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SOME NON NEWTONIAN CONTRACTIONS AND FIXED POINT RESULTS

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Abstract

In this paper, we have introduced the Kannan, Zamfirescu and Rhoades type contractions in the setting of non Newtonian Calculus. Also, some fixed point results are developed using these contractions.

Keywords: *Non-Newtonian contractions, fixed point theorems*

1. INTRODUCTION

The dawn of the fixed point theory starts when in 1912 Brouwer [1] proved a fixed point result for continuous self maps on a closed ball. In 1922, Banach [2] gave a very useful result known as the Banach Contraction Principle. Kannan [3], then relaxed the condition of continuity of the map considered in Banach Contraction Principle in his paper in 1968. Zamfirescu [4] and Rhoades[5], consequently developed more general contractions for a complete metric space. These contractions have been generalised to the other spaces also by various authors [6-10].

The study of non Newtonian calculi have been started in 1972 by Grossman and Katz [11]. These provide an alternative to the classical calculus and they include the geometric, anageometric and bigeometric calculi, etc. In 2002 Cakmac and Basar [12], have introduced the concept of non Newtonian metric space. Also they have given the triangle and Minkowski's inequalities in the sense of non-Newtonian calculus. Recently, Binbasioglu, denuriz and turkoglu [13] discussed some topological properties of the non Newtonian metric space and also introduced the concept of fixed point theory in the setting of non Newtonian Calculus. The non-Newtonian calculi are alternatives to the classical calculus of Newton and

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Leibnitz. They provide a wide variety of mathematical tools for use in science, engineering and mathematics.

2. PRELIMINARIES

Now, we define the non-Newtonian real field and we give the relevant properties due to Cakmak and Basar [12].

A *generator* is defined as an injective function with domain ℝ and the range of a generator is a subset of ℝ. Each generator generates one arithmetic if and only if each arithmetic is generated by one generator.

Let α be an exponential function defined as

$$\alpha: \mathbb{R} \to \mathbb{R}^+,$$

$$x \longmapsto \alpha(x) = e^x = y,$$

where, $\mathbb{R}^+$ is the set of positive real numbers.

Suppose that this function α is a generator, that is, if $\alpha = I, I(x) = x \ \forall \ x \in \mathbb{R}$, then β generates the classical arithmetic. If α = exp, then α generates geometrical arithmetic.

Define the set $\mathbb{R}(N)$ as

$$\mathbb{R}(N) \coloneqq \{\alpha(x): x \in \mathbb{R}\},$$

Where $\mathbb{R}(N)$ is the set of non-Newtonian real numbers.

All concepts of β-arithmetic have similar properties in classical arithmetic. α-zero, α-one and all α-integers are formed as

$$\dots, \alpha(-1), \alpha(0), \alpha(1), \dots.$$

Take any generator α with range A. Then define the operations α-addition, α-subtraction, α-multiplication, α-division and α-order in the following way for $x, y \in \mathbb{R}$, respectively:

α-addition $\quad x \dot{+} y = \alpha\{\alpha^{-1}(x) + \alpha^{-1}(y)\},$

α-subtraction $\quad x \dot{-} y = \alpha\{\alpha^{-1}(x) - \alpha^{-1}(y)\},$

α-multiplication $\quad x \dot{\times} y = \alpha\{\alpha^{-1}(x) \times \alpha^{-1}(y)\},$

α-division $\quad x \dot{/} y = \alpha\{\alpha^{-1}(x) \div \alpha^{-1}(y)\},$

α-order $\quad x \dot{<} y \Leftrightarrow \alpha(x) < \alpha(y).$

Proposition 2.1 [12]: $(\mathbb{R}(N), \dot{+}, \dot{\times})$ is a complete field.

