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Theory and Application of the Discrete Version of Generalizedα-Bernoullis Formula

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Abstract : In this paper, we derive the discrete version of the Bernoulli's formula according to the generalized α- difference operator $\Delta_{\alpha(\ell)}$ and to find the formula for the sum of the series of the product of polynomials and polynomial factorials in the field of Numerical Analysis. Suitable examples are provided to illustrate the series.

Keywords : Generalized α- Difference Operator, Polynomial, Polynomial factorial. Mathematics Subject Classification: 39A10, 39A11, 39A13.

1. INTRODUCTION

The theory of difference equations is based on the operator Δ defined as

$$\Delta u(k) = u(k+1) - u(k), k \in \mathbb{N}(0) = \{0,1,2,...\}. \tag{1}$$

Even though authors [1,10-13] have suggested the definition of Δ as

$$\Delta u(k) = u(k+\ell) - u(k),\ k \in [0,\infty), \ell \in (0,\infty), \tag{2}$$

and no significant progress took place on this line. Recently in [6] they took up the definition of Δ as given in (2), and developed the theory of difference equations in a different direction and many interesting results were obtained in number theory. For convenience, they labelled the operator Δ defined by (2) as Δ_ℓ and its inverse by Δ_ℓ^{-1}. When Δ_ℓ is operated on a complex function $u(k)$ and considering ℓ to be real, some new qualitative properties like rotatory, expanding, shrinking, spiral and weblike were noticed. The results obtained can be found in [2]. After that, we extend from the generalized difference operator Δ_ℓ to the generalized difference operator of the n^{th} kind Δ_ℓ and find to the formula for several types of arithmetic series using its inverse and striling numbers of first and second kinds respectively in the field of Numerical methods [8,9].

Jerzy Popenda [5], while discussing the behavior of solutions of a particular type of difference equation, defined Δ_α as $\Delta_\alpha u(k) = u(k+1) - \alpha u(k)$. This definition of Δ_α is being ignored for a long time. In [8] have generalized the definition of Δ_α by $\Delta_{\alpha(\ell)}$ defined as $u(k) = u(k+\ell) - \alpha u(k)$ for the real valued function $u(k)$ and $\ell \in (0,\infty)$ and also obtained the solutions of certain types of generalized α-difference equations, in particular, the generalized Clairaut's α-difference equation, generalized Euler α-difference equation and the generalized α-Bernoulli polynomial $B_{\alpha(n)}(k,\ell)$, which is a solution of the α-difference equation $u(k+\ell) - \alpha u(k) = nk^{n-1}$, for $n \in \mathbb{N}(1)$ ([7,9]).

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In [4], G.B.A.Xavier, et.al., extended from the definition of generalized α-difference operator of n^{th} kind and to obtain the formula for sum of partial sums of various types of arithmetic-geometric progression in the field of Numerical Analysis. Hence, in this paper we derive the generalized discrete α-Bernoulli's formula and to obtain the formula for sum of several types of arithmetic and geometric series using the Stirling numbers of first and second kind respectively.

Throughout this paper, we make use of the following notations:

1. $\left[\frac{k}{\ell}\right]$ means integer part of $\frac{k}{\ell}$,
2. $N_\ell(j) = \{j, \ell + j, 2\ell + j, \cdots\}$,
3. $L = \{\ell_1, \ell_2, ...\ell_n\}$, where $\ell_1, \ell_2, \cdots, \ell_n$ are positive reals,
4. $0(L) = \varphi, \varphi$ denotes the empty set
5. $\wp(L) = \bigcup_{t=0}^{n} t(L)$, power set of L and
6. $t(L)$ denotes the set of all subsets of size t from the set L

2. PRELIMINARIES

In this section, we present some basic definitions and preliminary results which will be useful for further subsequent discussions.

