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KRAWTCHOUK-GRIFFITHS SYSTEMS II: AS BERNOULLI SYSTEMS

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Abstract. We call Krawtchouk-Griffiths systems, KG-systems, systems of multivariate polynomials orthogonal with respect to corresponding multinomial distributions. The original Krawtchouk polynomials are orthogonal with respect to a binomial distribution. Here we present a Fock space construction with raising and lowering operators. The operators of "multiplication by X" are found in terms of boson operators and corresponding recurrence relations presented. The Riccati partial differential equations for the differentiation operators, Berezin transform and associated partial differential equations are found. These features provide the specifications for a Bernoulli system as a quantization formulation of multivariate Krawtchouk polynomials.

1. Introduction

The original paper of Krawtchouk [14] presents polynomials orthogonal with respect to a general binomial distribution and discusses the connection with Hermite polynomials. Krawtchouk polynomials are part of the legacy of Mikhail Kravchuk. A symposium in honor of his work and memory was held in Kiev and an accompanying volume was produced that is most highly recommended, Virchenko [18].

Krawtchouk polynomials appear in diverse areas of mathematics and science. Applications range from coding theory, [17], to image processing, [21]. Multivariable extensions are of interest and the field is very active. We cite works which have some connection to the approach in this paper.

The idea of extending to the multinomial distribution appears in the foundational work of Griffiths [2, 9, 10]. Connections with Lie theory have been studied more recently, [7, 8, 11, 12, 13, 16] as well as from the point of view of harmonic analysis [19, 20].

Bernoulli systems in one variable are explained in [5], with higher-dimensional Bernoulli systems appearing in [6], where the basic methods of this work appear initially. As a good resource, the Berezin approach was applied to the Schrödinger algebra in [4].

An analysis of the connections between orthogonal polynomials and probability distributions via properties of their generating function are accomplished in [1, 15].

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We summarize the contents of this work. Section 2 reviews the binomial case and introduces the matrix approach. This is followed up with a review of the basics of symmetric representations including the homomorphism and transpose properties. In §4, the matrix construction of Krawtchouk polynomial systems is presented. §§1-4 are a review of the basic material in KG-Systems I, [3].

Appell systems and Bernoulli systems are described next. Appell systems essentially turn out to have a generating function in the form of the exponential of raising operators acting on a vacuum state. Bernoulli systems are Appell systems with orthogonal polynomials as basis states. The Bernoulli systems provide models for Fock space constructions and for quantization with variables expressed in operator form. §6 discusses the form of the observables and lays out the associated constructions of interest, such as coherent states and the Leibniz function. After a review of the multinomial distribution in §7, in §8, we identify Krawtchouk polynomials in the context of Bernoulli systems. Especially, we find the canonical velocity (differentiation) operators and the form of the observables. Using coherent state techniques, the lowering operators are found via the Leibniz function. This rounds out a description of the Bernoulli system and associated quantities. To conclude, we find the X_j variables in selfadjoint form and present associated recurrence formulas for the basis Krawtchouk polynomials.

1.1. Basic notations and conventions. In this paper we will be working over $\mathbb{R}$.

(1) We consider polynomials in $d+1$ commuting variables.
(2) Multi-index notations for powers. With $n = (n_0, \dots, n_d)$, $x = (x_0, \dots, x_d)$:

$$x^n = x_0^{n_0} \cdots x_d^{n_d}$$

and the total degree $|n| = n_0 + \cdots + n_d$. Typically m and n will denote multi-indices, with i, j, k, ℓ for single indices. Running indices may be used as either type, determined from the context.
(3) We use the following *summation convention* repeated Greek indices, e.g., λ or μ, are summed from 0 to d.

Latin indices i, j, k, run from 1 to d unless explicitly indicated otherwise and are summed only when explicitly indicated, preferring ℓ for a single index running from 0 to d.

We will use the notation for standard basis e_ℓ as well for shifting multi-indices, e.g. $n \pm e_\ell$ shifts $n_\ell \pm 1$ accordingly.
(4) For simplicity, we will always denote identity matrices of the appropriate dimension by I.

The transpose of a matrix A is denoted $A^\top$.

We will use the notation $\mathcal{O}$ to denote a real orthogonal matrix.
(5) Given $N \geq 0$, B is defined as the multi-indexed matrix having as its only non-zero entries

$$B_{mm} = \binom{N}{m} = \frac{N!}{m_0! \dots m_d!}$$

the multinomial coefficients of order N.

(6) For a tuple of numbers, diag(...) is the diagonal matrix with the tuple providing the entries forming the main diagonal.
(7) Expectation with respect to a given underlying distribution is denoted $\langle\cdot\rangle$.

2. Krawtchouk Polynomials in One Variable

Krawtchouk polynomials may be defined via the generating function

$$(1+pv)^{N-x}(1-qv)^x = \sum_{0\le k\le N} v^k\, K_k(x,N).$$

The polynomials $K_k(x,N)$ are orthogonal with respect to the binomial distribution with parameters N,p. The associated probabilities have the form

$$\{\ \binom{N}{0}q^N p^0,\dots,\binom{N}{x}q^{N-x}p^x,\dots,\binom{N}{N}q^0p^N\ \}\ .$$

Let's verify this. Setting $G(v)=(1+pv)^{N-x}(1-qv)^x$, we have

$$\begin{aligned}\langle G(v)G(w)\rangle &= \sum_x \binom{N}{x} q^{N-x}p^x(1+pv)^{N-x}(1-qv)^x(1+pw)^{N-x}(1-qw)^x\\ &= (q+qp(v+w)+qp^2vw+p-pq(v+w)+pq^2vw)^N\\ &= (1+pqvw)^N\\ &= \sum_{k=0}^N \binom{N}{k}(pq)^k(vw)^k\end{aligned}$$

which shows orthogonality and identifies the squared norms

$$\langle K_iK_j\rangle = \delta_{ij}\binom{N}{i}(pq)^i\ .$$

with $0\le i,j\le N$.