For $x \in A \subset \mathbb{R}(N)$, a number α-square is described by $x \dot{\times} x$ and denoted by x^{2_N}. The symbol $\sqrt{x}^N$ denotes

$$t = \alpha\left\{\sqrt{\alpha^{-1}(x)}\right\}$$

which is the unique α nonnegative number whose α-square is equal to x and which means $t^{2_N} = x$, for each α nonnegative number t. Throughout this paper, x^{p_N} denotes the pth non-Newtonian exponent. Thus we have

$$x^{p_N} = x^{(p-1)_N} \dot{\times}\, x = \alpha\{[\alpha^{-1}(x)]^p\},$$

We denote by $|x|_N$ the α-absolute value of a number $x \in A \subset \mathbb{R}(N)$ defined as $\alpha(|\alpha^{-1}(x)|)$ and also

$$\sqrt{x^{2_N}}^N = |x|_N = \alpha\{|\alpha^{-1}(x)|\}$$

Thus,

$$|x|_N = \begin{cases} x, x \dot{>} \beta(0), \\ \alpha(0), x = \beta(0), \\ \alpha(0) \dot{-} x, x \dot{<} \beta(0). \end{cases}$$

For $x_1, x_2 \in A \subseteq \mathbb{R}(N)$, the non-Newtonian distance $|\cdot|_N$ is defined as

$$|x_1 \dot{-} x_2|_N = \alpha\{|\alpha^{-1}(x_1) - \alpha^{-1}(x_2)|\}.$$

This distance is commutative; i.e., $|x_1 \dot{-} x_2|_N = |x_2 \dot{-} x_1|_N$.

Take any $z \in \mathbb{R}(N)$, if $z \dot{>} \alpha(0)$, then z is called a positive non-Newtonian real number; if $z \dot{<} \alpha(0)$, then z is called a non-Newtonian negative real number and if $z = \alpha(0)$, then z is called an unsigned non-Newtonian real number. Non-Newtonian positive real numbers are denoted by $\mathbb{R}^+(N)$ and non-Newtonian negative real numbers by $\mathbb{R}^-(N)$[4].

The fundamental properties provided in the classical calculus are provided in non-Newtonian calculus, too.

Proposition 2.2 [12]: $|x \dot{\times} y|_N = |x|_N \dot{\times} |y|_N$ for any $x, y \in \mathbb{R}(N)$.

Proposition 2.3 [12]: The triangle inequality with respect to non-Newtonian distance $|\cdot|_N$, for any $x, y \in \mathbb{R}(N)$ is given by $|x \dot{+} y|_N \dot{\leq} |x|_N \dot{+} |y|_N$.

Definition 2.4 [12]: Let $X \neq \emptyset$ be a set. If a function $d_N: X \times X \to \mathbb{R}^+(N)$ satisfies the following axioms for all $x, y, z \in X$:

(NM1) $d_N(x, y) = \alpha(0) = \dot{0}$ if and only if $x = y$,

(NM2) $d_N(x, y) = d_N(y, x)$,

(NM3) $d_N(x, y) \dot{\leq} d_N(x, z) \dot{+} d_N(z, y)$,

then it is called a non-Newtonian metric on X and the pair (X, d_N) is called a non-Newtonian metric space.

Definition 2.5 [13]: Let (X, d_N) be a non-Newtonian metric space, $x \in X$ and $\varepsilon \dot{>} \dot{0}$, we now define a set $B_\varepsilon^N(x) = \{y \in X : d_N(x,y) \dot{<} \varepsilon\}$, which is called a non-Newtonian open ball of radius ε with center x. Similarly, one describes the non-Newtonian closed ball as $\bar{B}_\varepsilon^N(x) = \{y \in X : d_N(x,y) \dot{\leq} \varepsilon\}$.

Example 2.6: Consider the non-Newtonian metric space $(\mathbb{R}^+(N), d_N^*)$. From the definition of d_N^*, we can verify that the non-Newtonian open ball of radius $\varepsilon \dot{<} \dot{1}$ with center x_0 appears as $(x_0 \dot{-} \varepsilon, x_0 \dot{+} \varepsilon) \subset \mathbb{R}^+(N)$.

