ON THE SECOND FUNDAMENTAL THEOREM OF ASSET PRICING

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ABSTRACT. Let $X^1, \ldots, X^d$ be sigma-martingales on $(\Omega, \mathcal{F}, \mathsf{P})$. We show that every bounded martingale (with respect to the underlying filtration) admits an integral representation with respect to $X^1, \ldots, X^d$ if and only if there is no equivalent probability measure (other than P) under which $X^1, \ldots, X^d$ are sigma-martingales. From this we deduce the second fundamental theorem of asset pricing- that completeness of a market is equivalent to uniqueness of Equivalent Sigma-Martingale Measure (ESMM).

1. Introduction

The (first) fundamental theorem of asset pricing says that a market consisting of finitely many stocks satisfies the *No Arbitrage* (NA) property if and only there exists an *Equivalent Martingale Measure* (EMM)- *i.e. there exists an equivalent probability measure under which the (discounted) stocks are (local) martingales.* The No Arbitrage property has to be suitably defined when we are dealing in continuous time, where one rules out approximate arbitrage in the class of admissible strategies. For a precise statement in the case when the underlying processes are locally bounded, see Delbaen and Schachermayer [5]. Also see Bhatt and Karandikar [1] for an alternate formulation of a weaker result, where the approximate arbitrage is defined only in terms of simple strategies. For the general case, the result is true when local martingale in the statement above is replaced by sigma-martingale. See Delbaen and Schachermayer [6]. They have an example where the No Arbitrage property holds but there is no equivalent measure under which the underlying process is a local martingale. However, there is an equivalent measure under which the process is a sigma-martingale.

The second fundamental theorem of asset pricing says that the market is complete (*i.e. every contingent claim can be replicated by trading on the underlying securities*) if and only if the EMM is unique. Interestingly, this property was studied by probabilists well before the connection between finance and stochastic calculus was established (by Harrison–Pliska [9]). The completeness of market is same as the question: when is every martingale representable as a stochastic integral with respect to a given set of martingales $\{M^1, \ldots, M^d\}$. When $M^1, \ldots, M^d$ is the d-dimensional Wiener Process, this property was proven by Ito [10]. Jacod

and Yor [13] proved that if M is a P-local martingale, then every martingale N admits a representation as a stochastic integral with respect to M if and only if there is no probability measure Q (other than P) such that Q is equivalent to P and M is a Q-local martingale. The situation in higher dimension is more complex. The obvious generalisation to higher dimension is not true as was noted by Jacod–Yor [13].

To remedy the situation, a notion of vector stochastic integral was introduced-where a vector valued predictable process is the integrand and vector valued martingale is the integrator. The resulting integral yields a class larger than the linear space generated by component wise integrals. See [12], [2]. However, one has to prove various properties of the vector stochastic integrals once again.

Here we achieve the same objective in another fashion avoiding defining integration again from scratch. In the same breath, we also take into account the general case, when the underlying processes need not be bounded but satisfy the property NFLVR and thus one has an equivalent sigma-martingale measure (ESMM). To the best of our knowledge, the martingale representation property in the framework of sigma-martingales is not available in literature. Indeed, most treatments deal with square integrable martingales where the notion of orthogonality of martingales is available which simplifies the treatment.

For a semimartingale X, let $L(X)$ denote the class of predictable processes f such that the stochastic integral $\int f \, dX$ is defined. A semimartingale Z is said to admit an integral representation with respect to semimartingales $(X^1, X^2, \ldots, X^d)$ if there exists a semimartingale Y and predictable processes f, g^j such that $f \in \mathbb{L}(Y)$, $g^j \in \mathbb{L}(X^j)$

$$Y_t = Y_0 + \sum_{j=1}^{d} \int_0^t g_s^j \, dX_s^j, \ \ \forall t \geq 0$$

and

$$Z_t = Z_0 + \int_0^t f_s \, dY_s, \ \ \forall t \geq 0.$$

In Theorem 5.3 we will show that for a multidimensional sigma-martingale $(X^1, X^2, \ldots, X^d)$ all bounded martingales admit a representation with respect to X^j, $1 \leq j \leq d$ if and only if the ESMM is unique.

The most critical part of its proof is to show that the class of martingales that admit representation with respect to $(X^1, X^2, \ldots, X^d)$ is closed under L^1 convergence : If M^n are martingales such that there exist $f^n, g^{n,j}, Y^n$ with

$$Y_t^n = Y_0^n + \sum_{j=1}^{d} \int_0^t g_s^{n,j} \, dX_s^j, \ \ \forall t \geq 0$$

and

$$M_t^n = M_0^n + \int_0^t f_s^n \, dY_s^n, \ \ \forall t \geq 0$$

and if $\mathsf{E}[|M_t^n - M_t|] \to 0$, then we need to show that M also admits a representation with respect to $(X^1, X^2, \ldots, X^d)$. When $(X^1, X^2, \ldots, X^d)$ is d-dimensional Brownian motion, or when X^j and X^k are orthogonal as martingales, one can deduce that for each j, $\{g^{n,j} : n \geq 1\}$ is Cauchy in an appropriate norm and

thereby complete the proof. This step fails in general, as the Jacod–Yor example shows. So we need to do an orthogonalisation of $(X^1, X^2, \ldots, X^d)$ to achieve the same. However, $(X^1, X^2, \ldots, X^d)$ may not be square integrable and thus we need to change the measure to a measure Q (equivalent to the underlying probability measure P) to make it so. Under Q, $(X^1, X^2, \ldots, X^d)$ need not be martingales. This part is delicately managed by keeping both P, Q in the picture.

The rest of the argument is on the lines of Jacod–Yor [13].

When $X^1, X^2, \ldots, X^d$ represent (prices of) stocks, Y can be thought of as (the price of) a mutual fund or an index fund and the investor is trading on such a fund trying to replicate the security Z.

In this framework, Theorem 5.3 gives us the second fundamental theorem of asset pricing : Market is complete if and only if ESMM (equivalent sigma-martingale measure) is unique.

2. Preliminaries and Notation

Let us start with some notations. $(\Omega, \mathcal{F}, \mathsf{P})$ denotes a complete probability space with a filtration $(\mathcal{F}_\bullet) = \{\mathcal{F}_t : t \geq 0\}$ such that $\mathcal{F}_0$ consists of all P-null sets (in $\mathcal{F}$) and

$$\cap_{t>s} \mathcal{F}_t = \mathcal{F}_s \ \ \forall s \geq 0.$$

Thus, we can (and do) assume that all martingales have r.c.l.l. paths.

For various notions, definitions and standard results on stochastic integrals, we refer the reader to Meyer [15], Jacod [11] or Protter [16].

Let $\mathbb{M}$ denote the class of martingales and $\mathbb{M}_{\mathrm{loc}}$ denote the class of local martingales. For $M \in \mathbb{M}_{\mathrm{loc}}$, let $\mathbb{L}^1_m(M)$ be the class of predictable processes f such that there exists a sequence of stopping times $\sigma_k \uparrow \infty$ with

$$\mathsf{E}[\{\int_0^{\sigma_k} f_s^2 d[M,M]_s\}^{\frac{1}{2}}] < \infty.$$

For such an f, $N = \int f \, dM$ is defined and is a local martingale.

For a semimartingale X, let $L(X)$ denote the class of predictable process f such that X admits a decomposition $X = N + A$ with $N \in \mathbb{M}_{\mathrm{loc}}$, A being a process with finite variation paths with $f \in \mathbb{L}^1_m(N)$ and

$$\int_0^t |f_s| d|A|_s < \infty \ \ a.s. \ \ \forall t < \infty. \tag{2.1}$$

For $f \in \mathbb{L}(X)$, the stochastic integral $\int f \, dX$ is defined as $\int f \, dN + \int f \, dA$. It can be shown that the decomposition $X = N + A$ is not unique, the definition does not depend upon the decomposition. See [11].

For $M^1, M^2, \ldots, M^d \in \mathbb{M}$ let $\mathbb{C}(M^1, M^2, \ldots, M^d)$ denote the class of martingales $Z \in \mathbb{M}$ such that $\exists f^j \in \mathbb{L}^1_m(M^j), 1 \leq j \leq d$ with

$$Z_t = Z_0 + \sum_{j=1}^{d} \int_0^t f_s^j dM_s^j, \ \ \forall t \geq 0.$$

For $T < \infty$, let

$$\tilde{\mathbb{K}}_T(M^1, M^2, \ldots, M^d) = \{Z_T \ : \ Z \in \mathbb{C}(M^1, M^2, \ldots, M^d)\}.$$

For the case $d = 1$, Yor [18] had proved that $\tilde{\mathbb{K}}_T$ is a closed subspace of $\mathbb{L}^1(\Omega, \mathcal{F}, \mathsf{P})$. The problem in case $d > 1$ is that in general $\tilde{\mathbb{K}}_T(M^1, M^2, \ldots, M^d)$ need not be closed. Jacod–Yor [13] gave an example where M^1, M^2 are continuous square integrable martingales and $\tilde{\mathbb{K}}_T(M^1, M^2)$ is not closed. Jacod [12] defined the vector stochastic integral and Memin [14] proved that with the modified definition of the integral, this space is closed. We will follow a different path.