2.1. Matrix formulation.

Setting $\begin{pmatrix}y_0\\ y_1\end{pmatrix}=\begin{pmatrix}1 & p\\ 1 & -q\end{pmatrix}\begin{pmatrix}v_0\\ v_1\end{pmatrix}$ we have

$$y_0^{N-x}y_1^x = \sum_k v_0^{N-k}v_1^k\Phi_{kx}\ .$$

We call Φ a (the) *Kravchuk matrix* . The rows of the matrix Φ consist of the values taken on by the corresponding polynomials at the points x. The expression of orthogonality takes the form

$$\Phi B \mathrm{P} \Phi^\top = BD,$$

where B is the diagonal matrix with entries the binomial coefficients $\binom{N}{k}$, the matrix P is diagonal with entries $q^{N-k}p^k$ and D is the diagonal matrix with $D_{ii}=(pq)^i$, $0\le i\le N$.

3. Symmetric Tensor Powers

Given a $(d+1)\times(d+1)$ matrix A, the action on the symmetric tensor algebra of the underlying vector space defines its "second quantization" or *symmetric representation.*

Introduce commuting variables $v_0,\ldots,v_d$. Map

$$y_i=\sum_{j=0}^{d} A_{ij}v_j .$$

The induced matrix, $\bar{A}$, at level (homogeneous degree) $N=n_0+\cdots+n_d$ has entries $\bar{A}_{mn}$ determined by the expansion

$$y^m=y_0^{m_0}\cdots y_d^{m_d}=\sum_n \bar{A}_{mn}v^n\ .$$

Remark 3.1. Monomials are ordered according to dictionary ordering with 0 ranking first, followed by $1,2,\ldots,d$. Thus the first column of $\bar{A}$ gives the coefficients of v_0^N, etc.

The map $A\to\bar{A}$ is at each level a multiplicative homomorphism,

$$\overline{A_1A_2}=\bar{A}_1\,\bar{A}_2$$

thus implementing, for each $N\geq 0$, a representation of the multiplicative semigroup of $(1+d)\times(1+d)$ matrices into $\binom{N+d}{N}\times\binom{N+d}{N}$ matrices as well as a representation of the group $\mathrm{GL}(d+1)$ into $\mathrm{GL}(\binom{N+d}{N})$.

3.1. Transpose Lemma. An important lemma is the relation between the induced matrix of A with that of its transpose, $A^\top$.

Lemma 3.2. (Transpose Lemma) *The induced matrices at each level satisfy*

$$\overline{A^\top}=B^{-1}\bar{A}^\top B\ .$$

Remark 3.3. Proofs of the homomorphism property and Transpose Lemma are presented in [3].

Remark 3.4. (Diagonal matrices and multinomial distribution) Note that the N^{th} symmetric power of a diagonal matrix, D, is itself diagonal with homogeneous monomials of the entries of the original matrix along its diagonal. In particular, the trace will be the N^{th} homogeneous symmetric function in the diagonal entries of D.

Example 3.5. For $V=\begin{pmatrix} v_0 & 0 & 0\\ 0 & v_1 & 0\\ 0 & 0 & v_2\end{pmatrix}$ we have in degree 2,

$$\bar{V}=\begin{pmatrix} v_0^2 & 0 & 0 & 0 & 0 & 0\\ 0 & v_0v_1 & 0 & 0 & 0 & 0\\ 0 & 0 & v_0v_2 & 0 & 0 & 0\\ 0 & 0 & 0 & v_1^2 & 0 & 0\\ 0 & 0 & 0 & 0 & v_1v_2 & 0\\ 0 & 0 & 0 & 0 & 0 & v_2^2\end{pmatrix}$$

and so on.

Note that the special matrix B, diagonal with multinomial coefficients as entries along the diagonal may be obtained as the diagonal of the induced matrix at level N of the all 1's matrix.

We see that if p is a diagonal matrix with entries $p_\ell > 0$, $0 \le \ell \le d$, $\sum_\ell p_\ell = 1$, then the diagonal matrix

$$B\overline{\mathrm{p}}$$

yields the probabilities for the corresponding multinomial distribution.

4. Construction of Krawtchouk Polynomial Systems

We start with $\mathcal{O}$, a real orthogonal matrix with the extra condition that all entries in the first column are positive. Form the probability matrix thus

$$\mathrm{p} = \begin{pmatrix} \mathcal{O}_{00}^2 & & \\ & \ddots & \\ & & \mathcal{O}_{d0}^2 \end{pmatrix} = \begin{pmatrix} p_0 & & \\ & \ddots & \\ & & p_d \end{pmatrix}$$

row and column indices running from 0 to d.

Define

$$A = \frac{1}{\sqrt{\mathrm{p}}}\,\mathcal{O}\,\sqrt{D},$$

where D is diagonal with all positive entries on the diagonal, normalized by requiring $D_{00} = 1$. The essential property satisfied by A is

$$A^\top \mathrm{p} A = D,$$

while observing that the entries of the first column, label 0, are all 1's, i.e. $A_{\ell 0} = 1$, $0 \le \ell \le d$.

Definition 4.1. We say that A satisfies the K*-condition* if there exists a positive diagonal probability matrix p and a positive diagonal matrix D such that

$$A^\top \mathrm{p} A = D$$

with $A_{\ell 0} = 1$, $0 \le \ell \le d$.