Definition 2.7: Let (X, d_X^N) and (Y, d_Y^N) be two non-Newtonian metric spaces and let $f : X \to Y$ be a function. If f satisfies the requirement that, for every $\varepsilon \dot{>} \dot{0}$, there exists $\delta \dot{>} \dot{0}$ such that $f(B_\delta^N(x)) \subset B_\varepsilon^N(f(x))$, then f is said to be non-Newtonian continuous at $x \in X$.

Example 2.8: Given a non-Newtonian metric space (X, d_N), define a non Newtonian metric on $X \times X$ by $p\big((x_1, x_2), (y_1, y_2)\big) = d_N(x_1, y_1) \dot{+} d_N(x_2, y_2)$. Then the non-Newtonian metric $d_N : X \times X \to (\mathbb{R}^+(N), |\cdot|_N)$ is non-Newtonian continuous on $X \times X$. To show this, let us take the points, $(y_1, y_2), (x_1, x_2) \in X \times X$. Since we have $|d_N(y_1, y_2) \dot{-} d_N(x_1, x_2)|_N \dot{\leq} d_N(x_1, y_2) \dot{+} d_N(x_2, y_2)$, it is clear that d_N is non-Newtonian continuous on $X \times X$. Now, we emphasize some properties of convergent sequences in a non-Newtonian metric space.

Definition 2.9 [12]: A sequence (x_n) in a metric space $X = (X, d_N)$ is said to be convergent if for every given $\varepsilon \dot{>} \dot{0}$ there exist an $n_0 = n_0(\varepsilon) \in Nand\, x \in X$ such that $d_N(x_n, x) \dot{<} \varepsilon$ for all $n > n_0$, and it is denoted by ${}^N\lim_{n\to\infty} x_n = x$ or $x_n \overset{N}{\to} x$, as $n \to \infty$.

Definition 2.10 [13]: A sequence (x_n) in a non-Newtonian metric space $X = (X, d_N)$ is said to be non-Newtonian Cauchy if for every $\varepsilon \dot{>} \dot{0}$ there exists an $n_0 = n_0(\varepsilon) \in N$ such that $d_N(x_n, x_m) \dot{<} \varepsilon$ for all $m, n > n_0$. Similarly, if for every non-Newtonian open ball $B_\varepsilon^N(x)$, there exists a natural number n0 such that $n > n_0$, $x_n \in B_\varepsilon^N(x)$, then the sequence (x_n) is said to be non Newtonian convergent to x.

The space X is said to be non-Newtonian complete if every non-Newtonian Cauchy sequence in X converges [12].

Proposition 2.11. [12]: Let $X = (X, d_N)$ be a non-Newtonian metric space. Then

(i) a convergent sequence in X is bounded and its limit is unique,

(ii) a convergent sequence in X is a Cauchy sequence in X.

Lemma 2.12. [13]: Let (X, d_N) be a non-Newtonian metric space, (x_n) a sequence in X and $x \in X$. Then $x_n \overset{N}{\to} x$ $(n \to \infty)$ if and only if $d_N(x_n, x) \overset{N}{\to} \dot{0}$ $(n \to \infty)$.

Lemma 2.13 [13]: Let (X, d_N) be a non-Newtonian metric space and let (x_n) be a sequence in X. If the sequence (x_n) is non-Newtonian convergent, then the non-Newtonian limit point is unique.

Theorem 2.14 [13]: Let (X, d_N^X) and (Y, d_N^Y) be two non-Newtonian metric spaces, $f : X \rightarrow Y$ a mapping and (x_n) any sequence in X. Then, f is non-Newtonian continuous at the point $x \in X$ if and only if $f(x_n) \overset{N}{\rightarrow} f(x)$ for every sequence (x_n) with $x_n \overset{N}{\rightarrow} x$ $(n \rightarrow \infty)$.

Theorem 2.15 [13]: Let (X, d_N) be a non-Newtonian metric space and $S \subset X$. Then

(i) a point $x \in X$ belongs to $\bar{S}$ if and only if there exists a sequence (x_n) in S such that $x_n \overset{N}{\rightarrow} x$ $(n \rightarrow \infty)$,

(ii) the set S is non-Newtonian closed if and only if every non-Newtonian convergent sequence in S has a non-Newtonian limit point that belongs to S.