For martingales $M^1, M^2, \ldots, M^d$, let $\mathbb{F}(M^1, \ldots, M^d)$ be the class of martingales $Z \in \mathbb{M}$ such that $\exists Y \in \mathbb{C}(M^1, \ldots, M^d)$ and $f \in \mathbb{L}^1_m(Y)$ with

$$Z_t = Z_0 + \int_0^t f_s \, dY_s, \ \forall t \geq 0.$$

Let

$$\mathbb{K}_T(M^1, M^2, \ldots, M^d) = \{Z_T \ : \ Z \in \mathbb{F}(M^1, M^2, \ldots, M^d)\}.$$

The main result of the next section is

Theorem 2.1. *Let* $M^1, M^2, \ldots, M^d$ *be martingales. Then* $\mathbb{K}_T(M^1, M^2, \ldots, M^d)$ *is closed in* $\mathbb{L}^1(\Omega, \mathcal{F}, \mathsf{P})$.

This will be deduced from

Theorem 2.2. *Let* $M^1, M^2, \ldots, M^d$ *be martingales and* $Z^n \in \mathbb{F}(M^1, M^2, \ldots, M^d)$ *be such that* $\mathsf{E}[|Z_t^n - Z_t|] \to 0$ *for all* t. *Then* $Z \in \mathbb{F}(M^1, M^2, \ldots, M^d)$.

When $M^1, M^2, \ldots, M^d$ are square integrable martingales, the analogue of Theorem 2.1 for $\mathbb{L}^2$ follows from the work of Davis–Varaiya [4]. However, for the EMM characterisation via integral representation, one needs the $\mathbb{L}^1$ version, which we deduce using change of measure technique.

We will need the Burkholder–Davis–Gundy inequality (see [15]) (for $p = 1$) which states that there exist universal constants c^1, c^2 such that for all martingales M and $T < \infty$,

$$c^1 \mathsf{E}[([M, M]_T)^{\frac{1}{2}}] \leq \mathsf{E}[\sup_{0 \leq t \leq T} |M_t|] \leq c^2 \mathsf{E}[([M, M]_T)^{\frac{1}{2}}].$$

After proving Theorem 2.1, in the next section we will introduce sigma-martingales and give some elementary properties. Then we come to the main theorem on integral representation of martingales. This is followed by the second fundamental theorem of asset pricing.

3. Proof of Theorem 2.1

In this section, we fix martingales $M^1, M^2, \ldots, M^d$. We begin with a few auxiliary results.

Lemma 3.1. *Let* $\mathbb{C}_b(M^1, \ldots, M^d)$ *be the class of martingales* Z *such that* $\exists$ *bounded predictable processes* $f^j, 1 \leq j \leq d$ *with*

$$Z_t = Z_0 + \sum_{j=1}^d \int_0^t f_s^j \, dM_s^j, \ \ \forall t \geq 0.$$

Let $\mathbb{F}_b(M^1, \ldots, M^d)$ be the class of martingales $Z \in \mathbb{M}$ such that $\exists f \in \mathbb{L}_m^1(Y)$ and $Y \in \mathbb{C}_b(M^1, \ldots, M^d)$ with

$$Z_t = Z_0 + \int_0^t f_s \, dY_s, \ \forall t \geq 0.$$

Then $\mathbb{F}_b(M^1, \ldots, M^d) = \mathbb{F}(M^1, \ldots, M^d)$.

Proof. Since bounded predictable process belong to $\mathbb{L}_m^1(N)$ for every martingale N, it follows that $\mathbb{C}_b(M^1, \ldots, M^d) \subseteq \mathbb{C}(M^1, \ldots, M^d)$. Thus $\mathbb{F}_b(M^1, \ldots, M^d) \subseteq \mathbb{F}(M^1, \ldots, M^d)$.

For the other part, let $Z \in \mathbb{F}$ be given by

$$Z_t = Z_0 + \int_0^t f_s \, dY_s, \ f \in \mathbb{L}_m^1(Y),$$

where

$$Y_t = Y_0 + \sum_{j=1}^d \int_0^t g_s^j \, dM_s^j$$

with $g^j \in \mathbb{L}_m^1(M^j)$. Let

$$\xi_s = 1 + \sum_{j=1}^d |g_s^j|, \ h_s^j = \frac{g_s^j}{\xi_s}$$

and

$$V_t = \sum_{j=1}^d \int_0^t h_s^j \, dM_s^j.$$

Since h^j are bounded, it follows that $V \in \mathbb{C}_b(M^1, M^2, \ldots, M^d)$. Using $g_s^j = \xi_s h_s^j$ and $g^j \in \mathbb{L}_m^1(M^j)$, it follows that $\xi \in \mathbb{L}_m^1(V)$ and

$$Y_t = Y_0 + \int_0^t \xi_s \, dV_s.$$

Since $f \in \mathbb{L}_m^1(Y)$, it follows that $f\xi \in \mathbb{L}_m^1(V)$ and $\int f \, dY = \int f\xi \, dV$. $\qquad \square$

Lemma 3.2. *Let $Z \in \mathbb{M}$ be such that there exists a sequence of stopping times $\sigma_k \uparrow \infty$ with $\mathsf{E}_\mathsf{P}[\sqrt{[Z,Z]_{\sigma_k}}] < \infty$ and $X^k \in \mathbb{F}(M^1, M^2, \ldots, M^d)$ where $X_t^k = Z_{t \wedge \sigma_k}$. Then*

$$Z \in \mathbb{F}(M^1, M^2, \ldots, M^d).$$

Proof. Let $X^k = Z_0 + \int f^k \, dY^k$ for $k \geq 1$ with $Y^k \in \mathbb{C}_b(M^1, M^2, \ldots, M^d)$ and $f^k \in \mathbb{L}_m^1(Y^k)$. Let $\phi^{k,j}$ be bounded predictable processes such that

$$Y_t^k = Y_0^k + \sum_{j=1}^d \int_0^t \phi_s^{k,j} \, dM_s^j.$$

Let $c_k > 0$ be a common bound for $\phi^{k,1}, \phi^{k,2}, \ldots, \phi^{k,d}$. Let us define η^j, f by

$$\eta_t^j = \sum_{k=1}^\infty \frac{1}{c_k} \phi_t^{k,j} 1_{\{(\sigma_{k-1}, \sigma_k]\}}(t).$$

$$f_t = \sum_{k=1}^{\infty} c_k f_t^k 1_{\{(\sigma_{-1}, \sigma_k]\}}(t).$$

$$Y_t = \sum_{j=1}^{d} \int_0^t \eta_s^j dM_s^j.$$

By definition, η^j is bounded by 1 for every j and thus $Y \in \mathbb{C}_b(M^1, M^2, \ldots, M^d)$. We can note that

$$Z_{t \wedge \sigma_k} - Z_{t \wedge \sigma_{k-1}} = X_{t \wedge \sigma_k}^k - X_{t \wedge \sigma_{k-1}}^k$$

$$= \int_0^t f_s^k 1_{\{(\sigma_{k-1}, \sigma_k]\}}(s) dY_s^k$$

$$= \int_0^t f_s 1_{\{(\sigma_{k-1}, \sigma_k]\}}(s) dY_s.$$

Thus

$$Z_{t \wedge \sigma_k} = Z_0 + \int_0^t 1_{[0, \sigma_k]}(s) f_s dY_s$$

and hence

$$[Z, Z]_{\sigma_k} = \int_0^{\sigma_k} (f_s)^2 d[Y, Y]_s.$$

Since by assumption, for all k

$$\mathsf{E}_{\mathsf{P}}[\sqrt{[Z, Z]_{\sigma_k}}] < \infty$$

it follows that $f \in \mathbb{L}_m^1(Y)$. This proves the required result. $\qquad \square$

Lemma 3.3. *Let $Z^n \in \mathbb{M}$ be such that $\mathsf{E}[|Z_t^n - Z_t|] \to 0$ for all t. Then there exists a sequence of stopping times $\sigma_k \uparrow \infty$ and a subsequence $\{n^j\}$ such that for each $k \geq 1$,*

$$\mathsf{E}[\sqrt{[Z, Z]_{\sigma_k}}] < \infty$$

and writing $Y^j = Z^{n^j}$,

$$\mathsf{E}[\sqrt{[Y^j - Z, Y^j - Z]_{\sigma_k}}] \to 0 \ as \ j \uparrow \infty. \tag{3.1}$$

Proof. Let $n^0 = 0$. For each j, $\mathsf{E}[|Z_j^n - Z_j|] \to 0$ as $n \to \infty$ and hence we can choose $n^j > n^{(j-1)}$ such that

$$\mathsf{E}[|Z_j^{n^j} - Z_j|] \leq 2^{-j}.$$

Then using Doob's maximal inequality we have

$$\mathsf{P}(\sup_{t \leq j} |Z_t^{n^j} - Z_t| \geq \frac{1}{j^2}) \leq \frac{j^2}{2^j}.$$

As a consequence, writing $Y^j = Z^{n^j}$, we have

$$\eta_t = \sum_{j=1}^{\infty} \sup_{s \leq t} |Y_s^j - Z_s| < \infty \ a.s. \text{ for all } t < \infty. \tag{3.2}$$