Notation. Throughout the remainder of this work, if A satisfies the K-condition, we will denote its inverse by C. Thus,

$$C = A^{-1} = D^{-1} A^\top \mathrm{p}\ . \tag{4.1}$$

We note two useful properties

Proposition 4.2. *For A satisfying the K-condition we have*

1. $(p_0, p_1, \ldots, p_d)A = (1, 0, \ldots, 0) = e_0$. That is, the vector of probabilities $\{p_\ell\}$ times A yields e_0. We express this as

$$p_\mu A_{\mu\ell} = \delta_{0\ell}\ .$$

2. The first row of C is $(p_0, p_1, \ldots, p_d)$, i.e., $C_{0\ell} = p_\ell$.

Proof. Start with the observation that since the first column of A consists of all 1's, the first row of $A^\top$ is all 1's. So the first row of $A^\top \mathrm{p}$ is $(p_0, p_1, \dots, p_d)$.

Now, for #1, the row of probabilities times A is the first row of $A^\top \mathrm{p} A$, thus, the first row of D, which is precisely e_0.

For #2, using the form $D^{-1}A^\top \mathrm{p}$ for C, as in #1, the top row of $A^\top \mathrm{p}$ is the row of probabilities, and multiplication by D^{-1} leaves it unchanged, as $D_{00} = 1$. □

4.1. Krawtchouk systems. In any degree N, the induced matrix $\bar{A}$ satisfies

$$\overline{A^\top}\bar{\mathrm{p}}\bar{A} = \bar{D}\ .$$

Using the Transpose Lemma

$$B\overline{A^\top} = \bar{A}^\top B$$

where B is the special multinomial diagonal matrix yields

$$\Phi\, B\bar{\mathrm{p}}\, \Phi^\top = B\bar{D}$$

the Krawtchouk matrix Φ being thus defined as $\bar{A}^\top$.

The entries of Φ are the values of the *multivariate Krawtchouk polynomials* thus determined. $B\bar{D}$ is the diagonal matrix of squared norms according to the orthogonality of the Krawtchouk polynomial system with respect to the corresponding multinomial distribution.

Example 4.3. Start with the orthogonal matrix $\mathcal{O} = \begin{pmatrix} \sqrt{q} & \sqrt{p} \\ \sqrt{p} & -\sqrt{q} \end{pmatrix}$. Factoring out the squares from the first column yields

$$\mathrm{p} = \begin{pmatrix} q & 0 \\ 0 & p \end{pmatrix}$$

and we take

$$A = \begin{pmatrix} 1 & p \\ 1 & -q \end{pmatrix}$$

satisfying

$$A^\top \mathrm{p} A = \begin{pmatrix} 1 & 0 \\ 0 & pq \end{pmatrix} = D\ .$$

Take $N = 4$. We have the Kravchuk matrix $\Phi = \bar{A}^\top =$

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 4p & -q+3p & -2q+2p & -3q+p & -4q \\ 6p^2 & -3pq+3p^2 & q^2-4pq+p^2 & 3q^2-3pq & 6q^2 \\ 4p^3 & -3p^2q+p^3 & 2pq^2-2p^2q & -q^3+3pq^2 & -4q^3 \\ p^4 & -p^3q & p^2q^2 & -pq^3 & q^4 \end{pmatrix}.$$

p is promoted to the induced matrix

$$\bar{\mathrm{p}} = \begin{pmatrix} q^4 & 0 & 0 & 0 & 0 \\ 0 & q^3p & 0 & 0 & 0 \\ 0 & 0 & q^2p^2 & 0 & 0 \\ 0 & 0 & 0 & qp^3 & 0 \\ 0 & 0 & 0 & 0 & p^4 \end{pmatrix}.$$

and the binomial coefficient matrix $B = \mathrm{diag}(1, 4, 6, 4, 1)$.

Remark 4.4. This approach is presented in detail in [3]. Here we continue with an analytic approach based on operator calculus techniques.

5. Appell and Bernoulli Systems

An *Appell system* of polynomials is a sequence $\{\phi_n(x)\}_{n\geq 0}$ such that

(1) $\deg \phi_n = n$

(2) $\partial_x \phi_n = n\phi_{n-1}$ where $\partial_x = \dfrac{d}{dx}$.

Introduce the *raising operator*

$$\mathcal{R}\phi_n = \phi_{n+1} \ .$$

The pair $\partial_x, \mathcal{R}$ satisfy the commutation relations

$$[\partial_x, \mathcal{R}] = I$$

of the Heisenberg-Weyl algebra, i.e., boson commutation relations. Consider a convolution family of probability measures p_t, $p_t * p_s = p_{t+s}$, for $s, t \geq 0$, p_0 a point mass at 0, with corresponding family of moment generating functions

$$\int_{\mathbb{R}} e^{zx}\, p_t(dx) = e^{tH(z)}$$

where, extending z to complex values, we assume $H(z)$ to be analytic in a neighborhood of the origin in $\mathbb{C}$, with $H(0) = 0$.

Remark 5.1. Discrete values of t work in general, while for continuous $t \geq 0$ we require p_t to be infinitely divisible.

We have as generating function for the sequence $\{\phi_n\}$

$$e^{xz-tH(z)} = \sum_{n=0}^{\infty} \frac{z^n}{n!}\, \phi_n(x,t)$$

including the additional "time" variable. Note that $\phi_0(x,t) = 1$ with

$$\int_{\mathbb{R}} \phi_n(x,t)\, p_t(dx) = \delta_{0n}$$

for $n \geq 0$.

Remark 5.2. In the infinitely divisible case, we have the exponential martingale for the corresponding process with independent increments.

For the multivariate case, in the exponent, $xz = \sum x_i z_i$. We have $\partial_j = \partial/\partial x_j$, with $\mathcal{R}_i$ raising the index n_i to $n_i + 1$, satisfying

$$[\partial_j, \mathcal{R}_i] = \delta_{ij} 1$$

noting that the action of ∂_j is the same as multiplication by z_j and that the action of $\mathcal{R}_i$ is the same as $\partial/\partial z_i$.