Definition 2.1. [6] For a real valued function $u(k)$, the generalized difference operator Δ_ℓ and its inverse on $u(k)$ are respectively defined as

$$\Delta_\ell u(k) = u(k+\ell) - u(k), k \in [0, \infty), \ell \in (0, \infty), \tag{3}$$

and if $\Delta_\ell v(k) = u(k)$, then $v(k) = \Delta_\ell^{-1} u(k) + c_j$,

where c_j is constant for all $k \in N_\ell(j)$.

Definition 2.2. [8] If $u(k)$ is positive real valued function then the generalized α-difference operator is defined by

$$\Delta_{\alpha(\ell)} u(k) = u(k+\ell) - \alpha u(k), \alpha > 0, \ell \in (0, \infty) \tag{5}$$

and inverse is defined by

$$\Delta_{\alpha(\ell)}^{-1} u(k) = v(k) - \alpha^{\left[\frac{k}{\ell}\right]} v(j), \tag{6}$$

where $v(j)$ is constant for all $k \in N_\ell(j)$.

Theorem 2.3. [6] If $\ell \in (0, \infty)$ and $k \in N_\ell(j)$, then

$$\Delta_\ell^{-1} u(k)\Big|_j^k = \sum_{r=1}^{\left[\frac{k}{\ell}\right]} u(k - r\ell) \tag{7}$$

Lemma 2.4. [6] If s_r^n and S_r^n are the Stirling numbers of the first, second kinds an $k_\ell^{(n)} = k(k-\ell)(k-2\ell)\cdots(k-(n-1)\ell)$, then

$$k_\ell^{(n)} = \sum_{r=1}^{n} s_r^n \ell^{n-r} k^r, \; k^n = \sum_{r=1}^{n} S_r^n \ell^{n-r} k_\ell^{(r)} \text{ and } \Delta_\ell^{-1} k_\ell^{(n)} = \frac{k_\ell^{(n+1)}}{(n+1)\ell}. \tag{8}$$

Theorem 2.5. [3] If $n \in N(1)$ and $k \in [n\ell, \infty)$, then

$$\Delta_\ell^{-n} u(k)\Big\|_{(n-1)\ell+j}^{k} = \sum_{r=n}^{\left[\frac{k}{\ell}\right]} \frac{(r-1)^{(n-1)}}{(n-1)^{(n-1)}} u(k - n\ell + r\ell). \tag{9}$$

Lemma 2.6. [4] For $k \in [n\ell, \infty)$,

$$\Delta_{\alpha(\ell)}^{-n} u(k) \|_{(n-1)\ell+j}^{k} = \sum_{r=n}^{\left[\frac{k}{\ell}\right]} \frac{(r-1)^{(n-1)}}{(n-1)}^{(n-1)} \alpha^{r-n} u(k - r\ell). \quad (10)$$

Lemma 2.7. [8] Let $u(k)$ and $v(k) \neq 0$ be two real valued functions. Then,

$$\Delta_{\alpha(\ell)}[u(k)v(k)] = u(k)\Delta_{\alpha(\ell)}v(k) + v(k+\ell)\Delta_{\ell}u(k). \quad (11)$$

and inverse is defined by $$\Delta_{\alpha(\ell)}^{-1}[u(k)v(k)] = u(k)\Delta_{\alpha(\ell)}^{-1}v(k) - \Delta_{\alpha(\ell)}^{-1}[\Delta_{\alpha(\ell)}^{-1}v(k+\ell)\Delta_{\ell}u(k)]. \quad (12)$$

3. APPLICATIONS OF GENERALIZED DISCRETE Á-BERNOULLI'S FORMULA

In this section, we derive the discrete α-Bernoulli's formula establish the sum of general partial sums of products of polynomials and polynomial factorials using the inverse of generalized α-difference operator and stirling numbers of first kind and second kind respectively.

The following theorem is the generalized version of discrete α- Bernoulli's formula according to $\Delta_{\alpha(\ell)}$.