Definition 2.16 [13]: Let X be a set and T a map from X to X. A fixed point of T is a point $x \in X$ such that $Tx = x$. In other words, a fixed point of T is a solution of the functional equation $Tx = x, x \in X$.

Definition 2.17 [13]: Suppose that (X, d_N) is a non-Newtonian complete metric space and $T : X \rightarrow X$ is any mapping. The mapping T is said to satisfy a non-Newtonian Lipschitz condition with $k \in \mathbb{R}(N)$ if $d(T(x), T(y)) \dot{\leq} k \dot{\times} d(x, y)$ holds for all $x, y \in X$.

If $k \dot{<} \dot{1}$, then T is called a non-Newtonian contraction mapping.

Theorem 2.18 [13]: Let T be a non-Newtonian contraction mapping on a non-Newtonian complete metric space X. Then T has a unique fixed point.

Main results:

Theorem 3.1: (A generalisation of the Banach Contraction Principle): Let (X, d_N) be a complete non-Newtonian metric space and $T: X \rightarrow X$ be a self map. Assume that there exists a right continuous real function

$$\Delta: [\dot{0}, u] \overset{N}{\rightarrow} [\dot{0}, u]$$

where, u is sufficiently large number such that

$$\Delta(a) \dot{<} a \text{ if } a \dot{>} \dot{0}, \tag{3.1}$$

and let T satisfies

$$d_N(Tx_1, Tx_2) \dot{\leq} \Delta\big(d_N(x_1, x_2)\big) \tag{3.2}$$

For all $x_1, x_2 \in (X, d_N)$. Then T has a unique fixed point $c \in (X, d_N)$ and the sequence $T^n(x)$ converges to c for every $x \in X$.

Proof: Let us take a point $x_0 \in X$ and define the sequence $T(x_n) = x_{n+1}$. For $n \in \mathbb{N}$. Thus, the following sequence:

$$a_n = d_N(x_n, x_{n-1}).$$

Using (3.1) and (3.2), we obtain

$$a_{n+1} = d_N(x_{n+1}, x_n) \dot{\leq} \Delta\big(d_N(x_n, x_{n-1})\big) \dot{<} d_N(x_n, x_{n-1}) = a_n$$

for all $n \in \mathbb{N}$. Thus the sequence a_n is decreasing and so it has a limit a. If we assume that $a > 0$, we have

$$a_{n+1} \dot{\leq} \Delta(a_n)$$

from (3.2). Since Δ is right continuous, we get

$$a \dot{\leq} \Delta(a)$$

But it contradicts with (3.1). as a result, $a_n \xrightarrow{N} \dot{0}$ as $n \to \infty$.

We would like to show that x_n is Cauchy sequence. Then there exists $\epsilon \dot{>} \dot{0}$ and integers $m > n \geq k$ for every $k \geq 1$ such that

$$d_N(x_m, x_n) \dot{\geq} \epsilon.$$

For a smallest m, we can suppose that $d_N(x_m, x_n) < \epsilon$. If we use the triangle inequality, we obtain

$$\epsilon \dot{\leq} d_N(x_m, x_n) \dot{\leq} d_N(x_m, x_{m-1}) \dot{+} d_N(x_{m-1}, x_n) \dot{<} \epsilon \dot{+} d_N(x_m, x_{m-1}).$$

Since $d_N(x_n, x_{n-1}) \xrightarrow{N} \dot{0}$ as $n \to \infty$, we conclude that

$$\epsilon \dot{\leq} d_N(x_m, x_n) \dot{<} \epsilon \Rightarrow d_N(x_m, x_n) \xrightarrow{N} \epsilon \ as \ n \longrightarrow \infty.$$

From the fact that

$$m > n \Rightarrow d_N(x_{m+1}, x_m) \dot{\leq} d_N(x_{n+1}, x_n)$$

And (3.2), we have

$$\begin{aligned}\epsilon \dot{\leq} d_N(x_m, x_n) &\dot{\leq} d_N(x_m, x_{m+1}) \dot{+} d_N(x_{m+1}, x_{n+1}) \dot{+} d_N(x_{n+1}, x_n) \\ &\dot{\leq} \alpha(2) \dot{\times} d_N(x_{n+1}, x_n) \dot{+} \Delta\big(d_N(x_m, x_n)\big).\end{aligned}$$