5.1. Canonical Appell system. Now observe that if, in one variable, $V(z)$ is analytic in a neighborhood of $0 \in \mathbb{C}$, we can apply the operator $V(\partial)$ to polynomials in x and we have as well

$$V(\partial)\, e^{xz} = V(z)\, e^{xz}$$

acting on exponentials, for z in the domain of V. We have further the commutation relation

$$[V(\partial), x] = V'(\partial)$$

differentiating V. Next require that $V(0) = 0$, $V'(0) \neq 0$ so that V has a locally analytic inverse in a neighborhood of the origin as well, denoted by $U(v)$, $U(V(z)) = z$. This yields a *canonical pair*

$$\mathcal{V} = V(\partial) \qquad \text{and} \qquad \mathcal{R} = xW(\partial)$$

where $W(z) = 1/V'(z)$.

If we have an Appell system in several variables as above, we define canonical raising and *velocity* operators defined by

$$\mathcal{V}_j \phi_n = n_j \phi_{n-e_j} \qquad \text{and} \qquad \mathcal{R}_i \phi_n = \phi_{n+e_i}$$

satisfying

$$[\mathcal{V}_j, \mathcal{R}_i] = \delta_{ij}\, 1$$

where $\mathcal{V} = (\mathcal{V}_1, \dots, \mathcal{V}_d)$ is given by a function V of $\partial = (\partial_1, \dots, \partial_d)$, analytic in a neighborhood of 0 in $\mathbb{C}^d$, with a locally analytic inverse, U. The generating function becomes

$$e^{xz - tH(z)} = \sum_n \frac{V(z)^n}{n!}\, \phi_n(x,t)$$

with multi-index notation for the monomials in $V(z)$, and $n! = n_1! \cdots n_d!$ as usual. The generating function thus takes the equivalent form

$$e^{xU(v) - tH(U(v))} = \sum_n \frac{v^n}{n!}\, \phi_n(x,t)$$

with multiplication by v_j implemented as the operator $V_j(\partial)$ and $\partial/\partial v_j$ yielding the raising operator $\mathcal{R}_j$ after expressing the action in terms of ∂.

Example 5.3. For an example in one variable, take

$$V(z) = -\log(1-z)\,, \quad U(v) = 1 - e^{-v}\,, \quad W(z) = 1 - z$$

with no time variable we have

$$\exp\big(x(1 - e^{-v})\big) = \sum_{n \geq 0} \frac{\phi_n(x)}{n!}\, v^n$$

with action of the raising operator

$$\mathcal{R}\phi_n = x(1-\partial)\phi_n = \phi_{n+1}$$

and

$$\mathcal{V} = -\log(1-\partial) = \sum_{n \geq 1} \frac{\partial^n}{n}\ .$$

The coefficients of the polynomials ϕ_n are (up to sign) Stirling numbers of the second kind.

With $p(dx) = e^{-x}\,dx$ on $x \geq 0$, we have

$$\int_0^\infty e^{zx-x}dx = (1-z)^{-1} = e^{H(z)}$$

so $H(z) = -\log(1-z)$, which happens to equal $V(z)$. We get

$$e^{xz-tH(z)} = \exp\bigl(x(1-e^{-v})\bigr)\,e^{-tv} = \sum_{n\geq 0} \frac{\phi_n(x,t)}{n!}\,v^n$$

where now the raising operator is

$$\mathcal{R} = x(1-\partial) - t$$

leaving $\mathcal{V}$ unchanged. And we have for $t > 0$,

$$\int_0^\infty e^{-x}x^{t-1}\phi_n(x,t)\,dx/\Gamma(t) = \delta_{0n}$$

for $n \geq 0$, the family of measures p_t given by

$$p_t(dx) = e^{-x}x^{t-1}\,dx/\Gamma(t)$$

on $[0,\infty)$.

5.2. Bernoulli systems. A Bernoulli system is a canonical Appell system such that, for each t, the polynomials $\{\phi_n(x,t)\}$ form an orthogonal system with respect to the measure p_t. To indicate this, write J_n generically for the corresponding canonical Appell sequence, thus

$$e^{xU(v)-tH(U(v))} = \sum_{n\geq 0} \frac{v^n}{n!}\,J_n(x,t)\ .$$

Example 5.4. Probably the most well-known example are Hermite polynomials, $\{H_n\}$, with generating function

$$e^{xz-z^2t/2} = \sum_{n\geq 0} \frac{z^n}{n!}\,H_n(x,t)$$

orthogonal with respect to the Gaussian distribution with mean zero and variance t. Thus $H(z) = z^2/2$,

$$\mathcal{R} = x - t\partial \qquad \text{and} \qquad \mathcal{V} = \partial\ .$$

For an example with nontrivial V, consider a family of Poisson-Charlier polynomials with generating function $(1+v)^x\,e^{-tv}$. So

$$U(v) = \log(1+v)\,, \quad V(z) = e^z - 1\,, \quad W(z) = e^{-z}$$

with $H(z) = e^z - 1$, equal to $V(z)$ in this case. Thus

$$\mathcal{R} = xe^{-\partial} - t \qquad \text{and} \qquad \mathcal{V} = e^{\partial} - 1\ .$$

The polynomials are orthogonal with respect to the Poisson distribution on the nonnegative integers with mean t.

5.2.1. *Operator formulation.* We construct a representation space for the boson commutation relations starting with a *vacuum state*, Ω, satisfying $\mathcal{V}_j\Omega = 0$, $\forall j$. The basis states are built by acting with the raising operators $\mathcal{R}_i$ on the vacuum state, thus they are of the form $\mathcal{R}^n\Omega$, for multi-indices n, $n_i \geq 0$.