Theorem 3.1. Let $u_i(k)$, for $i = 1, 2, \cdots, m$ be the positive real valued functions. Then

$$\Delta_{\alpha(\ell)}^{-n}\left[\prod_{i=1}^{m} u_i(k)\right] \|_{(n-1)\ell+j}^{k} = \Delta_{\alpha(\ell)}^{-n}\left[\prod_{i=1}^{m} u_i(k)\right] |_{(n-1)\ell+j}^{k}$$

$$+ \sum_{t=1}^{n-1} \sum_{\{n_1, n_2, \cdots, n_t\} \in t(L_{n-1})} \Delta_{\alpha(\ell)}^{-t}\left[\prod_{i=1}^{m} u_i(t-1)\ell + j\right] \frac{(-1)^t \left([\frac{k}{\ell}]\right)^{(n-n_t)} \alpha^{[\frac{k}{\ell}]-(n-1)}}{(n-n_t)!} |_{(n-1)\ell+j}^{k}. \quad (13)$$

Proof. From the Definition 2.2, we have

$$\Delta_{\alpha(\ell)}^{-1}\left[\prod_{i=1}^{m} u_i(k)\right] |_{j}^{k} = \Delta_{\alpha(\ell)}^{-1}\left[\prod_{i=1}^{m} u_i(k)\right] - \alpha^{\left[\frac{k}{\ell}\right]} \Delta_{\alpha(\ell)}^{-1}\left[\prod_{i=1}^{m} u_i(j)\right].$$

where $\Delta_{\alpha(\ell)}^{-1}\left[\prod_{i=1}^{m} u_i(k)\right]$ is a function of *k* and $\Delta_{\alpha(\ell)}^{-1}\left[\prod_{i=1}^{m} u_i(j)\right]$ is constant. Again taking $\Delta_{\alpha(\ell)}^{-1}$ and applying the limit from $\ell + j$ to *k*, we obtain

$$\Delta_{\alpha(\ell)}^{-2}\left[\prod_{i=1}^{m} u_i(k)\right] \|_{\ell+j}^{k} = \Delta_{\alpha(\ell)}^{-2}\left[\prod_{i=1}^{m} u_i(k)\right] |_{\ell+j}^{k} - \Delta_{\alpha(\ell)}^{-1}\left[\prod_{i=1}^{m} u_i(j)\right] [\frac{k}{\ell}] \alpha^{[\frac{k}{\ell}]-1} |_{\ell+j}^{k},$$

which is same as $$\Delta_{\alpha(\ell)}^{-2}\left[\prod_{i=1}^{m} u_i(k)\right] \|_{\ell+j}^{k} = \Delta_{\alpha(\ell)}^{-2}\left[\prod_{i=1}^{m} u_i(k)\right] - \alpha^{\left[\frac{k}{\ell}\right]} \Delta_{\alpha(\ell)}^{-2}\left[\prod_{i=1}^{m} u_i(\ell + j)\right]$$

$$+ \sum_{\{n_1\} \in 1(L_1)} \Delta_{\alpha(\ell)}^{-1}\left[\prod_{i=1}^{m} u_i(j)\right] \frac{[\frac{k}{\ell}]^{(2-n_1)} \alpha^{[\frac{k}{\ell}]-(2-1)}}{(2-n_1)!} |_{\ell+j}^{k}.$$

Similarly again operating $\Delta_{\alpha(\ell)}^{-1}$ on both sides and applying the limit from $2\ell + j$ to *k* and which can be expressed as

$$\Delta_{\alpha(\ell)}^{-3}\left[\prod_{i=1}^{m} u_i(k)\right] \|_{2\ell+j}^{k} = \Delta_{\alpha(\ell)}^{-3}\left[\prod_{i=1}^{m} u_i(k)\right] - \alpha^{\left[\frac{k}{\ell}\right]} \Delta_{\alpha(\ell)}^{-3}\left[\prod_{i=1}^{m} u_i(2\ell + j)\right]$$

$$+ \sum_{t=1}^{2} \sum_{\{n_1, n_2, \cdots, n_t\} \in t(L_{n-1})} \Delta_{\alpha(\ell)}^{-t}\left[\prod_{i=1}^{m} u((t-1)\ell + j)\right] \frac{(-1)^t \left([\frac{k}{\ell}]\right)^{(3-n_t)} \alpha^{[\frac{k}{\ell}]-(3-1)}}{(3-n_t)!} |_{2\ell+j}^{k}.$$

The proof completes by continuing this process.