Now define

$$U_t = |Z_t| + \sum_{j=1}^{\infty} |Y_t^j - Z_t|.$$

In view of (3.2), it follows that U is r.c.l.l. adapted process. For any stopping time $\tau \leq m$, we have

$$E[U_\tau] = E[|Z_\tau|] + \sum_{j=1}^{\infty} E[|Y_\tau^j - Z_\tau|]$$

$$\leq E[|Z_m|] + \sum_{j=1}^{\infty} E[|Y_m^j - Z_m|]$$

$$\leq E[|Z_m|] + \sum_{j=1}^{m} E[|Y_m^j - Z_m|] + \sum_{j=m+1}^{\infty} 2^{-j}$$

$$< \infty.$$

Here, we have used that Z, $Y^j - Z$ being martingales, $|Z|$, $|Y^j - Z|$ are submartingales and $\tau \leq m$. Now defining

$$\sigma_k = \inf\{t : U_t \geq k \text{ or } U_{t-} \geq k\} \wedge k$$

it follows that σ_k are bounded stop times increasing to ∞ with

$$\sup_{s \leq \sigma_k} U_s \leq k + U_{\sigma_k}$$

and hence

$$E[\sup_{s \leq \sigma_k} U_s] < \infty. \tag{3.3}$$

Thus, for each k fixed $E[\sup_{s \leq \sigma_k} |Z_s|] < \infty$ and thus by Burkholder–Davis–Gundy inequality ($p = 1$ case), we have $E[\sqrt{[Z,Z]_{\sigma_k}}] < \infty$. In view of (3.2)

$$\lim_{j \to \infty} \sup_{s \leq \sigma_k} |Y_s^j - Z_s| = 0 \quad a.s.$$

and is dominated by $(\sup_{s \leq \sigma_k} U_s)$ which in turn is integrable as seen in (3.3). Thus by dominated convergence theorem, we have

$$\lim_{j \to \infty} E[\sup_{s \leq \sigma_k} |Y_s^j - Z_s|] = 0.$$

The result (3.1) now follows from the Burkholder–Davis–Gundy inequality ($p = 1$ case). $\qquad\square$

Lemma 3.4. *Let $V \in \mathbb{F}(M^1, M^2, \ldots, M^d)$ and τ be a bounded stopping time such that*

$$E[\sqrt{[V,V]_\tau}] < \infty.$$

Then for $\epsilon > 0$, there exists $U \in \mathbb{C}_b(M^1, M^2, \ldots, M^d)$ such that

$$E[\sqrt{[V-U, V-U]_\tau}] \leq \epsilon.$$

Proof. Let $V_t = V_0 + \int_0^t f\, dX$ where $f \in \mathbb{L}_m^1(X)$ and $X \in \mathbb{C}_b(M^1, M^2, \ldots, M^d)$. Since

$$[V, V]_t = \int_0^t |f_s|^2 d[X, X]_s,$$

the assumption on V gives

$$\mathsf{E}[\sqrt{\int_0^\tau |f_s|^2 d[X, X]_s}] < \infty. \tag{3.4}$$

Defining $f_s^k = f_s 1_{\{|f_s| \leq k\}}$, let

$$U^k = \int f^k\, dX.$$

Since $X \in \mathbb{C}_b(M^1, M^2, \ldots, M^d)$ and f^k is bounded, it follows that

$$U^k \in \mathbb{C}_b(M^1, M^2, \ldots, M^d).$$

Note that as $k \to \infty$,

$$\mathsf{E}[\sqrt{[V - U^k, V - U^k]_\tau}] = \mathsf{E}[\sqrt{\int_0^\tau |f_s|^2 1_{\{|f_s| > k\}} d[X, X]_s}] \to 0$$

in view of (3.4). The result now follows by taking $U = U^k$ with k large enough so that

$$\mathsf{E}[\sqrt{[V - U^k, V - U^k]_\tau}] < \epsilon.$$

$\square$

Lemma 3.5. *Suppose $Z \in \mathbb{M}$ and τ is a bounded stopping time such that $Z_t = Z_{t \wedge \tau}$ for all $t \geq 0$ and $\mathsf{E}[\sqrt{[Z, Z]_\tau}\,] < \infty$. Let $U^n \in \mathbb{C}_b(M^1, M^2, \ldots, M^d)$ with $U_0^n = 0$ be such that*

$$\mathsf{E}[\sqrt{[U^n - Z, U^n - Z]_\tau}\,] \leq 4^{-n}.$$

Then there exists $X \in \mathbb{C}_b(M^1, M^2, \ldots, M^d)$ and $f \in \mathbb{L}_m^1(X)$ such that

$$Z_t = Z_0 + \int_0^t f_s\, dX_s. \tag{3.5}$$

Proof. Since $U^n \in \mathbb{C}_b(M^1, M^2, \ldots, M^d)$ with $U_0^n = 0$, get bounded predictable processes $\{f^{n,j} : n \geq 1, 1 \leq j \leq d\}$ such that

$$U_t^n = \sum_{j=1}^d \int_0^t f_s^{n,j} dM_s^j. \tag{3.6}$$

Without loss of generality, we assume that $U_t^n = U_{t \wedge \tau}^n$ and $f_s^{n,j} = f_s^{n,j} 1_{[0,\tau]}(s)$. Let

$$\zeta = \sum_{n=1}^\infty 2^n \sqrt{[U^n - Z, U^n - Z]_\tau}.$$

Then $\mathsf{E}[\zeta] < \infty$ and hence $\mathsf{P}(\zeta < \infty) = 1$. Let

$$\eta = \zeta + \sqrt{[Z, Z]_\tau} + \sum_{j=1}^d \sqrt{[M^j, M^j]_\tau}$$

Let $c = \mathsf{E}[\exp\{-\eta\}]$ and let Q be the probability measure on $(\Omega, \mathcal{F})$ defined by

$$\frac{d\mathsf{Q}}{d\mathsf{P}} = \frac{1}{c}\exp\{-\eta\}.$$

Then it follows that $\alpha = \mathsf{E}_{\mathsf{Q}}[\eta^2] < \infty$. Noting that

$$\eta^2 \geq [Z, Z]_\tau + \sum_{j=1}^{d}[M^j, M^j]_\tau + \sum_{n=1}^{\infty} 2^{2n}[U^n - Z, U^n - Z]_\tau$$

we have $\mathsf{E}_{\mathsf{Q}}[[Z, Z]_\tau] < \infty$, $\mathsf{E}_{\mathsf{Q}}[[M^j, M^j]_\tau] < \infty$ for $1 \leq j \leq d$. Likewise, $\mathsf{E}_{\mathsf{Q}}[[U^n - Z, U^n - Z]_\tau] < \infty$ and so $\mathsf{E}_{\mathsf{Q}}[[U^n, U^n]_\tau] < \infty$. Note that Z, M^j are no longer martingales under Q, but we do not need that.

Let $\widetilde{\Omega} = [0, \infty) \times \Omega$. Recall that the predictable σ-field $\mathcal{P}$ is the smallest σ field on $\widetilde{\Omega}$ with respect to which all continuous adapted processes are measurable. We will define signed measures Γ_{ij} on $\mathcal{P}$ as follows: for $E \in \mathcal{P}$, $1 \leq i, j \leq d$ let

$$\Gamma_{ij}(E) = \int_\Omega \int_0^\tau 1_E(s, \omega)\, d[M^i, M^j]_s(\omega)\, d\mathsf{Q}(\omega).$$

Let $\Lambda = \sum_{j=1}^{d} \Gamma_{jj}$. From the properties of quadratic variation $[M^i, M^j]$, it follows that for all $E \in \mathcal{P}$, the matrix $((\Gamma_{ij}(E)))$ is non-negative definite. Further, Γ_{ij} is absolutely continuous with respect to Λ $\forall i, j$. It follows (see appendix) that we can get predictable processes c^{ij} such that

$$\frac{d\Gamma_{ij}}{d\Lambda} = c^{ij}$$

and that $C = ((c^{ij}))$ is a non-negative definite matrix. By construction $|c^{ij}| \leq 1$ and are predictable. We can diagonalise C (i.e. obtain singular value decomposition) in a measurable way (see appendix) to obtain predictable processes b^{ij}, d^j such that for all i, k, (writing $\delta_{ik} = 1$ if $i = k$ and $\delta_{ik} = 0$ if $i \neq k$),

$$\sum_{j=1}^{d} b_s^{ij} b_s^{kj} = \delta_{ik}, \tag{3.7}$$

$$\sum_{j=1}^{d} b_s^{ji} b_s^{jk} = \delta_{ik}, \tag{3.8}$$

$$\sum_{j,l=1}^{d} b_s^{ij} c_s^{jl} b_s^{kl} = \delta_{ik} d_s^i. \tag{3.9}$$

Since $((c_s^{ij}))$ is non-negative definite, it follows that $d_s^i \geq 0$. For $1 \leq j \leq d$, let

$$N^k = \sum_{l=1}^{d} \int b_s^{kl}\, dM^l.$$

Then N^k are P- martingales since b^{ik} is are bounded predictable processes. Further,

$$[N^i, N^k] = \sum_{j,l=1}^{d} b_s^{ij} \int b_s^{kl}\, d[M^j, M^l]$$

and hence for any bounded predictable process h and $i \neq k$, we have

$$
\begin{aligned}
\mathsf{E}_{\mathsf{Q}}\Big[\int_0^\tau h_s \, d[N^i, N^k]]\Big] &= \int_\Omega \int_0^\tau h_s \sum_{j,l=1}^d b_s^{ij} b_s^{kl} \, d[M^j, M^l] \, d\mathsf{Q}(\omega) \\
&= \int_{\bar\Omega} h \sum_{j,l=1}^d b^{ij} b^{kl} \, d\Gamma_{jl} \\
&= \int_{\bar\Omega} h \sum_{j,l=1}^d b^{ij} b^{kl} c^{jl} \, d\Lambda \\
&= 0,
\end{aligned}
$$

where the last step follows from (3.9). As a consequence, for bounded predictable processes h^i, $1 \leq i \leq d$, it follows that

$$
\mathsf{E}_{\mathsf{Q}}\Big[\sum_{i,k=1}^d \int_0^\tau h_s^i h_s^k \, d[N^i, N^k]_s\Big] = \mathsf{E}_{\mathsf{Q}}\Big[\sum_{k=1}^d \int_0^\tau (h_s^k)^2 \, d[N^k, N^k]_s\Big]. \tag{3.10}
$$