The operator form of the generating function is the exponential of the raising operators acting on the vacuum state, using the abbreviated notation $V(z)\mathcal{R} = \sum_i V_i(z)\mathcal{R}_i$:

$$e^{V(z)\mathcal{R}}\Omega = e^{xz-tH(z)} = \sum_{n\geq 0} \frac{V(z)^n}{n!}\, J_n(x,t)$$

where the vacuum state Ω is here $J_0(x,t)$, the constant function equal to 1.

Introducing the inverse function U, the generating function takes the form

$$e^{v\mathcal{R}}\Omega = e^{xU(v)-tH(U(v))} = \sum_{n\geq 0} \frac{v^n}{n!}\, J_n(x,t)$$

with the actions of $\{\mathcal{R}_i\}$ and $\{\mathcal{V}_j\}$ as

$$\mathcal{R}_i J_n = J_{n+e_i}\,, \qquad \mathcal{V}_j J_n = n_j J_{n-e_j}\ .$$

6. Quantization

We want a commuting family of selfadjoint operators to serve as quantum observables. Introduce the operators X_j, multiplication by the variables x_j. These will provide the desired operators.

Rewrite the generating function in the form

$$e^{zX}\Omega = e^{tH(z)}\, e^{V(z)\mathcal{R}}\,\Omega\ .$$

We start by differentiating with respect to z_j yielding the relation

$$X_j = t\,\frac{\partial H}{\partial z_j} + \sum_i \mathcal{R}_i\,\frac{\partial V_i}{\partial z_j}\ .$$

These act as operators by converting the z_j to the partial differentiation operators, ∂_j.

6.1. Specification of the system. Let's consider the various operators and features involved in specifying the Bernoulli system.

First, since we have a Hilbert space (in the present context, over $\mathbb{R}$), we want to find *lowering* operators $\{\mathcal{L}_1,\ldots,\mathcal{L}_d\}$, where, for each i, $\mathcal{L}_i$ is adjoint to $\mathcal{R}_i$. We wish to express all operators in terms of the canonical raising and velocity operators $\mathcal{R}_i$, $\mathcal{V}_j$.

Since we are working with noncommuting operators, it is of interest to study the Lie algebra generated by the raising and lowering operators.

With the $\mathcal{L}_j$ in hand, we will express X_j in manifestly self-adjoint form.

Some related constructions of interest will be considered as well, notably the Berezin transform, based on the inner product of coherent states generated by the raising operators. This information is summarized in the *Leibniz function* to be explained subsequently. These will provide tools to study the relationships among the lowering operators and the raising and velocity operators. We will find as well the Riccati partial differential equations satisfied by the velocity operators, a hallmark feature of Bernoulli systems and equations related to the Leibniz function/Berezin transform.

We start in the next two sections reviewing properties of multinomial distributions and the details of the Krawtchouk polynomials providing the basis states for the Bernoulli system.

7. Multinomial Distribution

First we describe the multinomial process we are interested in. The process is a counting process keeping track of d possible results, with the possibility that none of them occurs. Thus, at each time step the process makes one of $d+1$ choices:

1. With probability p_0, none of the levels 1 through d increase.
2. With probability p_i, $1 \le i \le d$, level i increases by 1.

The corresponding moment generating function for one time step is

$$\begin{aligned} p_0 + \sum_i p_i e^{z_i} &= 1 + \sum_i p_i (e^{z_i} - 1) \\ &= p_\mu e^{z_\mu} \end{aligned}$$

where we set $z_0 = 0$. The moment generating function for N steps is thus

$$e^{tH(z)} = (p_\mu e^{z_\mu})^N$$

where we identify

$$t = N \qquad \text{and} \qquad H(z) = \log\left(p_\mu e^{z_\mu}\right) \ .$$

8. Multivariate Krawtchouk Polynomials as Bernoulli Systems

We are given a matrix A satisfying the K-condition $A^\top \mathrm{p} A = D$. The Kravchuk matrix Φ is the transpose of the symmetric power of A. In degree N, we replace the index m by the variables $\{N - \sum x_i, x_1, \ldots, x_d\}$, where the system has d variables, the variable x_0 being determined by homogeneity, equivalently, in terms of the process, after N steps if you know $x_1, \ldots, x_d$, then x_0 is known. Thus,

$$(Av)^x = \sum_n v^n \Phi_{nx} = \sum_n \frac{v^n}{n!}\, K_n(x, N) \ .$$

More explicitly,

$$\begin{aligned} &(A_{0\mu} v_\mu)^{N - \sum x_i} (A_{1\mu} v_\mu)^{x_1} \cdots (A_{d\mu} v_\mu)^{x_d} \\ &\qquad = \sum \frac{v^n}{n!}\, K_n(x, N) \ . \end{aligned}$$

Recall that the first column of A consists of all 1's, and set $\alpha_0 = A_{00} = 1$, $\alpha_i = A_{0i}$, $1 \le i \le d$. We get

$$(\alpha_\lambda v_\lambda)^N \prod_i \left(\frac{A_{i\mu} v_\mu}{\alpha_\nu v_\nu} \right)^{x_i} = e^{xU(v) - NH(U(v))}$$

as the generating function for a Bernoulli system.