Theorem 3.2. Let $L_{n-1}=\{1,2,\cdots,(n-1)\}$ and $t(L_{n-1})$ be the size t from the set L_{n-1}. Then

$$\sum_{r=n}^{[\frac{k}{\ell}]}\frac{(r-1)^{(n-1)}}{(n-1)!}\alpha^{r-n}\left[\prod_{i=i}^{m}u_i(k-r\ell)\right] = \Delta_{\alpha(\ell)}^{-n}\left[\prod_{i=1}^{m}u_i(k)\right]\Big|_{(n-1)\ell+j}^{k}$$

$$+\sum_{t=1}^{n-1}\sum_{\{n_1,n_2,\cdots,n_t\}\in t(L_{n-1})}\Delta_{\alpha(\ell)}^{-t}\left[\prod_{i=1}^{m}u_i(t-1)\ell+j\right]\frac{(-1)^t\left([\frac{k}{\ell}]\right)^{(n-n_t)}\alpha^{[\frac{k}{\ell}]-(n-1)}}{(n-n_t)!}\Big|_{(n-1)\ell+j}^{k}. \quad (14)$$

Proof. The proof follows by equating (10) and (13).

Theorem 3.3. If $n_i, i=1,2,\cdots,m$ and $t_r, r=1,2,\cdots,n$ are the positive integers and $t_0=0$, then

$$\Delta_{\alpha(\ell)}^{-n}\left[\prod_{i=1}^{m}(k+r_i\ell)_\ell^{(n_i)}\right]\Big|\Big|_{((n-1)\ell+j)}^{k} = \prod_{r=1}^{n}\sum_{t_r=0}^{n_1-t_{r-1}}(-1)^{t_n}(n_1-r)^{(t_r)}\ell^{t_r+t_n}(n_1-t_n)^{(t_n)}$$

$$(k+r_1\ell)_\ell^{(n_1-\sum_{p=1}^{n}t_p)}\Delta_{\alpha(\ell)}^{-(n+\sum_{p=1}^{n}t_p)}\prod_{i=1}^{m-1}\left[(k+(r_i+\sum_{p=1}^{n}t_p)\ell)_\ell^{(n_i)}\right]\Big|_{(n-1)\ell+j}^{k}$$

$$+\sum_{t=1}^{n-1}\sum_{\{n_1,\cdots,n_t\}\in t(L_{n-1})}\Delta_{\alpha(\ell)}^{-t}\left[\prod_{i=1}^{m}\left(k+(r_i+(t-1))\ell+j\right)\right](-1)^t\frac{\left(\left[\frac{k}{\ell}\right]\right)^{(n-n_t)}\alpha^{\left[\frac{k}{\ell}\right]-(n-1)}}{(n-n_t)!}\Big|_{(n-1)\ell+j}^{k}. \quad (15)$$

Proof. The proof follows by taking $u_i(k)=(k+r_i\ell)_\ell^{(n_i)}$, for $i=1,2,\cdots,m$ in (13).

Theorem 3.4. Let $k_\ell^{(n)}$ be the generalized polynomial factorial. Then,

$$\sum_{r=n}^{[\frac{k}{\ell}]}\frac{(r-1)^{(n-1)}}{(n-1)!}\alpha^{r-n}\left[\prod_{i=1}^{m}(k+r_i\ell-r\ell)_\ell^{(n_i)}\right] = \prod_{r=1}^{n}\sum_{t_r=0}^{n_1-t_{r-1}}(-1)^{t_n}(n_1-r)^{(t_r)}\ell^{t_r+t_n}(n_1-t_n)^{(t_n)}$$