Taking the limit as $n \to \infty$, from these inequalities we get $\epsilon \dot{\leq} \Delta(\epsilon)$ but contradicts with (3.1) because $\epsilon \dot{>} 0$. As a result, x_n is a Cauchy sequence and since (X, d_N) is a complete digital metric space, $T^n(x)$ converges in (X, d_N).

Now, we prove the uniqueness. Let u_1, u_2 be two fixed point of T. By (3.1) and (3.2), we get

$$d_N(u_1, u_2) = d_N\big(T(u_1), T(u_2)\big) \dot{\leq} \Delta\big(d_N(u_1, u_2)\big) \Rightarrow u_1 = u_2.$$

Definition 3.2: (Kannan type non-Newtonian contraction): Suppose that (X, d_N) is a non-Newtonian metric space and $T: X \to X$ is any mapping. If there exist an $\mu \in (\dot{0}, \alpha(1/2)]$, such that for all $x, y \in X$, $d_N(Tx, Ty) \dot{\leq} \mu \dot{\times} [d_N(x, Tx) \dot{+} d_N(y, Ty)]$, then T is called a non-Newtonian Kannan Contraction.

Remark 3.3: The non-Newtonian contraction mapping defined in [13] requires T to be non-Newtonian continuous mapping. By defining non-Newtonian Kannan Contraction, we relax the condition of continuity on T.

Definition 3.4: (non-Newtonian Zamfirescu type Contraction): Let, (X, d_N) be any non-Newtonian metric space and $T: X \to X$ be a self map. If there exists $\lambda \in (\dot{0}, \dot{1})$ such that for all $x, y \in X$,

$$d_N(Tx, Ty) \dot{\leq} \lambda \dot{\times} max\left\{d_N(x, y), \frac{\{d_N(x, Tx) \dot{+} d_N(y, Ty)\}}{\alpha(2)}, \frac{\{d_N(x, Ty) \dot{+} d_N(y, Tx)\}}{\alpha(2)}\right\},$$

then T is called a non-Newtonian Zamfirescu type contraction.

Definition 3.5: (non-Newtonian Rhoades type contraction) Let, (X, d_N) be any non Newtonian metric space and $T: X \to X$ be a self map. If there exists $\lambda \in (\dot{0}, \dot{1})$ such that for all $x, y \in X$,

$$d_N(Tx, Ty) \dot{\leq} \lambda \dot{\times} max\left\{d_N(x, y), \frac{\{d_N(x, Tx) \dot{+} d_N(y, Ty)\}}{\alpha(2)}, d_N(x, Ty), d_N(y, Tx)\right\},$$

then T is called a non-Newtonian Rhoades type contraction.

Theorem 3.6: Let, T be a non-Newtonian Kannan type contraction mapping on a complete non-Newtonian metric space (X, d_N). Then T has a unique fixed point.

Proof: Let, x_0 be any point of X. Consider the iterate sequence $Tx_n = x_{n+1}$. Using induction on n, we obtain

$$d_N(x_{n+1}, x_n) \dot{\leq} \mu \dot{\times} [d_N(x_n, x_{n-1}) \dot{+} d_N(x_{n-1}, x_{n-2})] \dot{\leq} (\alpha(2) \dot{\times} \mu) \dot{\times} d_N(x_n, x_{n-1})$$

$$\Rightarrow d_N(x_{n+1}, x_n) \dot{\leq} (2\mu) \dot{\times} d_N(x_n, x_{n-1}) \dot{\leq} (2\mu)^{n_N} \dot{\times} d_N(Tx_0, x_0).$$