8.1. Identification of Bernoulli constituents. Let's determine the various Bernoulli parameters.

As seen in §7, we have $t = N$ and $H(z) = \log p_\mu e^{z_\mu}$. From the generating function, the coefficient of N shows that

$$H(z) = \log p_\mu e^{z_\mu} = \log(1/\alpha_\mu V_\mu(z)) \tag{8.1}$$

or

$$p_\mu e^{z_\mu} = \frac{1}{\alpha_\mu V_\mu(z)} \tag{8.2}$$

Looking at the coefficients of the variables x_i in the exponent, we identify U, the inverse to V, such that

$$U_k(v) = \log \frac{A_{k\mu} v_\mu}{\alpha_\nu v_\nu} \tag{8.3}$$

8.2. Canonical velocity operators. Now we can solve for the velocity operators $V_k(z)$. We have the inverse matrix

$$C = A^{-1} = D^{-1} A^\top \mathrm{p} \ .$$

Combine equations (8.3) and (8.2) to get

$$z_k = U_k(V) = \log\left(p_\lambda e^{z_\lambda} A_{k\mu} V_\mu\right).$$

Now exponentiate and move the p factor across

$$\frac{e^{z_k}}{p_\mu e^{z_\mu}} = A_{k\mu} V_\mu \tag{8.4}$$

and applying C to both sides we have

Proposition 8.1. *For the Krawtchouk Bernoulli system we have the canonical velocity operators*

$$V_k(z) = \frac{1}{p_\mu e^{z_\mu}}\, C_{k\lambda} e^{z_\lambda}$$

satisfying the Riccati partial differential equations

$$\frac{\partial V_i}{\partial z_j} = \left(C_{ij} - p_j V_i\right) A_{j\mu} V_\mu$$

for $1 \le i, j \le d$.

Proof. We need only verify the form of the differential equations. We have

$$\begin{aligned}\frac{\partial V_i}{\partial z_j} &= -\frac{p_j e^{z_j}}{p_\mu e^{z_\mu}}\, V_i + \frac{C_{ij} e^{z_j}}{p_\mu e^{z_\mu}} \\ &= (-p_j V_i + C_{ij}) \frac{e^{z_j}}{p_\mu e^{z_\mu}}\end{aligned}$$

and the result follows upon substituting the relation from equation (8.4). □

It is convenient to assign/adjoin *projective coordinates*, $v_0 = V_0 = 1$, and we have previously set $z_0 = 0$. To verify consistency, substitute $k = 0$ in the formula for V_k:

$$V_0(z) = \frac{1}{p_\mu e^{z_\mu}}\, C_{0\lambda} e^{z_\lambda} \ .$$

Now invoke Proposition 4.2, #2, to see that V_0 is identically equal to one.

8.3. Observables. We can now express the observables X_j in terms of the raising and velocity operators. Recall the relation

$$X_j = t\,\frac{\partial H}{\partial z_j} + \sum_i \mathcal{R}_i\,\frac{\partial V_i}{\partial z_j}$$

resulting by differentiating the generating function with respect to z_j.

Proposition 8.2. *The observables X_j have the form*

$$X_j = \left(N\,p_j + \sum_i \mathcal{R}_i(C_{ij} - p_j\mathcal{V}_i)\right)\,A_{j\mu}\mathcal{V}_\mu\ .$$

Proof. Observe that

$$\frac{\partial H}{\partial z_j} = \frac{p_j e^{z_j}}{p_\mu e^{z_\mu}} = p_j A_{j\mu}\mathcal{V}_\mu$$

by equation (8.4). Now apply Proposition 8.1 to get the result. □

9. Coherent States, Leibniz Function, and Lie Algebra

Now we want to find the lowering operators, the operators adjoint to the raising operators. $\mathcal{L}_i$ denotes the adjoint of $\mathcal{R}_i$. We employ techniques involving coherent states.

The generating function $e^{V\mathcal{R}}\Omega$ is a type of coherent state. The inner product of coherent states has the form

$$\Upsilon = \langle e^{B\mathcal{R}}\Omega, e^{V\mathcal{R}}\Omega\rangle = \phi(B_1V_1, \ldots, B_dV_d)$$

by orthogonality. Working with this we can find the lowering operators.

We have

$$\Upsilon = \langle \Omega, e^{B\mathcal{L}}e^{V\mathcal{R}}\Omega\rangle$$

equal to the vacuum expectation value of the group element $e^{B\mathcal{L}}e^{V\mathcal{R}}$. Comparing with the Heisenberg-Weyl group

$$e^{B\partial}e^{VX} = e^{VX}e^{BV}e^{B\partial}$$

we call Υ the *Leibniz function* of the system.

9.1. Finding the lowering operators. If we know the Leibniz function, we have the differential relations

$$\frac{\partial\Upsilon}{\partial V_i} = \langle e^{B\mathcal{R}}\Omega, \mathcal{R}_i e^{V\mathcal{R}}\Omega\rangle \quad \text{and} \quad \frac{\partial\Upsilon}{\partial B_i} = \langle e^{B\mathcal{R}}\Omega, \mathcal{L}_i e^{V\mathcal{R}}\Omega\rangle\ .$$

These are effectively the *Berezin transforms* of $\mathcal{R}_i$ and $\mathcal{L}_i$ respectively.

Thus to find the lowering operators, we wish to express the partial derivatives $\frac{\partial\Upsilon}{\partial B_i}$ in terms of V_i and $\frac{\partial\Upsilon}{\partial V_i}$. With the correspondence

$$\frac{\partial\Upsilon}{\partial V_i} \longleftrightarrow \mathcal{R}_i$$

we will have found the lowering operators in terms of the canonical raising and velocity operators.

9.2. Leibniz function for the Krawtchouk system. In our case, the generating function for the Krawtchouk polynomials is the coherent state we will use:

$$e^{V\mathcal{R}}\Omega = e^{xU(V)-tH(U(V))}$$

Multiplying by $e^{B\mathcal{R}}\Omega$ and averaging, we have

$$\langle e^{xU(V)+xU(B)}\rangle e^{-t(H(U(B))+H(U(V)))}$$

Recalling the moment generating function, using averaging notation,

$$\langle e^{xz}\rangle = e^{tH(z)}$$

we find in the exponent t times

$$H(U(B)+U(V)) - H(U(B)) - H(U(V)) = \psi(BV) = \psi(B_1V_1,\ldots,B_dV_d)$$

thus defining ψ, where we use the fact that we have an orthogonal system.