$$(k+r_1\ell)_\ell^{(n_1-\sum_{p=1}^{n}t_p)}\Delta_{\alpha(\ell)}^{-(n+\sum_{p=1}^{n}t_p)}\prod_{i=1}^{m-1}\left[(k+(r_i+\sum_{p=1}^{n}t_p)\ell)_\ell^{(n_i)}\right]\Big|_{(n-1)\ell+j}^{k}$$

$$+\sum_{t=1}^{n-1}\sum_{\{n_1,\cdots,n_t\}\in t(L_{n-1})}\Delta_{\alpha(\ell)}^{-t}\left[\prod_{i=1}^{m}\left(k+r_i\ell+(t-1)\ell+j\right)\right](-1)^t\frac{\left(\left[\frac{k}{\ell}\right]\right)^{(n-n_t)}\alpha^{\left[\frac{k}{\ell}\right]-(n-1)}}{(n-n_t)!}\Big|_{(n-1)\ell+j}^{k}. \quad (16)$$

Proof. Substituting $u(k)=\prod_{i=1}^{m}(k+r_i\ell)_\ell^{(n_i)}$ in (10), we get

$$\Delta_{\alpha(\ell)}^{-n}\left[\prod_{i=1}^{m}(k+r_i\ell)_\ell^{(n_i)}\right]\Big|\Big|_{(n-1)\ell+j}^{k} = =\sum_{r=n}^{[\frac{k}{\ell}]}\frac{(r-1)^{(n-1)}}{(n-1)^{(n-1)}}\alpha^{r-n}\left[\prod_{i=1}^{m}(k+r_i\ell-r\ell)_\ell^{(n_i)}\right] \quad (17)$$

The proof follows by equating (15) and (17)

Corollary 3.5. Let $k \in [0,\infty)$ and $j = k-[k/\ell]\ell$. Then,

$$\sum_{r=3}^{[\frac{k}{\ell}]}\frac{(r-1)^{(2)}}{2}\alpha^{r-3}(k+(r_1-r)\ell)_\ell^{(2)}(k+(r_2-r)\ell)_\ell^{(3)} = \sum_{p=0}^{2}(-1)^p 6(p!)\ell^p$$

$$\Bigg\{\left[\frac{(k+(r_2+p)\ell)^3-3\ell(k+(r_2+p)\ell)^2+2\ell^2(k+(r_2+p)\ell)}{(1-\alpha)^{3+p}}\right]$$

$$-\left[\frac{3\ell(3+p)(k+(r_2+p)\ell)^2-3\ell^2(3+p)(k+(r_2+p)\ell)}{(1-\alpha)^{4+p}}\right]$$

$$+\left[\frac{3\ell^2(4+p)^{(2)}(k+(r_2+p)\ell)-6\ell^3}{(1-\alpha)^{5+t}}\right]-\left[\frac{9\ell^3(3+p)^{(2)}}{(1-\alpha)^{6+p}}\right]\Bigg\}\Big|_{2\ell+j}^{k}. \quad (18)$$

Proof. Substituting $n=3, m=2, n_1=2, n_2=3,$ in (16), we get (18).

Example 3.6. Taking $k=21$, $\ell=2$, $j=1$, $r_1=4$, $r_2=5$ and $\alpha=3$, in (18), we arrive

$$\sum_{r=3}^{10}\frac{(r-1)^{(2)}}{2}3^{r-3}(29-2r)_2^{(2)}(31-2r)_2^{(3)} = (521372.25)-3^{10}(-173470.3148)$$

$$= 1.024376999\times 10^{10}$$

Theorem 3.7. Let $k \in [0,\infty)$ and $j = k-[k/\ell]\ell$. Then,

$$\Delta_{\alpha(\ell)}^{-n}\left[\prod_{i=1}^{m}(k+r_i\ell)^{n_i}\right] = \prod_{r=1}^{n}\sum_{t_r=0}^{n_1-t_{r-1}}(-1)^{t_n}(n_1-r)^{(t_r)}\ell^{(t_r+t_n)}(n_1-t_n)^{(t_n)}$$