Let us observe that (3.10) holds for any predictable processes $\{h^i : 1 \leq i \leq d\}$ provided the RHS is finite: we can first note that it holds for $\tilde{h}^i = h^i 1_{\{|h| \leq c\}}$ where $|h| = \sum_{i=1}^d |h^i|$ and then let $c \uparrow \infty$. Note that for $n \geq m$

$$
\begin{aligned}
\sqrt{[U^n - U^m, U^n - U^m]_\tau} &\leq \sqrt{[U^n - Z, U^n - Z]_\tau} + \sqrt{[U^m - Z, U^m - Z]_\tau} \\
&\leq 2^{-m} \eta
\end{aligned}
$$

and hence

$$
\mathsf{E}_{\mathsf{Q}}[[U^n - U^m, U^n - U^m]_\tau] \leq 4^{-m} \alpha. \tag{3.11}
$$

Let us define $g^{n,k} = \sum_{j=1}^d f^{n,j} b^{kj}$. Then note that

$$
\begin{aligned}
\sum_{k=1}^d \int g^{n,k} \, dN^k &= \sum_{k=1}^d \sum_{j=1}^d \int f^{n,j} b^{kj} \, dN^k \\
&= \sum_{k=1}^d \sum_{j=1}^d \sum_{l=1}^d \int f^{n,j} b^{kj} b_s^{kl} \, dM^l \\
&= \sum_{j=1}^d \int f^{n,j} \, dM^j \\
&= U^n,
\end{aligned} \tag{3.12}
$$

where in the last but one step, we have used (3.8). Using (3.10) for $n \geq m$ we have

$$
\mathsf{E}_{\mathsf{Q}}[[U^n - U^m, U^n - U^m]_\tau] = \mathsf{E}_{\mathsf{Q}}\Big[\sum_{k=1}^d \int_0^\tau (g_s^{n,k} - g_s^{m,k})^2 \, d[N^k, N^k]_s\Big] \tag{3.13}
$$

and using (3.11), we conclude

$$\mathsf{Q}(\sum_{k=1}^{d} \int_0^\tau (g_s^{n,k} - g_s^{m,k})^2 d[N^k, N^k] \geq \frac{1}{m^4}) \leq m^4 \mathsf{E}_{\mathsf{Q}}[[U^n - U^m, U^n - U^m]_\tau]$$

$$\leq \alpha m^4 4^{-m}.$$

(3.14)

Since $\mathsf{E}_{\mathsf{Q}}[[M^i, M^i]_\tau] < \infty$ and $g^{n,i}$ are bounded for all n, i, it follows that $\mathsf{E}_{\mathsf{Q}}[[N^i, N^i]_\tau] < \infty$. Thus defining a measure Θ on $\mathcal{P}$ by

$$\Theta(E) = \int \Big[\sum_{k=1}^{d} \int_0^\tau 1_E(s, \omega) d[N^k, N^k]_s(\omega) \Big] d\mathsf{Q}(\omega)$$

we get (using (3.11) and (3.13))

$$\int (g^{m+1,k} - g^{m,k})^2 d\Theta \leq \alpha 4^{-m}$$

and as a consequence, using Cauchy–Schwartz inequality, we get

$$\int \sum_{m=1}^{\infty} |g^{m+1,k} - g^{m,k}| d\Theta \leq \sqrt{\Theta(\bar{\Omega})\alpha} < \infty.$$

Defining

$$g_s^k = \limsup_{m \to \infty} g_s^{m,k},$$

it follows that $g^{m,k} \to g^k$ a.s. Θ and as a consequence, taking limit in (3.14) as $n \to \infty$, we get

$$\mathsf{Q}(\sum_{k=1}^{d} \int_0^\tau (g_s^k - g_s^{m,k})^2 d[N^k, N^k]_s \geq \frac{1}{m^4}) \leq m^4 4^{-m}.$$

(3.15)

Since Q and P and equivalent measures, it follows that

$$\mathsf{P}(\sum_{k=1}^{d} \int_0^\tau (g_s^k - g_s^{m,k})^2 d[N^k, N^k]_s \geq \frac{1}{m^4}) \to 0 \text{ as } m \to \infty.$$

(3.16)

In view of (3.12), we have for $m \leq n$

$$[U^n, U^n]_\tau = \sum_{i,j=1}^{d} \int_0^\tau g_s^{n,i} g_s^{n,j} d[N^i, N^j]_s$$

(3.17)

and

$$[U^n - U^m, U^n - U^m]_\tau = \sum_{i,j=1}^{d} \int_0^\tau (g_s^{n,i} - g_s^{m,i})(g_s^{n,j} - g_s^{m,j}) d[N^i, N^j]_s.$$

(3.18)

Taking limit in (3.17) as $n \to \infty$, we get (using Fatou's lemma)

$$\mathsf{E}_{\mathsf{P}}[\sqrt{\sum_{i,j=1}^{d} \int_0^\tau g_s^i g_s^j d[N^i, N^j]}] \leq \mathsf{E}_{\mathsf{P}}[\sqrt{[Z, Z]_\tau}]$$

(3.19)

(since (3.1) implies $\mathsf{E}_\mathsf{P}[\sqrt{[U^n, U^n]_\tau}\,] \to \mathsf{E}_\mathsf{P}[\sqrt{[Z, Z]_\tau}\,]$). Let us define bounded predictable processes ϕ^j and predictable process h^n, h and a P-martingale X as follows:

$$h_s = 1 + \sum_{i=1}^{d} |g_s^i|, \tag{3.20}$$

$$\phi_s^j = \frac{g_s^j}{h_s}, \tag{3.21}$$

$$X_t = \sum_{j=1}^{d} \int_0^t \phi_s^j \, dN_s^j. \tag{3.22}$$

Since ϕ^j is predictable, $|\phi^j| \le 1$ it follows that $X \in \mathbb{C}_b(M^1, M^2, \ldots, M^d)$ and

$$[X, X]_t = \sum_{j,k=1}^{d} \int_0^t \phi_s^j \phi_s^k \, d[N^j, N^k]_s. \tag{3.23}$$

Noting that $g_s^j = h_s \phi_s^j$ by definition, we conclude using (3.19) that

$$\int_0^t (h_s)^2 \, d[X, X]_s = \sum_{j,k=1}^{d} \int_0^t g_s^j g_s^k \, d[N^j, N^k]_s$$

and hence that

$$\mathsf{E}_\mathsf{P}[\sqrt{\int_0^\tau (h_s)^2 \, d[X, X]_s}\,] \le \mathsf{E}_\mathsf{P}[\sqrt{[Z, Z]_\tau}\,]. \tag{3.24}$$

Since $h = h 1_{[0,\tau]}$, we conclude that $h \in \mathbb{L}_m^1(X)$ and $Y = \int h \, dX$ is a martingale with $Y_t = Y_{t \wedge \tau}$ for all t. Observe that

$$[U^n, X]_t = \sum_{k,j=1}^{d} \int_0^t g_s^{n,k} \phi_s^j \, d[N^k, N^j]_s$$

and hence

$$
\begin{aligned}
[U^n, Y]_t &= \int_0^t h_s \, d[U^n, X]_s \\
&= \sum_{k,j=1}^{d} \int_0^t g_s^{n,k} h_s \phi_s^j \, d[N^k, N^j]_s \\
&= \sum_{k,j=1}^{d} \int_0^t g_s^{n,k} g_s^j \, d[N^k, N^j]_s.
\end{aligned}
$$

As a consequence

$$[U^n - Y, U^n - Y]_t = [U^n, U^n]_t - 2[U^n, Y]_t + [Y, Y]_t$$

$$= \sum_{k,j=1}^{d} \int_0^t g_s^{n,k} g_s^{n,j} d[N^k, N^j]_s - 2 \sum_{k,j=1}^{d} \int_0^t g_s^{n,k} g_s^j d[N^k, N^j]_s$$

$$+ \sum_{k,j=1}^{d} \int_0^t g_s^k g_s^j d[N^k, N^j]_s$$

$$= \sum_{k,j=1}^{d} \int_0^t (g_s^{n,k} - g_s^k)(g_s^{n,j} - g_s^j) d[N^k, N^j]_s$$

and thus using (3.10) we get

$$\mathsf{E}_\mathsf{Q}[[U^n - Y, U^n - Y]_\tau] = \mathsf{E}_\mathsf{Q}[\sum_{k=1}^{d} \int_0^\tau (g_s^{n,k} - g_s^k)^2 d[N^k, N^k]_s. \qquad (3.25)$$

Taking $\liminf$ as $n \to \infty$ on the RHS in (3.13) and using (3.11), we conclude

$$\mathsf{E}_\mathsf{Q}[\sum_{k=1}^{d} \int_0^\tau (g_s^k - g_s^{m,k})^2 d[N^k, N^k]_s] \le \alpha 4^{-m}$$

and hence (3.25) yields

$$\mathsf{E}_\mathsf{Q}[[U^n - Y, U^n - Y]_\tau] \le \alpha 4^{-n}.$$

Thus $[U^n - Y, U^n - Y]_\tau \to 0$ in Q-probability and hence in P-probability. By assumption, $[U^n - Z, U^n - Z]_\tau \to 0$ in P-probability. Since

$$[Y - Z, Y - Z]_\tau \le 2([Y - U^n, Y - U^n]_\tau + [Z - U^n, Z - U^n]_\tau)$$

for every n, it follows that

$$[Y - Z, Y - Z]_\tau = 0 \ \ a.s. \ \mathsf{P}. \qquad (3.26)$$

Since Y, Z are P-martingales such that $Z_t = Z_{t \wedge \tau}$ and $Y_t = Y_{t \wedge \tau}$, (3.26) implies $Y_t - Y_0 = Z_t - Z_0$ for all t. Recall that by construction, $Y_0 = 0$, $Y = \int h \, dX$ and $h \in \mathbb{L}_m^1(X)$, $X \in \mathbb{C}_b(M^1, M^2, \ldots, M^d)$. Thus (3.5) holds. $\qquad \square$

We now come to the proof of Theorem 2.2. Let $Z^n \in \mathbb{F}(M^1, M^2, \ldots, M^d)$ be such that $\mathsf{E}[|Z_t^n - Z_t|] \to 0$ for all t. We have to show that $Z \in \mathbb{F}(M^1, M^2, \ldots, M^d)$.