Proposition 9.1. *The Leibniz function for the Krawtchouk system is given by*

$$\Upsilon = \langle e^{B\mathcal{R}}\Omega, e^{V\mathcal{R}}\Omega\rangle = (B_\mu D_\mu V_\mu)^N$$

where $B_0 = V_0 = 1$ and $D_i = D_{ii}$ are the diagonal entries of D.

Proof. We will show that the function ψ above is given by

$$\psi(BV) = \log B_\mu D_\mu V_\mu\ .$$

By equation (8.1), we have

$$H(U(V)) = \log(1/\alpha_\mu V_\mu) \qquad \text{and} \qquad H(U(B)) = \log(1/\alpha_\mu B_\mu)\ . \tag{9.1}$$

Now, using equations (8.1) and (8.3), we have

$$\begin{aligned} H(U(V)+U(B)) &= \log(p_\mu e^{U_\mu(V)} e^{U_\mu(B)}) \\ &= \log\left(p_\mu \frac{A_{\mu\lambda}V_\lambda}{\alpha_\sigma V_\sigma}\frac{A_{\mu\nu}B_\nu}{\alpha_\epsilon B_\epsilon}\right) \end{aligned}$$

Rewriting this last in the form

$$\log\left(\frac{V_\lambda (A^\top \mathrm{p} A)_{\lambda\nu} B_\nu}{\alpha_\sigma V_\sigma\, \alpha_\epsilon B_\epsilon}\right)$$

invoke the K-condition, $A^\top \mathrm{p}A = D$ and bring in equation (9.1) yielding

$$\log(V_\lambda D_\lambda B_\lambda) + H(U(V)) + H(U(B))$$

from which the form of ψ follows. Exponentiating and raising to the power N then gives the result. □

9.3. Lowering operators for the Krawtchouk system and Lie algebra. We are now in a position to determine the lowering operators $\mathcal{L}_i$.

Proposition 9.2. *The Leibniz function Υ satisfies the partial differential equations*

$$\frac{1}{D_i}\frac{\partial\Upsilon}{\partial B_i} = NV_i\Upsilon - V_i\sum V_j\frac{\partial\Upsilon}{\partial V_j} \ .$$

Proof. First, for the left hand side

$$\frac{1}{D_i}\frac{\partial\Upsilon}{\partial B_i} = \frac{NV_i}{B_\mu V_\mu D_\mu}\Upsilon \ .$$

Now calculate

$$\sum_j V_j\frac{\partial\Upsilon}{\partial V_j} = N\sum_j \frac{B_jD_jV_j}{B_\mu V_\mu D_\mu}\Upsilon = N\,\frac{B_\nu V_\nu D_\nu - 1}{B_\mu V_\mu D_\mu}\Upsilon = N(1-\frac{1}{B_\mu V_\mu D_\mu})\Upsilon$$

taking out the term $B_0V_0D_0 = 1$. Hence

$$N\Upsilon - \sum_j V_j\frac{\partial\Upsilon}{\partial V_j} = \frac{N}{B_\mu V_\mu D_\mu}\Upsilon$$

and multiplying through by V_i yields the result. □

Re-interpreting the derivatives $\partial\Upsilon/\partial V_i$ as raising operators $\mathcal{R}_i$ yields

Corollary 9.3. *The lowering operators have the form*

$$\mathcal{L}_i = D_i\big(N - \sum_{j=1}^{d}\mathcal{R}_j\mathcal{V}_j\big)\,\mathcal{V}_i \ .$$

9.3.1. *Lie algebra.* Introduce the *number operator* $\mathcal{N} = \sum_k \mathcal{R}_k\mathcal{V}_k$, satisfying

$$\mathcal{N}\mathcal{R}^n\Omega = |n|\mathcal{R}^n\Omega \ .$$

We can write the above result in a convenient form.

Proposition 9.4. *In terms of the number operator $\mathcal{N}$, we have*

$$\mathcal{L}_i = D_i(N-\mathcal{N})\mathcal{V}_i \ .$$

Note the commutation relations

$$[\mathcal{N},\mathcal{R}_i] = \mathcal{R}_i \qquad \text{and} \qquad [\mathcal{V}_j,\mathcal{N}] = \mathcal{V}_j \ .$$

Next, form the d^2 operators

$$\rho_{ij} = [\mathcal{L}_i,\mathcal{R}_j] \ .$$

Proposition 9.5. *We have*

1. $\rho_{ii} = D_i(N - \mathcal{R}_i\mathcal{V}_i - \mathcal{N})$.
2. *For $i \neq j$, $\rho_{ij} = -D_i\mathcal{R}_j\mathcal{V}_i$.*

Proof. If $i \neq j$, then $\mathcal{R}_j$ and $\mathcal{V}_i$ commute so that

$$[\mathcal{L}_i,\mathcal{R}_j] = -D_i[\mathcal{N},\mathcal{R}_j]\mathcal{V}_i = -D_i\mathcal{R}_j\mathcal{V}_i$$

as stated. For $i = j$, we get

$$[\mathcal{L}_i,\mathcal{R}_i] = D_i(N - [\mathcal{N},\mathcal{R}_i]\mathcal{V}_i - \mathcal{N}) = D_i(N - \mathcal{R}_i\mathcal{V}_i - \mathcal{N})$$

as required. □

Denoting adjoint by * we note that

$$\rho_{ij}* = \rho_{ji}$$

and that ρ_{ii}, $\mathcal{N}$, and $\mathcal{R}_i\mathcal{V}_i$ are all selfadjoint.