For natural numbers $n \in \mathbb{N}$ and $m \geq 1$, we conclude that

$$d_N(x_{n+m}, x_n) \dot{\leq} d_N(x_{n+m}, x_{n+m-1}) \dot{+} \dots \dot{+} d_N(x_{n+1}, x_n)$$

$$\dot{\leq} \left[(\alpha(2) \dot{\times} \mu)^{(n+m)_N} \dot{+} (\alpha(2) \dot{\times} \mu)^{(n+m-1)_N} \dot{+} \dots \dot{+} (\alpha(2) \dot{\times} \mu)^{n_N}\right] d_N(Tx_0, x_0)$$

$$\dot{\leq} \frac{(\alpha(2) \dot{\times} \mu)^{n_N}}{1 - \alpha(2) \dot{\times} \mu} d_N(Tx_0, x_0)$$

As a result, x_n is a Cauchy sequence. There is a limit point of x_n because (X, d_N) is a non-Newtonian metric space. Let c be the limit of x_n. From the continuity of T we get

$$T(c) = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} x_{n+1} = c.$$

Therefore, T has a unique fixed point.

Assume that $a, b \in X$ are fixed points of T. Then we have the following:

$$d_N(a,b) = d_N(Ta,Tb) \dot{\leq} \mu \dot{\times} [d_N(a,a) \dot{+} d_N(b,b)]$$

$$\Rightarrow \quad d_N(a,b) \dot{\leq} 0$$

$$\Rightarrow \quad a = b$$

So, our theorem is proved.

Theorem 3.7: (Zamfirescu contraction principle) Let (X, d_N) be a non-Newtonian complete metric space, and $T: (X, d_N) \to (X, d_N)$ be a Zamfirescu type non-Newtonian contraction mapping. Then T has a unique fixed point in X.

Proof: Let x_0 be any point of X. Consider the iterate sequence $Tx_n = x_{n+1}$. Using induction on n, we obtain

$$d_N(x_{n+1}, x_n) \dot{\leq} \lambda \dot{\times} max$$

$$\left\{d_N(x_n, x_{n-1}), \frac{\{d_N(x_n, x_{n-1}) \dot{+} d_N(x_{n-1}, x_{n-2})\}}{\alpha(2)}, \frac{\{d_N(x_n, x_{n-2}) \dot{+} d_N(x_{n-1}, x_{n-1})\}}{\alpha(2)}\right\},$$

Case 1: $d_N(x_{n+1}, x_n) \dot{\leq} \lambda \dot{\times} d_N(x_n, x_{n-1}) \dot{\leq} \dots \dot{\leq} \lambda^{n_N} \dot{\times} d_N(Tx_0, x_0)$.

For natural numbers $n \in \mathbb{N}$ and $m \geq 1$, we conclude that

$$d_N(x_{n+m}, x_n) \dot{\leq} d_N(x_{n+m}, x_{n+m-1}) \dot{+} \dots \dot{+} d_N(x_{n+1}, x_n)$$

$$\dot{\leq} \left[\lambda^{(n+m)_N} \dot{+} \lambda^{(n+m-1)_N} \dot{+} \dots \dot{+} \lambda^{n_N}\right] \dot{\times} d_N(Tx_0, x_0) \dot{\leq} \frac{\lambda^{n_N}}{1-\lambda} \dot{\times} d_N(Tx_0, x_0)$$

Case 2: $d_N(x_{n+1}, x_n) \dot{\leq} \lambda \dot{\times} \frac{\{d_N(x_n, x_{n-1}) \dot{+} d_N(x_{n-1}, x_{n-2})\}}{\alpha(2)}$

$$\dot{\leq} \lambda \dot{\times} d_N(x_n, x_{n-1}) \dot{\leq} \dots \dot{\leq} \lambda^{n_N} \dot{\times} d_N(Tx_0, x_0)$$

For natural numbers $n \in \mathbb{N}$ and $m \geq 1$, we conclude that

$$d_N(x_{n+m}, x_n) \dot{\leq} d_N(x_{n+m}, x_{n+m-1}) \dot{+} \dots \dot{+} d_N(x_{n+1}, x_n)$$

$$\dot{\leq} \left[\lambda^{(n+m)_N} \dot{+} \lambda^{(n+m-1)_N} \dot{+} \dots \dot{+} \lambda^{n_N}\right] \dot{\times} d_N(Tx_0, x_0) \dot{\leq} \frac{\lambda^{n_N}}{1-\lambda} \dot{\times} d_N(Tx_0, x_0)$$

