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SOME MORE RESULTS ON THE ALGEBRA (C_0, C_0) OF INFINITE MATRICES IN NON-ARCHIMEDEAN FIELDS

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ABSTRACT: In this note, we record briefly some more results regarding the algebra (c_0, c_0) of infinite matrices in complete, non-trivially valued, non-archimedean fields.

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In this note, we supplement a few more results to an earlier paper of the author [1]. Throughout this note, K denotes a complete, non-trivially valued, non-archimedean field. Sequences and infinite matrices have entries in K. Following the notations and techniques used in the proofs of the theorems in [1], we prove the following theorems which are worth recording in the context of the algebra (c_0, c_0) considered in [1].

Theorem 1: (c_0, c_0) is a non-archimedean Banach algebra, with identity, under the norm

$$\| A \| = \sup_{n,k} |a_{nk}|, \quad A = (a_{nk}) \in (c_0, c_0) \tag{1}$$

under the usual matrix addition, scalar multiplication and multiplication.

Proof: Let $A = (a_{nk})$, $B = (b_{nk}) \in (c_0, c_0)$. Let $AB = (c_{nk})$ and $\{x_k\} \in c_0$.

Then,

$$\lim_{n \to \infty} y'_n = 0, \quad \text{where} \quad y'_n = \sum_{k=0}^{\infty} b_{nk} x_k,$$

$$\lim_{n \to \infty} y''_n = 0, \quad \text{where} \quad y''_n = \sum_{k=0}^{\infty} a_{nk} y'_k .$$

Now,

$$\sum_{k=0}^{\infty} c_{nk} x_k = \sum_{k=0}^{\infty} \left(\sum_{i=0}^{\infty} a_{ni} b_{ik} \right) x_k$$

$$= \sum_{i=0}^{\infty} a_{ni} \left(\sum_{k=0}^{\infty} b_{ik} x_k \right),$$

rearranging the double sum (see [2], p.133)

$$= \sum_{i=0}^{\infty} a_{ni} y'_i$$

$$= y''_n,$$

so that $\lim_{n\to\infty} \sum_{k=0}^{\infty} c_{nk} x_k = \lim_{n\to\infty} y''_n = 0$. Thus $AB \in (c_0, c_0)$ and so (c_0, c_0) is closed under matrix multiplication. Also,

$$|c_{nk}| = \left| \sum_{i=0}^{\infty} a_{ni} b_{ik} \right|$$

$$\leq \left(\sup_{n,k} |a_{nk}|\right)\left(\sup_{n,k} |b_{nk}|\right)$$

$$= \|A\| \, \|B\|, n, k = 0, 1, 2, \ldots,$$

so that

$$\sup_{n,k} |c_{nk}| \leq \|A\| \, \|B\|,$$

i.e., $$\|AB\| \leq \|A\| \, \|B\|.$$

The identity matrix $I = (e_{nk})$, where

$$e_{nk} = \begin{cases} 1, & \text{if} \quad k = n; \\ 0, & \text{if} \quad k \neq n, \end{cases}$$

is in (c_0, c_0) and it is the identity element of (c_0, c_0) under matrix multiplication. It was proved in Theorem 1 of [1] that (c_0, c_0) is complete under the norm defined by (1). Thus (c_0, c_0) is an algebra, completing the proof of the theorem. □

Theorem 2: $(c_0, c_0; P)$, as a subset of (c_0, c_0), is a closed K-convex semigroup with identity.

Proof: It was shown in Theorem 2 of [1] that $(c_0, c_0; P)$ is a closed, K-convex subset of (c_0, c_0). The identity matrix I is the identity element of $(c_0, c_0; P)$. We shall now prove that $(c_0, c_0; P)$ is closed under matrix multiplication. If $A = (a_{nk})$, $B = (b_{nk}) \in (c_0, c_0; P)$, we have proved in Theorem 1 that $AB \in (c_0, c_0)$.

Further,

$$\sum_{n=0}^{\infty} (AB)_{nk} = \sum_{n=0}^{\infty} \left(\sum_{i=0}^{\infty} a_{ni} b_{ik} \right)$$

$$= \sum_{i=0}^{\infty} b_{ik} \left(\sum_{n=0}^{\infty} a_{ni} \right), \text{ rearranging the double sum as before}$$

$$= \sum_{i=0}^{\infty} b_{ik}$$

$$= 1,$$

since $\sum_{n=0}^{\infty} a_{nk} = \sum_{n=0}^{\infty} b_{nk} = 1, \quad k = 0, 1, 2, \ldots$, which proves that $AB \in (c_0, c_0; P)$, completing the proof.

We now prove a Mercerian theorem for the algebra (c_0, c_0) under matrix multiplication.

Theorem 3: If $y_n = x_n + \lambda(\alpha^n x_0 + \alpha^{n-1} x_1 + \cdots + \alpha x_{n-1} + x_n)$, $\alpha \in K$, $|\alpha| < 1$ and $\{y_n\} \in c_0$, then $\{x_a\} \in c_0$, provided $|\lambda| < 1$.

Proof: Since (c_0, c_0) is an algebra under matrix multiplication, if $|\lambda| < \frac{1}{\|A\|}$, then $I + \lambda A$ has an inverse in (c_0, c_0). We note that the equations

$$y_n = x_n + \lambda(\alpha^n x_0 + \alpha^{n-1} x_1 + \cdots + \alpha x_{n-1} + x_n), \quad n = 0, 1, 2, \cdots$$

can be written as

$$(I + \lambda A)x' = y',$$

where

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ \alpha & 1 & 0 & 0 & \cdots \\ \alpha^2 & \alpha & 1 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix},$$

$$x' = \begin{pmatrix} x_0 & 0 & 0 & 0 & \cdots \\ x_1 & 0 & 0 & 0 & \cdots \\ x_2 & 0 & 0 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix},$$

$$y' = \begin{pmatrix} y_0 & 0 & 0 & 0 & \cdots \\ y_1 & 0 & 0 & 0 & \cdots \\ y_2 & 0 & 0 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}.$$

Note that $A \in (c_0, c_0)$ with $\|A\| = 1$: So, if $|\lambda| < 1$; as observed earlier, $I + \lambda A$ has an inverse in (c_0, c_0). Thus

$$x' = (I + \lambda A)^{-1} y'.$$

Since $(I + \lambda A)^{-1}$, $y' \in (c_0, c_0)$, we have $x' \in (c_0, c_0)$. It now follows (see [1], Theorem A) that $\lim_{n \to \infty} x_n = 0$, i.e, $\{x_n\} \in c_0$. Proof of the theorem is now complete.

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