Using Lemma 3.3, get a sequence of stopping times $\sigma_k \uparrow \infty$ and a subsequence $\{n^j\}$ such that $Y^j = Z^{n^j}$ satisfies for each $k \ge 1$, $\mathsf{E}[\sqrt{[Z, Z]_{\sigma_k}}] < \infty$ and

$$\mathsf{E}[\sqrt{[Y^j - Z, Y^j - Z]_{\sigma_k}}] \to 0 \text{ as } j \uparrow \infty.$$

Fix k and let $\tilde{Z}_t = Z_{t \wedge \sigma_k}$. We will show that $\tilde{Z} \in \mathbb{F}(M^1, M^2, \ldots, M^d)$. This will complete the proof in view of Lemma 3.2. By relabelling, we assume that

$$\mathsf{E}[\sqrt{[Z^n - Z, Z^n - Z]_{\sigma_k}}] \to 0 \text{ as } n \uparrow \infty. \qquad (3.27)$$

Since $Z^n \in \mathbb{F}(M^1, M^2, \ldots, M^d)$ and $\mathsf{E}[\sqrt{[Z^n, Z^n]_{\sigma_k}}] < \infty$ (at least for large n, say $n \geq n^*$), using Lemma 3.4, for $n \geq n^*$ we can get $U^n \in \mathbb{C}_b(M^1, M^2, \ldots, M^d)$ such that

$$\mathsf{E}[\sqrt{[U^n - Z^n, U^n - Z^n]_{\sigma_k}}] \leq \frac{1}{n}. \tag{3.28}$$

(3.27) and (3.28) give

$$\mathsf{E}[\sqrt{[U^n - \tilde{Z}, U^n - \tilde{Z}]_{\sigma_k}}] \to 0.$$

Thus $\tilde{Z} \in \mathbb{F}(M^1, M^2, \ldots, M^d)$ in view of Lemma 3.5

Now we turn to proof of Theorem 2.1. Let $\xi^n \in \mathbb{K}_T$ be such that $\xi^n \to \xi$ in $\mathbb{L}^1(\Omega, \mathcal{F}, \mathsf{P})$. Let $\xi^n = X_T^n$ where $X^n \in \mathbb{F}(M^1, M^2, \ldots, M^d)$. Let us define $Z_t^n = X_{t \wedge T}^n$. Then $Z^n \in \mathbb{F}(M^1, M^2, \ldots, M^d)$ and the assumption on ξ^n implies

$$Z_t^n \to Z_t = \mathsf{E}[\xi \mid \mathcal{F}_t] \text{ in } \mathbb{L}^1(\Omega, \mathcal{F}, \mathsf{P}) \ \forall t.$$

Thus Theorem 2.2 implies $Z \in \mathbb{F}(M^1, M^2, \ldots, M^d)$ and thus $\xi = Z_T$ belongs to $\mathbb{K}_T$.

4. Sigma-martingales

Let M be a martingale, $f \in \mathbb{L}(M)$ and $Z = \int f \, dM$. Then Z is a local martingale if and only if $f \in \mathbb{L}_m^1(M)$. In answer to a question raised by P. A. Meyer, Chou [3] introduced a class Σ_m of semimartingales consisting of $Z = \int f \, dM$ for $f \in \mathbb{L}(M)$. Emery [7] constructed example of f, M such that $f \in \mathbb{L}(M)$ but $Z = \int f \, dM$ is not a local martingale. Such processes occur naturally in mathematical finance and have been called sigma-martingales by Delbaen and Schachermayer[6].

Definition 4.1. A semimartingale X is said to be a *sigma-martingale* if $\exists \phi \in \mathbb{L}(X)$ such that ϕ is $(0, \infty)$ valued and

$$M_t = \int_0^t \phi_s \, dX_s \tag{4.1}$$

is a martingale. Our first observation is:

Lemma 4.2. *Every local martingale N is a sigma-martingale.*

Proof. Let $\eta_n \uparrow \infty$ be a sequence of stopping times such that $N_{t \wedge \eta_n}$ is a martingale,

$$\sigma_n = \inf\{t \geq 0 : |N_t| \geq n \text{ or } |N_{t-}| \geq n\} \wedge n$$

and $\tau_n = \sigma_n \wedge \eta_n$. It follows that $N_{t \wedge \tau_n}$ is a uniformly integrable martingale and $a_n = \mathsf{E}[\sqrt{[N, N]_{\tau_n}}] < \infty$.

Define

$$h_s = \frac{1}{1 + |N_0|} 1_{\{0\}}(s) + \sum_{n=1}^\infty \frac{1}{2^n (1 + a_n)} 1_{(\tau_{n-1}, \tau_n]}(s).$$

Then h being bounded belongs to $\mathbb{L}(N)$ and $M = \int h \, dN$ is a local martingale with

$$\sup_{t < \infty} \mathsf{E}[\sqrt{[M, M]_t}] < \infty. \tag{4.2}$$

Thus M is a uniformly integrable martingale. Since h is $(0, \infty)$ valued by definition, it follows that N is a sigma-martingale. $\square$

This leads to

Lemma 4.3. *A semimartingale X is a sigma-martingale if and only if there exists a uniformly integrable martingale M satisfying (4.2) and a predictable process $\psi \in \mathbb{L}(M)$ such that*

$$X_t = \int_0^t \psi_s \, dM_s. \tag{4.3}$$

Proof. Let X be given by (4.3) with M being a martingale satisfying (4.2) and $\psi \in \mathbb{L}(M)$, then defining

$$g_s = \frac{1}{(1 + (\psi_s)^2)}, \quad N_t = \int_0^t g_s \, dX_s$$

it follows that $N = \int g\psi \, dM$. Since $g\psi$ is bounded by 1 and M satisfies (4.2), it follows that N is a martingale. Thus X is a sigma-martingale.

Conversely, given a sigma-martingale X and a $(0, \infty)$ valued predictable process ϕ such that $N = \int \phi \, dX$ is a martingale, get h as in Lemma 4.2 and let $M = \int h \, dN = \int h\phi \, dX$. Then M is a uniformly integrable martingale that satisfies (4.2) and $h\phi$ is a $(0, \infty)$ valued predictable process. $\square$

From the definition, it is not obvious that sum of sigma-martingales is also a sigma-martingale, but this is so as the next result shows.

Theorem 4.4. *Let X^1, X^2 be sigma-martingales and a_1, a_2 be real numbers. Then $Y = a_1 X^1 + a_2 X^2$ is also a sigma-martingale.*

Proof. Let ϕ^1, ϕ^2 be $(0, \infty)$ valued predictable processes such that

$$M_t^i = \int_0^t \phi_s^i \, dX_s^i, \quad i = 1, 2$$

are uniformly integrable martingales. Then, writing $\xi = \min(\phi^1, \phi^2)$ and $\eta_s^i = \frac{\xi_s}{\phi_s^i}$, it follows that

$$N_t^i = \int_0^t \eta_s^i \, dM_s^i = \int_0^t \xi_s \, dX_s^i$$

are uniformly integrable martingales since η^i is bounded by one. Clearly, $Y = a_1 X^1 + a_2 X^2$ is a semimartingale and $\xi \in \mathbb{L}(X^i)$ for $i = 1, 2$ implies $\xi \in \mathbb{L}(Y)$ and

$$\int_0^t \xi_s \, dY_s = a_1 N_s^1 + a_2 N_s^2$$

is a uniformly integrable martingale. Since ξ is $(0, \infty)$ valued predictable process, it follows that Y is a sigma-martingale. $\square$

The following result gives conditions under which a sigma-martingale is a local martingale.

Lemma 4.5. *Let X be a sigma-martingale with $X_0 = 0$. Then X is a local martingale if and only if there exists a sequence of stopping times $\tau_n \uparrow \infty$ such that*

$$\mathsf{E}[\sqrt{[X,X]_{\tau_n}}] < \infty \ \ \forall n. \tag{4.4}$$

Proof. Let X be a sigma-martingale and ϕ, ψ, M be such that (4.1), (4.2) holds. Let $\psi_s = \frac{1}{\phi_s}$ and as noted above, (4.3) holds. Then

$$[X,X]_t = \int_0^t (\psi_s)^2 d[M,M]_s.$$

Defining $\psi_s^k = \psi_s 1_{\{|\psi_s| \le k\}}$, it follows that

$$X^k = \int_0^t \psi_s^k dM_s$$

is a uniformly integrable martingale. Noting that for $k \ge 1$

$$[X - X^k, X - X^k]_t = \int_0^t (\psi_s)^2 1_{\{k < |\psi_s|\}} d[M,M]_s$$

the assumption (4.4) implies that for each n fixed,

$$\mathsf{E}[\sqrt{[X - X^k, X - X^k]_{\tau_n}}] \to 0 \text{ as } k \to \infty.$$