Case 3: $d_N(x_{n+1}, x_n) \dot{\leq} \lambda \dot{\times} \left\{\frac{d_N(x_n, x_{n-2}) \dot{+} d_N(x_{n-1}, x_{n-1})}{\alpha(2)}\right\}$

$$\dot{\leq} \lambda \dot{\times} \frac{d_N(x_n, x_{n-2})}{\alpha(2)} \dot{\leq} \frac{\lambda}{\alpha(2)} \dot{\times} \{d_N(x_n, x_{n-1}) + d_N(x_{n-1}, x_{n-2})\} \dot{\leq} \lambda \dot{\times} d_N(x_n, x_{n-1})$$

For natural numbers $n \in \mathbb{N}$ and $m \geq 1$, we conclude that

$$d_N(x_{n+m}, x_n) \dot{\leq} d_N(x_{n+m}, x_{n+m-1}) \dot{+} \ldots \dot{+} d_N(x_{n+1}, x_n)$$

$$\dot{\leq} \left[\lambda^{(n+m)_N} \dot{+} \lambda^{(n+m-1)_N} \dot{+} \ldots \dot{+} \lambda^{n_N}\right] \dot{\times} d_N(Tx_0, x_0) \dot{\leq} \frac{\lambda^{n_N}}{1-\lambda} \dot{\times} d_N(Tx_0, x_0)$$

As a result, x_n is a Cauchy sequence. There is a limit point of x_n because (X, d_N) is a complete non-Newtonian metric space. Let c be the limit of x_n. From the continuity of T we get

$$T(c) = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} x_{n+1} = c.$$

Therefore, T has a fixed point.

Assume that $a, b \in X$ are fixed points of T. Then we have the following:

$$d_N(a, b) = d_N(Ta, Tb) \dot{\leq} \lambda \dot{\times} [d_N(a, a) \dot{+} d_N(b, b)]$$

$$\Rightarrow \qquad d_N(a, b) \dot{\leq} 0$$

$$\Rightarrow \qquad a = b$$

So, our theorem is proved.

Theorem 3.8: Let, T be a non-Newtonian Rhoades type contraction mapping on a complete non-Newtonian metric space (X, d_N). Then T has a unique fixed point.

Proof: Let, x_0 be any point of X. Consider the iterate sequence $Tx_n = x_{n+1}$. Using induction on n, we obtain

$$d_N(x_{n+1}, x_n) \dot{\leq} \lambda \dot{\times} max \begin{Bmatrix} d_N(x_n, x_{n-1}), \dfrac{\{d_N(x_n, x_{n-1}) \dot{+} d_N(x_{n-1}, x_{n-2})\}}{\alpha(2)}, \\ d_N(x_n, x_{n-2}), d_N(x_{n-1}, x_{n-1}) \end{Bmatrix},$$

Case 1: $d_N(x_{n+1}, x_n) \dot{\leq} \lambda \dot{\times} d_N(x_n, x_{n-1}) \dot{\leq} \ldots \dot{\leq} \lambda^{n_N} \dot{\times} d_N(Tx_0, x_0)$.

For natural numbers $n \in \mathbb{N}$ and $m \geq 1$, we conclude that

$$d_N(x_{n+m}, x_n) \dot{\leq} d_N(x_{n+m}, x_{n+m-1}) \dot{+} \ldots \dot{+} d_N(x_{n+1}, x_n)$$

$$\dot{\leq} \left[\lambda^{(n+m)_N} \dot{+} \lambda^{(n+m-1)_N} \dot{+} \ldots \dot{+} \lambda^{n_N}\right] \dot{\times} d_N(Tx_0, x_0) \dot{\leq} \frac{\lambda^{n_N}}{1-\lambda} \dot{\times} d_N(Tx_0, x_0)$$