$$\Delta_{\alpha(\ell)}^{-\left(n+\sum_{p=1}^{n}t_p\right)}\left(\sum_{q=1}^{n_1}s_q^{n_1}\ell^{n_1-q}(t_1)^{(t_r)}\ell^{t_r}(k+r_1\ell)_\ell^{(q-\sum_{p=1}^{n}t_p)}\right)\prod_{i=1}^{m-1}\left(k+(r_i+\sum_{p=1}^{n}t_p)\ell\right)^{n_i}\Big|_{(n-1)\ell+j}^{k}$$

$$+\sum_{t=1}^{n-1}\sum_{\{n_1,\cdots,n_t\}\in t(L_{n-1})}\Delta_{\alpha(\ell)}^{-t}\left[\prod_{i=1}^{m}\big(k+(r_i+(t-1)\ell)\ell+j\big)\right](-1)^t$$

$$\frac{\left(\left[\frac{k}{\ell}\right]\right)^{(n-n_t)}\alpha^{\left[\frac{k}{\ell}\right]-(n-1)}}{(n-n_t)!}\Big|_{(n-1)\ell+j}^{k}. \quad (19)$$

Proof. (19) follows by substituting $u_i(k)=(k+r_i\ell)^{n_i}$, for $i=1,2,\cdots,m$ in (13).

Theorem 3.8. If k^n is the polynomial of degree n, then

$$\sum_{r=n}^{[\frac{k}{\ell}]}\frac{(r-1)^{(n-1)}}{(n-1)^{(n-1)}}\alpha^{r-n}\left[\prod_{i=1}^{m}(k+(r_i-r)\ell)^{n_i}\right] = \prod_{r=1}^{n}\sum_{t_r=0}^{n_1-t_{r-1}}(-1)^{t_n}(n_1-r)^{(t_r)}\ell^{(t_r+t_n)}(n_1-t_n)^{(t_n)}$$

$$\left(\sum_{q=1}^{n_1}s_q^{n_1}\ell^{n_1-q}(t_1)^{(t_r)}\ell^{t_r}(k+r_1\ell)_\ell^{(q-\sum_{p=1}^{n}t_p)}\right)\Big|_{(n-1)\ell+j}^{k}$$

$$+\sum_{t=1}^{n-1}\sum_{\{n_1,\cdots,n_t\}\in t(L_{n-1})}\Delta_{\alpha(\ell)}^{-\left(n+\sum_{p=1}^{n}t_p\right)}\prod_{i=1}^{m-1}\left(k+(r_i+\sum_{p=1}^{n}t_p)\ell\right)^{n_i}\Big|_{(n-1)\ell+j}^{k} \quad (20)$$

Corollary 3.9. Let us assume $m = 3$ in (20).Then

$$\sum_{r=n}^{[\frac{k}{\ell}]}\frac{(r-1)^{(n-1)}}{(n-1)^{(n-1)}}\alpha^{r-n}\left[\prod_{i=1}^{3}(k+(r_i-r)\ell)^{n_i}\right] = \sum_{p=0}^{n}(k+r_1\ell)^{n_1}(k+r_2\ell)^{n_2}\Delta_{\alpha(\ell)}^{-(2+p)}(k+(r_3+p)\ell)^{n_3}$$

$$+(n+1)\sum_{p=n+1}^{\infty}(-1)^p\Delta_\ell^p(k+r_1\ell)^{n_1}(k+r_2\ell)^{n_2}\Delta_{\alpha(\ell)}^{-(2+p)}(k+(r_3+p)\ell)^{n_3}\Big|_{(n-1)\ell+j}^{k} \quad (21)$$