The Burkholder–Davis–Gundy inequality $(p = 1)$ implies that for each n fixed,

$$\mathsf{E}[\sup_{0 \le t \le \tau_n} |X_t - X_t^k|] \to 0 \text{ as } k \to \infty.$$

and as a consequence $X_t^{[n]} = X_{t \wedge \tau_n}$ is a martingale for all n and so X is a local martingale. Conversely, if X is a local martingale with $X_0 = 0$, and σ_n are stop times increasing to ∞ such that $X_{t \wedge \sigma_n}$ are martingales then defining $\zeta_n = \inf\{t : |X_t| \ge n\}$ and $\tau_n = \sigma_n \wedge \zeta_n$, it follows that $\mathbb{E}[|X_{\tau_n}|] < \infty$ and since

$$\sup_{t \le \tau_n} |X_t| \le n + |X_{\tau_n}|$$

it follows that $\mathbb{E}[\sup_{t \le \tau_n} |X_t|] < \infty$. Thus, (4.4) holds in view of Burkholder–Davis–Gundy inequality $(p = 1)$. $\qquad \square$

The previous result gives us:

Corollary 4.6. *A bounded sigma-martingale X is a martingale.*

Proof. Since X is bounded, say by K, it follows that jumps of X are bounded by $2K$. Thus jumps of the increasing process $[X,X]$ are bounded by $4K^2$ and thus X satisfies (4.4) for

$$\tau_n = \inf\{t \ge 0 \ : \ [X,X]_t \ge n\}.$$

Thus X is a local martingale and being bounded, it is a martingale. $\qquad \square$

Essentially the same argument also gives that if a sigma-martingale X satisfies $|X_t| \le \xi$ where ξ is square integrable, then X is a martingale, in fact a square integrable martingale.

Here is a variant of the example given by Emery [7] of a sigma-martingale that is not a local martingale. Let τ be a random variable with exponential

distribution (assumed to be $(0, \infty)$ valued without loss of generality) and ξ with $\mathsf{P}(\xi = 1) = \mathsf{P}(\xi = -1) = 0.5$, independent of τ. Let

$$M_t = \xi 1_{[\tau, \infty)}(t)$$

and $\mathcal{F}_t = \sigma(M_s : s \le t)$. Easy to see that M is a martingale. Let $f_t = \frac{1}{t} 1_{(0, \infty)}(t)$ and $X_t = \int_0^t f dM$. Then X is a sigma-martingale and

$$[X, X]_t = \frac{1}{\tau^2} 1_{[\tau, \infty)}(t).$$

For any stopping time σ, it can be checked that σ is a constant on $\sigma < \tau$ and thus if σ is not identically equal to 0, $\sigma \ge (\tau \wedge a)$ for some $a > 0$. Thus, $\sqrt{[X, X]_\sigma} \ge \frac{1}{\tau} 1_{\{\tau < a\}}$. It follows that for any stop time σ, not identically zero, $\mathsf{E}[\sqrt{[X, X]_\sigma}] = \infty$ and so X is not a local martingale.

The next result shows that $\int f dX$ is a sigma-martingale if X is one.

Lemma 4.7. *Let X be a sigma-martingale, $f \in \mathbb{L}(X)$ and let $U = \int f dX$. Then U is a sigma-martingale.*

Proof. Let M be a martingale and $\psi \in \mathbb{L}(M)$ be such that $X = \int \psi dM$ (as in Lemma 4.3). Now $U = \int f dX = \int f \psi dM$. Thus, once again invoking Lemma 4.3, one concludes that X is a sigma-martingale. $\qquad \square$

We now introduce the class of equivalent sigma-martingale measures (ESMM) and show that it is a convex set. Let $X^1, \dots, X^d$ be r.c.l.l. adapted processes and let $\mathbb{E}^s(X^1, \dots, X^d)$ denote the class of probability measures Q such that $X^1, \dots, X^d$ are sigma-martingales with respect to Q. Let

$$\mathbb{E}_\mathsf{P}^s(X^1, \dots, X^d) = \{\mathsf{Q} \in \mathbb{E}^s(X^1, \dots, X^d) : \mathsf{Q} \text{ is equivalent to } \mathsf{P}\}$$

and

$$\tilde{\mathbb{E}}_\mathsf{P}^s(X^1, \dots, X^d)$$
$$= \{\mathsf{Q} \in \mathbb{E}^s(X^1, \dots, X^d) : \mathsf{Q} \text{ is absolutely continuous with respect to } \mathsf{P}\}.$$

Theorem 4.8. *For semimartingales $X^1, \dots, X^d$,*

$$\mathbb{E}^s(X^1, \dots, X^d), \quad \mathbb{E}_\mathsf{P}^s(X^1, \dots, X^d) \quad and \quad \tilde{\mathbb{E}}_\mathsf{P}^s(X^1, \dots, X^d)$$

are convex sets.

Proof. Let us consider the case $d = 1$. Let $\mathsf{Q}^1, \mathsf{Q}^2 \in \mathbb{E}^s(X)$. Let ϕ^1, ϕ^2 be $(0, \infty)$ valued predictable processes such that

$$M_t^i = \int_0^t \phi_s^i dX_s$$

are martingales under Q_i, $i = 1, 2$. Let $\phi_s = min(\phi_s^1, \phi_s^2)$ and let

$$M_t = \int_0^t \phi_s dX_s.$$

Noting that $M_t = \int_0^t \xi_s^i dM_s^i$ where $\xi_s = \phi_s(\phi_s^i)^{-1}$ is bounded, it follows that M is a martingale under $\mathsf{Q}^i, i = 1, 2$. Now if Q is any convex combination of $\mathsf{Q}^1, \mathsf{Q}^2$, it follows that M is a Q martingale and hence $X_t = \int_0^t (\phi_s)^{-1} dM_s$ is a

sigma-martingale under Q. Thus $\mathbb{E}^s_\mathsf{P}(X)$ is a convex set. Since $\mathbb{E}^s(X^1, \ldots, X^d) = \cap^d_{j=1} \mathbb{E}^s(X^j)$ it follows that $\mathbb{E}^s(X^1, \ldots, X^d)$ is convex.

Convexity of $\mathbb{E}^s_\mathsf{P}(X^1, \ldots, X^d)$ and $\tilde{\mathbb{E}}^s_\mathsf{P}(X^1, \ldots, X^d)$ follows from this. $\qquad \square$

For sigma-martingales $M^1, M^2, \ldots, M^d$ the classes $\mathbb{C}(M^1, M^2, \ldots, M^d)$ and $\mathbb{F}(M^1, \ldots, M^d)$ are defined exactly as they were in case $M^1, \ldots, M^d$ are martingales. In particular, they remains subsets of the class of martingales $\mathbb{M}$.

Lemma 4.9. *Let $M^1, \ldots, M^d$ be sigma-martingales and let ϕ^j be $(0, \infty)$ valued predictable processes such that*

$$N^j_t = \int_0^t \phi^j_s \, dM^j_s \tag{4.5}$$

are uniformly integrable martingales. Then

$$\mathbb{C}(M^1, M^2, \ldots, M^d) = \mathbb{C}(N^1, N^2, \ldots, N^d), \tag{4.6}$$

$$\mathbb{F}(M^1, M^2, \ldots, M^d) = \mathbb{F}(N^1, N^2, \ldots, N^d). \tag{4.7}$$

Proof. Let $\psi^j_s = (\phi^j_s)^{-1}$. Note that $M^j = \int \psi^j \, dN^j$. If $Y \in \mathbb{C}(M^1, M^2, \ldots, M^d)$ is given by

$$Y_t = \sum_{j=1}^d \int_0^t f^j_s \, dM^j_s, \quad f^j \in \mathbb{L}(M^j) \tag{4.8}$$

then defining $g^j = f^j \psi^j$, we can see that $g^j \in \mathbb{L}(N^j)$ and $\int f^j \, dM^j = \int g^j \, dN^j$. Thus

$$Y_t = \sum_{j=1}^d \int_0^t g^j_s \, dN^j_s, \quad g^j \in \mathbb{L}(N^j). \tag{4.9}$$

Similarly, if $Y \in \mathbb{C}(N^1, N^2, \ldots, N^d)$ is given by (4.9), then defining $f^j = \phi^j g^j$, we can see that Y satisfies (4.8). Thus (4.6) is true. Now (4.7) follows from (4.6). $\quad \square$

5. Integral Representation with Respect to Martingales

Let $M^1, \ldots, M^d$ be sigma-martingales.

Definition 5.1. A sigma-martingale N is said to have an *integral representation* with respect to $M^1, \ldots, M^d$ if $N \in \mathbb{F}(M^1, M^2, \ldots, M^d)$ or in other words, $\exists Y \in \mathbb{C}(M^1, M^2, \ldots, M^d)$ and $f \in \mathbb{L}(Y)$ such that

$$N_t = N_0 + \int_0^t f_s \, dY_s \quad \forall t. \tag{5.1}$$

Here is an observation needed later.

Lemma 5.2. *Let M be a P-martingale. Let Q be a probability measure equivalent to P. Let $\xi = \frac{d\mathsf{Q}}{d\mathsf{P}}$ and let Z be the r.c.l.l. martingale given by $Z_t = \mathsf{E}_\mathsf{P}[\xi \mid \mathcal{F}_t]$. Then*

(i) M is a Q-martingale if and only if MZ is a P-martingale.
(ii) M is a Q-local martingale if and only if MZ is a P-local martingale.
(iii) If M is a Q-local martingale then $[M, Z]$ is a P-local martingale.
(iv) If M is a Q-sigma-martingale then $[M, Z]$ is a P-sigma-martingale.