Case 2: $d_N(x_{n+1}, x_n) \dot{\leq} \lambda \dot{\times} \frac{\{d_N(x_n, x_{n-1}) \dot{+} d_N(x_{n-1}, x_{n-2})\}}{\alpha(2)}$

$$\dot{\leq} \lambda \dot{\times} d_N(x_n, x_{n-1}) \dot{\leq} \ldots \dot{\leq} \lambda^{n_N} \dot{\times} d_N(Tx_0, x_0)$$

For natural numbers $n \in \mathbb{N}$ and $m \geq 1$, we conclude that

$$d_N(x_{n+m}, x_n) \dot{\leq} d_N(x_{n+m}, x_{n+m-1}) \dot{+} \ldots \dot{+} d_N(x_{n+1}, x_n)$$

$$\dot{\leq} \left[\lambda^{(n+m)_N} \dot{+} \lambda^{(n+m-1)_N} \dot{+} \ldots \dot{+} \lambda^{n_N}\right] \dot{\times} d_N(Tx_0, x_0) \dot{\leq} \frac{\lambda^{n_N}}{1-\lambda} \dot{\times} d_N(Tx_0, x_0)$$

Case 3: $d_N(x_{n+1}, x_n) \dot{\leq} \lambda \dot{\times} d_N(x_n, x_{n-2}) \dot{\leq} \lambda \dot{\times} d_N(x_n, x_{n-1})$
$\dot{\leq} \ldots \dot{\leq} \lambda^{n_N} \dot{\times} d_N(Tx_0, x_0)$,

For natural numbers $n \in \mathbb{N}$ and $m \geq 1$, we conclude that

$$d_N(x_{n+m}, x_n) \dot{\leq} d_N(x_{n+m}, x_{n+m-1}) \dot{+} \ldots \dot{+} d_N(x_{n+1}, x_n)$$

$$\dot{\leq} \left[\lambda^{(n+m)_N} \dot{+} \lambda^{(n+m-1)_N} \dot{+} \ldots \dot{+} \lambda^{n_N}\right] \dot{\times} d_N(Tx_0, x_0) \dot{\leq} \frac{\lambda^{n_N}}{1-\lambda} \dot{\times} d_N(Tx_0, x_0)$$

Case 4: $d_N(x_{n+1}, x_n) \dot{\leq} \lambda \dot{\times} d_N(x_{n-1}, x_{n-1}) = 0 \dot{\leq} \lambda \dot{\times} d_N(x_n, x_{n-1}) \dot{\leq} \ldots$
$\dot{\leq} \lambda^{n_N} \dot{\times} d_N(Tx_0, x_0)$,

For natural numbers $n \in \mathbb{N}$ and $m \geq 1$, we conclude that

$$d_N(x_{n+m}, x_n) \dot{\leq} d_N(x_{n+m}, x_{n+m-1}) \dot{+} \ldots \dot{+} d_N(x_{n+1}, x_n)$$

$$\dot{\leq} \left[\lambda^{(n+m)_N} \dot{+} \lambda^{(n+m-1)_N} \dot{+} \ldots \dot{+} \lambda^{n_N}\right] \dot{\times} d_N(Tx_0, x_0) \dot{\leq} \frac{\lambda^{n_N}}{1-\lambda} \dot{\times} d_N(Tx_0, x_0)$$

As a result, x_n is a Cauchy sequence. There is a limit point of x_n because (X, d_N) is a non-Newtonian metric space. Let c be the limit of x_n. From the continuity of T we get

$$T(c) = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} x_{n+1} = c.$$

Therefore, T has a fixed point.

Assume that $a, b \in X$ are fixed points of T. Then we have the following:

$$d_N(a, b) = d_N(Ta, Tb) \dot{\leq} \lambda \dot{\times} [d_N(a, a) \dot{+} d_N(b, b)]$$

$$\Rightarrow \quad d_N(a, b) \dot{\leq} 0$$

$$\Rightarrow \quad a = b$$

So, our theorem is proved.

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