For a dimension count, we have d^2 operators ρ_{ij} plus the $2d$ raising and lowering operators which yields a Lie algebra of dimension $d^2 + 2d = (d+1)^2 - 1$. Thus, we have a copy of $\mathfrak{sl}(d+1)$.

10. Observables

Going back to the observables, we can express the operators X_j in manifestly selfadjoint form.

Proposition 10.1. *For $1 \le j \le d$, we have*

$$X_j = \sum_{1\le i\le d} (\mathcal{R}_i + \mathcal{L}_i)\, C_{ij} + (N - \mathcal{N}) - \frac{1}{p_j} \sum_{\substack{1\le i\le d \\ 1\le k\le d}} C_{ij}C_{kj}\rho_{ik}.$$

First we need some basic identities

Lemma 10.2. *With $C = A^{-1}$, we have*

$$p_j A_{ji} = D_i C_{ij}$$

Note that this is an explicit form of the matrix relation $DC = A^\top \mathrm{p}$, cf. equation (4.1).

Proof. Recall, Proposition 8.2,

$$X_j = \left(N\, p_j + \sum_i \mathcal{R}_i(C_{ij} - p_j\mathcal{V}_i) \right) A_{j\mu}\mathcal{V}_\mu$$

and note that

$$A_{j\mu}\mathcal{V}_\mu = A_{j0}\mathcal{V}_0 + \sum_{k=1}^{d} A_{jk}\mathcal{V}_k = 1 + \sum_k A_{jk}\mathcal{V}_k \ .$$

We get, using the above Lemma, and Proposition 9.4,

$$\begin{aligned}
&(p_j(N-\mathcal{N}) + \sum_i \mathcal{R}_i C_{ij})(\sum_k A_{jk}\mathcal{V}_k + 1) \\
&\qquad = p_j(N-\mathcal{N}) + (N-\mathcal{N})\sum_i D_iC_{ij}\mathcal{V}_i + (\sum_i \mathcal{R}_iC_{ij})(\sum_k A_{jk}\mathcal{V}_k + 1) \\
&\qquad = \sum_i (\mathcal{R}_i + \mathcal{L}_i)C_{ij} + p_j(N-\mathcal{N}) + \sum_{i,k} C_{ij}A_{jk}\mathcal{R}_i\mathcal{V}_k \ . \qquad (10.1)
\end{aligned}$$

In the last sum, for $i \ne k$, we have $\mathcal{R}_i\mathcal{V}_k = -(1/D_k)\rho_{ki}$. We get

$$-\sum_{i\ne k} \frac{1}{D_k} C_{ij}A_{jk}\rho_{ki} = -\frac{1}{p_j}\sum_{i\ne k} C_{ij}C_{kj}\rho_{ki} \qquad (10.2)$$

as in Lemma 10.2. For $i = k$, we have from Proposition 9.5,

$$\mathcal{R}_i \mathcal{V}_i = N - \mathcal{N} - \frac{1}{D_i}\,\rho_{ii}$$

and

$$\begin{aligned}\sum_i C_{ij} A_{ji} \mathcal{R}_i \mathcal{V}_i &= \sum_i C_{ij} A_{ji} (N - \mathcal{N} - \frac{1}{D_i}\,\rho_{ii}) \\ &= \sum_i C_{ij} A_{ji} (N - \mathcal{N}) - \frac{1}{p_j} \sum_i C_{ij} C_{ij} \rho_{ii} \; . \end{aligned} \tag{10.3}$$

Finally, observe that

$$\sum_i C_{ij} A_{ji} = C_{\mu j} A_{j\mu} - C_{0j} A_{j0} = 1 - p_j$$

recalling Proposition 4.2. Combining equations (10.1), (10.2) and (10.3) we arrive at the desired form. □

10.1. Recurrence formulas. Now returning to the form of the X_j in terms of the canonical raising and velocity operators

$$X_j = \big(N\,p_j + \sum_i \mathcal{R}_i (C_{ij} - p_j \mathcal{V}_i)\big)\, A_{j\mu} \mathcal{V}_\mu$$

we see that these yield recurrence formulas for the multivariate Krawtchouk polynomial system.

Proposition 10.3. *The Krawtchouk polynomials satisfy the following recurrence relations*

$$\begin{aligned} x_j\, K_n(x,N) = p_j (N - |n|)\, K_n &+ \sum_i C_{ij} K_{n+e_i} + p_j N \sum_k A_{jk} n_k\, K_{n-e_k} \\ &- p_j \sum_k A_{jk} (|n|-1) n_k K_{n-e_k} + \sum_{i,k} C_{ij} A_{jk} n_k\, K_{n-e_k+e_i} \; . \end{aligned}$$

Proof. Starting with $A_{j\mu}\mathcal{V}_\mu = 1 + \sum_k A_{jk} \mathcal{V}_k$ as in the above proof, expand out the formula for X_j. Applying to K_n yields the result. □

11. Conclusion

The Krawtchouk polynomials and their multivariable generalizations provide models involving a wide spectrum of mathematical objects. Among the most interesting aspects from the present point of view are the representations of Lie algebras on spaces of polynomials and the quantization aspects, especially the connections with quantum probability.

The first part of this work, KG-Systems I, emphasizes the linear algebra and numerical aspects of these systems, while KG-Systems II, shows these systems in the analytic setting of a discrete quantum system, albeit over the reals.

One expects these classes of polynomials to be useful in coding theory while applications in image compression as well as quantum computation would certainly not be wholly unexpected. And from the theoretical point of view how these systems behave under various limit theorems analogous to the classical Poisson and Central limit theorems provides an area for interesting further study.

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