Proof. For a stopping time σ, let η be a non-negative $\mathcal{F}_\sigma$ measurable random variable. Then

$$\mathsf{E}_\mathsf{Q}[\eta] = \mathsf{E}_\mathsf{P}[\eta\xi] = \mathsf{E}_\mathsf{P}[\eta\mathsf{E}[\xi \mid \mathcal{F}_\sigma]] = \mathsf{E}_\mathsf{P}[\eta Z_\sigma].$$

Thus M_s is Q integrable if and only if $M_s Z_s$ is P-integrable. Further, for any stopping time σ,

$$\mathsf{E}_\mathsf{Q}[M_\sigma] = \mathsf{E}_\mathsf{P}[M_\sigma Z_\sigma]. \tag{5.2}$$

Thus (i) follows from the observation that an integrable adapted process N is a martingale if and only if $\mathsf{E}[N_\sigma] = \mathsf{E}[N_0]$ for all bounded stopping times σ. For (ii), if M is a Q-local martingale, then get stopping times $\tau_n \uparrow \infty$ such that for each n, $M_{t \wedge \tau_n}$ is a martingale. Then we have

$$\mathsf{E}_\mathsf{Q}[M_{\sigma \wedge \tau_n}] = \mathsf{E}_\mathsf{P}[M_{\sigma \wedge \tau_n} Z_{\sigma \wedge \tau_n}]. \tag{5.3}$$

Thus $M_{t \wedge \tau_n} Z_{t \wedge \tau_n}$ is a P-martingale and thus MZ is a P- local martingale. The converse follows similarly.

For (iii), note that

$$M_t Z_t = M_0 Z_0 + \int_0^t M_{s-} \, dZ_s + \int_0^t Z_{s-} \, dM_s + [M, Z]_t \tag{5.4}$$

and the two stochastic integrals appearing in (5.4) are P local martingales, the result follows from (ii). For (iv), representing the Q sigma-martingale M as $M = \int \psi \, dN$, where N is a Q martingale and $\psi \in \mathbb{L}(N)$, we see

$$[M, Z] = \int_0^t \psi_s \, d[N, Z]_s.$$

By (iii), $[N, Z]$ is a Q sigma-martingale and hence $[M, Z]$ is a Q sigma-martingale.
$\square$

The main result on integral representation is:

Theorem 5.3. *Let $M^1, \ldots, M^d$ be sigma-martingales on $(\Omega, \mathcal{F}, \mathsf{P})$. Suppose $\mathcal{F}_0$ is trivial. Then the following are equivalent.*

(i) *All bounded martingales admit representation with respect to $M^1, \ldots, M^d$.*
(ii) *All uniformly integrable martingales admit representation with respect to $M^1, \ldots, M^d$.*
(iii) *All sigma-martingales admit representation with respect to $M^1, \ldots, M^d$.*
(iv) *P is an extreme point of the convex set $\mathbb{E}^s(M^1, \ldots, M^d)$.*
(v) *$\tilde{\mathbb{E}}^s_\mathsf{P}(M^1, \ldots, M^d) = \{\mathsf{P}\}$.*
(vi) *$\mathbb{E}^s_\mathsf{P}(M^1, \ldots, M^d) = \{\mathsf{P}\}$.*

Proof. Since every bounded martingale is uniformly integrable and a uniformly integrable martinagle is a sigma-martingale, we have

$$\text{(iii)} \Rightarrow \text{(ii)} \Rightarrow \text{(i)}.$$

(i) $\Rightarrow$ (ii) is an easy consequence of Theorem 2.2- given a uniformly integrable martingale Z, let $\xi = \lim_{t \to \infty} Z_t$, so that $Z_t = \mathsf{E}[\xi \mid \mathcal{F}_t]$. For $n \geq 1$, let us define martingales Z^n as follows:

$$Z_t^n = \mathsf{E}[\xi 1_{\{|\xi| \leq n\}} \mid \mathcal{F}_t]$$

where take the r.c.l.l. version of the martingale. It is easy to see that Z^n are bounded martingales and in view of (i), $Z^n \in \mathbb{F}(M^1, M^2, \ldots, M^d)$. Moreover, for $n \geq t$

$$\mathsf{E}[|Z_t^n - Z_t|] \leq \mathsf{E}[\xi 1_{\{|\xi| > n\}}]$$

and hence for all t, $\mathsf{E}[|Z_t^n - Z_t|] \to 0$. Thus $Z \in \mathbb{F}(M^1, M^2, \ldots, M^d)$ by Theorem 2.2. This proves (ii).

We next prove (ii) $\Rightarrow$ (iii). Let X be a sigma-martingale. In view of Lemma 4.3, get a uniformly integrable martingale N and a predictable process ψ such that

$$X = \int \psi dN.$$

Let $N_t = N_0 + \int_0^t f_s dY_s$ where $Y \in \mathbb{C}(M^1, M^2, \ldots, M^d)$. Then we have

$$X_t = X_0 + \int_0^t f_s \psi dY_s$$

and thus X admits an integral representation with respect to $M^1, \ldots, M^d$.

Suppose (v) holds and suppose $\mathsf{Q}_1, \mathsf{Q}_2 \in \mathbb{E}^s(M^1, M^2, \ldots, M^d)$ and $\mathsf{P} = \alpha \mathsf{Q}_1 + (1 - \alpha)\mathsf{Q}_2$. It follows that $\mathsf{Q}_1, \mathsf{Q}_2$ are absolutely continuous with respect to P and hence $\mathsf{Q}_1, \mathsf{Q}_2 \in \tilde{\mathbb{E}}_\mathsf{P}^s(M^1, M^2, \ldots, M^d)$. In view of (v), $\mathsf{Q}_1 = \mathsf{Q}_2 = \mathsf{P}$ and thus P is an extreme point of $\mathbb{E}^s(M^1, M^2, \ldots, M^d)$ and so (iv) is true.

Since $\mathbb{E}_\mathsf{P}^s(M^1, M^2, \ldots, M^d) \subseteq \tilde{\mathbb{E}}_\mathsf{P}^s(M^1, M^2, \ldots, M^d)$, it follows that (v) implies (vi).

On the other hand, suppose (vi) is true and $\mathsf{Q} \in \tilde{\mathbb{E}}_\mathsf{P}^s(M^1, M^2, \ldots, M^d)$. Then $\mathsf{Q}_1 = \frac{1}{2}(\mathsf{Q} + \mathsf{P}) \in \mathbb{E}_\mathsf{P}^s(M^1, M^2, \ldots, M^d)$. Then uniqueness in (vi) implies $\mathsf{Q}_1 = \mathsf{P}$ and hence $\mathsf{Q} = \mathsf{P}$ and thus (v) holds.

Till now we have proved (i) $\iff$ (ii) $\iff$ (iii) and (iv) $\Leftarrow$ (v) $\iff$ (vi). To complete the proof, we will show (iii) $\Rightarrow$ (v) and (iv) $\Rightarrow$ (i).

Suppose that (iii) is true and let $\mathsf{Q} \in \tilde{\mathbb{E}}_\mathsf{P}^s(M^1, M^2, \ldots, M^d)$. Now let ξ be the Radon–Nikodym derivative of Q with respect to P. Let R denote the r.c.l.l. martingale: $R_t = \mathsf{E}[\xi \mid \mathcal{F}_t]$. Since $\mathcal{F}_0$ is trivial, $N_0 = 1$. In view of property (iii), we can get $Y \in \mathbb{C}(M^1, M^2, \ldots, M^d)$ and a predictable processes $f \in \mathbb{L}(Y)$ such that

$$R_t = 1 + \int_0^t f_s dY_s. \tag{5.5}$$

Note that

$$[R, R]_t = \int_0^t f_s^2 d[Y, Y]_s. \tag{5.6}$$

Since M^j is a sigma-martingale under Q for each j, it follows that Y is a Q sigma-martingale. By Lemma 5.2, this gives $[Y, R]$ is a P sigma-martingale and hence

$$V_t^k = \int_0^t f_s 1_{\{|f_s| \leq k\}} d[Y, R]_s \tag{5.7}$$

is a P sigma-martingale. Noting that

$$[Y, R]_t = \int_0^t f_s d[Y, Y]_s$$

we see that

$$V_t^k = \int_0^t f_s^2 1_{\{|f_s| \leq k\}} d[Y,Y]_s. \tag{5.8}$$

Thus we can get $(0, \infty)$ valued predictable processes ϕ^j such that

$$U_t^k = \int_0^t \phi_s^k dV_s^k$$

is a martingale. But U^k is a non-negative martingale with $U_0^k = 0$. As a result U^k is identically equal to 0 and thus so is V^k. It then follows that (see (5.6)) $[R,R] = 0$ which yields R is identical to 1 and so $\mathsf{Q} = \mathsf{P}$. Thus $\tilde{\mathbb{E}}_\mathsf{P}^s(M^1, M^2, \ldots, M^d)$ is a singleton. Thus (iii) $\Rightarrow$ (v).

