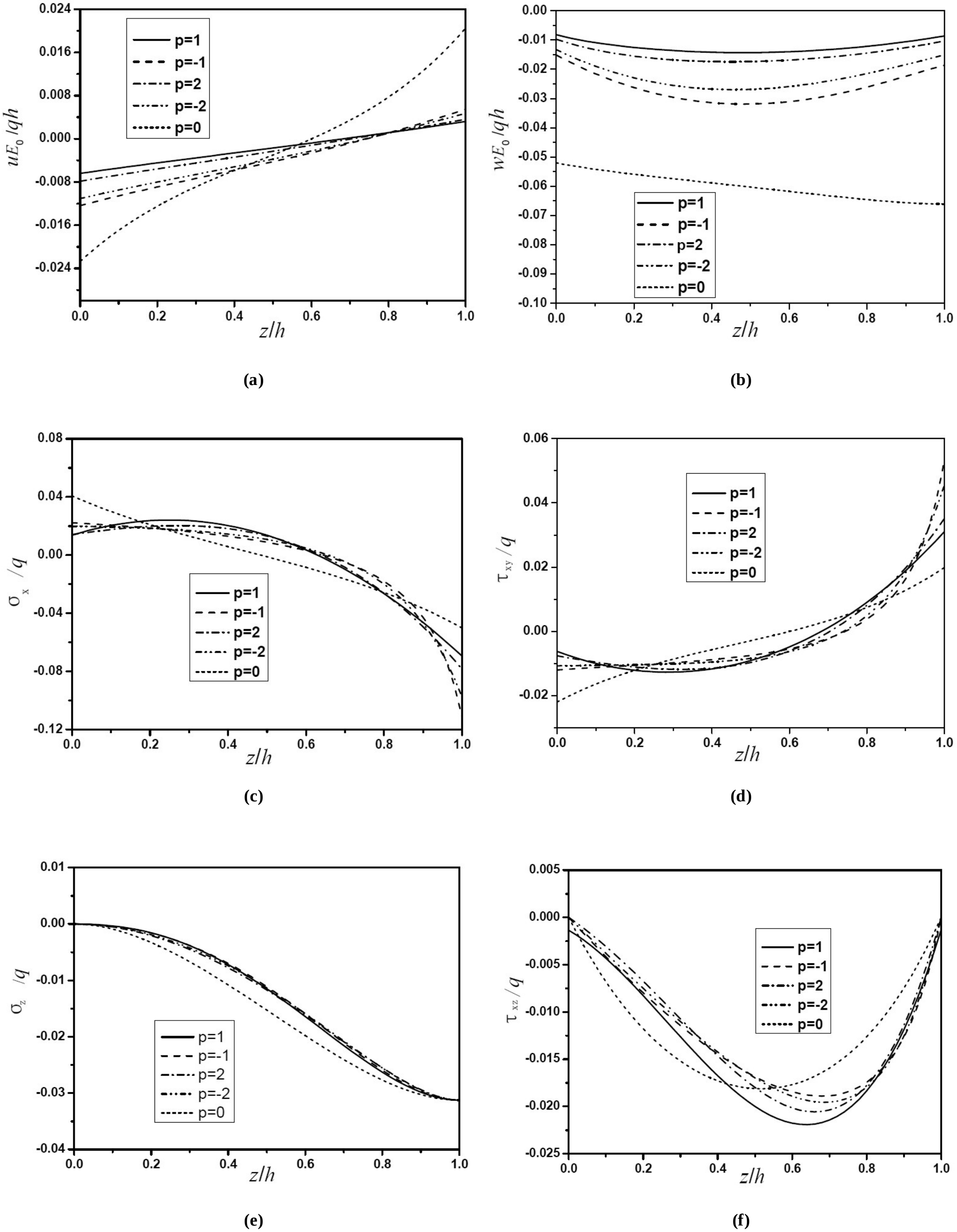

Figure 4: Variation of Physical Quantities with *z*-coordinate at a Chosen Location (x/a = 0.25, y/b = 0.25) for a Thick Plate (h/a = 0.4): (a) displacement *u*, (b) displacement *w*, (c) stress σ_x, (d) stress τ_{xy}, (e) stress σ_z, (f) stress τ_{zx}

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