Example 3.10. In (20), by taking $n = 2$, $n_1 = 3$, $n_2 = 4$ and $n_3 = 5$, we have

$$\sum_{r=2}^{[\frac{k}{\ell}]}(r-1)\alpha^{r-2}(k+(r_1-r)\ell)^3(k+(r_2-r)\ell)^4(k+(r_3-r)\ell)^5 = (k+r_1\ell)^3(k+r_2\ell)^4\Delta_{\alpha(\ell)}^{-2}(k+r_3\ell)^5$$

$$+\Delta_{\alpha(\ell)}^{-1}\left[\sum_{t=1}^{2}\Delta_\ell^{2-t}\left((k+r_1\ell)^3(k+r_2\ell)^4\right)\Delta_{\alpha(\ell)}^{-(3-t)}(k+(r_3+2)\ell)^5\right]$$

$$+\Delta_{\alpha(\ell)}^{-2}\left[\Delta_\ell^2\left((k+r_1\ell)^3(k+r_2\ell)^4\right)\Delta_{\alpha(\ell)}^{-2}(k+(r_3+2)\ell)^5\right]\Big|_{\ell+j}^{k} \quad (22)$$

Particularly, when $k = 31$, $\ell = 2$, $j = 1$, $r_1 = 5$, $r_2 = 6$, $r_3 = 7$ and $\alpha = 4$, we get

$$\sum_{r=2}^{15}(r-1)4^{r-2}(41-2r)^3(43-2r)^4(45-2r)^5 = (-1.201230534\times10^{23})-4^{15}(-1.698881063\times10^{14})$$

$$= 1.824159651\times10^{23}.$$

Theorem 3.11. Let $k\in[0,\infty)$ and $j = k-[k/\ell]\ell$. Then,

$$\sum_{r=4}^{[\frac{k}{\ell}]}\frac{(r-1)^{(3)}}{3^{(3)}}\alpha^{r-4}(k+(r_1-r)\ell)^2(k+(r_2-r)\ell)_\ell^{(3)}$$

$$= \Big\{(k+r_1\ell)^2\Delta_{\alpha(\ell)}^{-4}(k+r_2\ell)_\ell^{(3)}-\Delta_\ell(k+r_1\ell)^2\Delta_{\alpha(\ell)}^{-5}(k+(r_2+1)\ell)_\ell^{(3)}$$

$$\Delta_{\alpha(\ell)}^{-1}\left[\Delta_\ell^2(k+r_1\ell)^2\Delta_{\alpha(\ell)}^{-5}((k+(r_2+1)\ell)_\ell^{(3)})-\Delta_\ell(k+r_1\ell)^2\Delta_{\alpha(\ell)}^{-4}(k+(r_2+1)\ell)_\ell^{(3)}\right]$$

$$+\Delta_{\alpha(\ell)}^{-2}\left[\Delta_{\alpha(\ell)}^{-2}\Delta\ell^2(k+r_1\ell)^2\Delta_{\alpha(\ell)}^{-2}(k+(r_2+2)\ell)_\ell^{(3)}\right]$$

$$+\Delta_{\alpha(\ell)}^{-3}\left[\Delta_{\alpha(\ell)}^{-2}\Delta\ell^2(k+r_1\ell)^2\Delta_{\alpha(\ell)}^{-2}(k+(r_2+2)\ell)_\ell^{(3)}\right]$$

$$+\Delta_{\alpha(\ell)}^{-4}\left[\Delta_{\alpha(\ell)}^{-2}\Delta\ell^2(k+r_1\ell)^2\Delta_{\alpha(\ell)}^{-2}(k+(r_2+2)\ell)_\ell^{(3)}\right]\Big\}\Big|_{3\ell+j}^{k} \quad (23)$$

Example 3.12. In (23), substituting $k = 51$, $\ell = 4$, $j = 1$, $r_1 = 6$, $r_2 = 7$, and $\alpha = 5$, we get

$$\sum_{r=4}^{12}\frac{(r-1)^{(3)}}{3^{(3)}}5^{r-4}(75-4r)^2(79-4r)_4^{(3)} = 1.711318365\times10^{15}$$

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