To complete the proof, we will now prove that (iv) $\Rightarrow$ (i).
Suppose P is an extreme point of $\mathbb{E}^s(M^1, M^2, \ldots, M^d)$. Since M^j is a sigma-martingale under P, we can choose $(0, \infty)$ valued predictable ϕ^j such that

$$N_t^j = \int_0^t \phi_s^j dM_s^j$$

is a uniformly integrable martingale under P and as seen in Lemma 4.9, we then have

$$\mathbb{F}(M^1, M^2, \ldots, M^d) = \mathbb{F}(N^1, N^2, \ldots, N^d).$$

Suppose (i) is not true. We will show that this leads to a contradiction. So suppose S is a bounded martingale that does not admit representation with respect to $M^1, M^2, \ldots, M^d$, i.e. $S \notin \mathbb{F}(M^1, M^2, \ldots, M^d) = \mathbb{F}(N^1, N^2, \ldots, N^d)$, then for some T,

$$S_T \notin \mathbb{K}_T(N^1, N^2, \ldots, N^d)$$

We have proven in Theorem 2.1 that $\mathbb{K}_T(N^1, N^2, \ldots, N^d)$ is closed in $\mathbb{L}^1(\Omega, \mathcal{F}, \mathsf{P})$. Since $\mathbb{K}_T$ is not equal to $\mathbb{L}^1(\Omega, \mathcal{F}_T, \mathsf{P})$, by the Hahn–Banach Theorem, there exists $\xi \in \mathbb{L}^\infty(\Omega, \mathcal{F}_T, \mathsf{P})$, $\mathsf{P}(\xi \neq 0) > 0$ such that

$$\int \eta \xi d\mathsf{P} = 0 \quad \forall \eta \in \mathbb{K}_T.$$

Then for all constants c, we have

$$\int \eta(1 + c\xi) d\mathsf{P} = \int \eta d\mathsf{P} \quad \forall \eta \in \mathbb{K}_T. \tag{5.9}$$

Since ξ is bounded, we can choose a $c > 0$ such that

$$\mathsf{P}(c|\xi| < \frac{1}{2}) = 1.$$

Now, let Q be the measure with density $\eta = (1 + c\xi)$. Then Q is a probability measure. Thus (5.9) yields

$$\int \eta d\mathsf{Q} = \int \eta d\mathsf{P} \quad \forall \eta \in \mathbb{K}_T. \tag{5.10}$$

For any bounded stop time τ and $1 \leq j \leq d$, $N_{\tau \wedge T}^j \in \mathbb{K}_T$ and hence

$$\mathsf{E}_\mathsf{Q}[N_{\tau \wedge T}^j] = \mathsf{E}_\mathsf{P}[N_{\tau \wedge T}^j] = N_0^j \tag{5.11}$$

On the other hand,

$$
\begin{aligned}
\mathsf{E}_{\mathsf{Q}}[N^j_{\tau \vee T}] &= \mathsf{E}_{\mathsf{P}}[\eta N^j_{\tau \vee T}] \\
&= \mathsf{E}_{\mathsf{P}}[\mathsf{E}_{\mathsf{P}}[\eta N^j_{\tau \vee T} \mid \mathcal{F}_T]] \\
&= \mathsf{E}_{\mathsf{P}}[\eta \mathsf{E}_{\mathsf{P}}[N^j_{\tau \vee T} \mid \mathcal{F}_T]] \\
&= \mathsf{E}_{\mathsf{P}}[\eta N_T] \\
&= \mathsf{E}_{\mathsf{Q}}[N_T] \\
&= N^j_0.
\end{aligned}
\tag{5.12}
$$

where we have used the facts that η is $\mathcal{F}_T$ measurable, N^j is a P martingale and (5.11). Now

$$
\mathsf{E}_{\mathsf{Q}}[N^j_\tau] = \mathsf{E}_{\mathsf{Q}}[N^j_{\tau \wedge T}] + \mathsf{E}_{\mathsf{Q}}[N^j_{\tau \vee T}] - \mathsf{E}_{\mathsf{Q}}[N^j_T] = N^j_0.
$$

Thus N^j is a Q martingale and since

$$
M^j = \int_0^t \frac{1}{\phi^j_s} \, dN^j_s
$$

it follows that M^j is a Q sigma-martingale. Thus $\mathsf{Q} \in \mathbb{E}^s(M^1, \ldots, M^d)$. Similarly, if $\tilde{\mathsf{Q}}$ is the measure with density $\eta = (1 - c\xi)$, we can prove that $\tilde{\mathsf{Q}} \in \mathbb{E}^s(M^1, \ldots, M^d)$. Since $\mathsf{P} = \frac{1}{2}(\mathsf{Q} + \tilde{\mathsf{Q}})$, this contradicts the assumption that P is an extreme point of $\mathbb{E}^s(M^1, \ldots, M^d)$. Thus (iv) $\Rightarrow$ (i). This completes the proof. $\qquad\square$

6. Completeness of Markets

Let the (discounted) prices of d securities be given by $X^1, \ldots, X^d$. We assume that X^j are semimartingales and that they satisfy the property NFLVR so that an ESMM exists. See [6].

Theorem 6.1. (The Second Fundamental Theorem Of Asset Pricing)
Let $X^1, \ldots, X^d$ be semimartingales on $(\Omega, \mathcal{F}, \mathsf{P})$ such that $\mathbb{E}^s_{\mathsf{P}}(X^1, \ldots, X^d)$ is non-empty. Suppose $\mathcal{F}_0$ is trivial. Then the following are equivalent:

(a) For all $T < \infty$, for all $\mathcal{F}_T$ measurable bounded random variables ξ (bounded by say K), there exist $g^j \in \mathbb{L}(X^j)$ with

$$
Y_t = \sum_{j=1}^d \int_0^t g^j_s \, dX^j_s
\tag{6.1}
$$

a constant c and $f \in \mathbb{L}(Y)$ such that $|\int_0^t f_s \, dY_s| \leq 2K$ and

$$
\xi = c + \int_0^T f_s \, dY_s.
\tag{6.2}
$$

(b) The set $\mathbb{E}^s_{\mathsf{P}}(X^1, \ldots, X^d)$ is a singleton.

Proof. First suppose that $\mathbb{E}^s_{\mathsf{P}}(X^1, \ldots, X^d) = \{\mathsf{Q}\}$. Given a bounded $\mathcal{F}_T$ measurable random variable ξ, consider the martingale

$$
M_t = \mathsf{E}_{\mathsf{Q}}[\xi \mid \mathcal{F}_t].
$$

Note that M is bounded by K. In view of the equivalence of (i) and (v) in Theorem 5.3, we get that M admits a representation with respect to $X^1, \ldots, X^d$ - thus we get $g^j \in \mathbb{L}(X^j)$ and $f \in \mathbb{L}(Y)$ where Y is given by by (6.1), with

$$M_t = M_0 + \int_0^t f_s \, dY_s.$$

Since $\mathcal{F}_0$ is trivial, M_0 is a constant. Since M is bounded by K, it follows that $\int_0^t f_s \, dY_s$ is bounded by $2K$. Thus (b) implies (a).

Now suppose (a) is true. Let Q be an ESMM. Let M_t be a martingale. We will show that $M \in \mathbb{F}(X^1, \ldots, X^d)$, $i.e.$ M admits integral representation with respect to $X^1, \ldots, X^d$. In view of Lemmas 3.2 and 4.9, suffices to show that for each $T < \infty$, $N \in \mathbb{F}(X^1, \ldots, X^d)$, where N is defined by $N_t = M_{t \wedge T}$.

Let $\xi = N_T$. Then in view of assumption (a), we have

$$\xi = c + \int_0^T f_s \, dY_s$$

with Y given by (6.1), a constant c and $f \in \mathbb{L}(Y)$ such that $U_t = \int_0^t f_s \, dY_s$ is bounded. Since U is a sigma-martingale that is bounded, it follows that U is a martingale. It follows that

$$N_t = c + \int_0^t f_s \, dY_s, \quad 0 \le t \le T.$$

Thus $N \in \mathbb{F}(X^1, \ldots, X^d)$.

We have proved that (i) in Theorem 5.3 holds and hence (v) holds, $i.e.$ the ESMM is unique. $\qquad\square$

Appendix

I: For a non-negative definite $d \times d$ symmetric matrix C, the eigenvalue-eigenvector decomposition gives us a representation

$$C = B^T D B, \tag{A.1}$$

where

$$B \text{ is a } d \times d \text{ orthogonal matrix} \tag{A.2}$$

and

$$D \text{ is a } d \times d \text{ diagonal matrix.} \tag{A.3}$$

This decomposition is not unique. Note that for each non-negative definite symmetric matrix C, the set Γ_C of pairs (B, D) of $d \times d$ matrices satisfying (A.1)-(A.3) is compact. Thus it admits a measurable selection - in other words, for each C, we can pick $\theta(C) = (B, D) \in \Gamma_C$ in such a way so that θ is a measurable mapping. (See [8] or Corollary 5.2.6 [17]). Of course, (A.1) implies that BCB^T is a diagonal matrix.

II: Let $\mathcal{D}$ be a σ-field on a non-empty set Γ and for $1 \le i, j \le d$, λ_{ij} be signed measures on $(\Gamma, \mathcal{D})$ such that for all $E \in \mathcal{D}$, the matrix $((\lambda_{ij}(E)))$ is a symmetric non-negative definite matrix. Let $\mu(E) = \sum_{i=1}^d \lambda_{ii}(E)$. Then for $1 \le i, j \le d$ there

exist versions c^{ij} of the Radon–Nikodym derivate $\frac{d\lambda_{ij}}{d\mu}$ such that for all $\gamma \in \Gamma$, the matrix $((c^{ij}(\gamma)))$ is non-negative definite.

To see this, for $1 \leq i \leq j \leq d$ let f^{ij} be a version of the Radon–Nikodym derivative $\frac{d\lambda_{ij}}{d\mu}$ and let $f^{ji} = f^{ij}$. For rationals $r_1, r_2, \ldots, r_d$, let

$$A_{r_1, r_2, \ldots, r_d} = \{\gamma : \sum_{ij} r_i r_j f^{ij}(\gamma) < 0\}.$$

Then $\mu(A_{r_1, r_2, \ldots, r_d}) = 0$ and hence $\mu(A) = 0$ where

$$A = \cup \{A_{r_1, r_2, \ldots, r_d} : \ r_1, r_2, \ldots, r_d \text{ rationals}\}.$$

The required version is now given by

$$c^{ij}(\gamma) = f^{ij}(\gamma) 1_{A^c}(\gamma).$$

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