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SEMIMARTINGALES IN LOCALLY COMPACT ABELIAN GROUPS AND THEIR CHARACTERISTIC TRIPLES

M. S. BINGHAM

ABSTRACT. The concepts of semimartingales and their characteristic triples are introduced for stochastic processes taking their values in a locally compact second countable abelian group. It is proved that the third characteristic always exists and that the first two characteristics always exist when the group is compact. Any continuous additive Gaussian process in a locally compact second countable abelian group is shown to be a semimartingale with characteristics that agree with its Lévy-Khinchine canonical triple.

1. Introduction

In order to motivate the new developments in this paper, we first recall the definitions of the characteristics of a real-valued semimartingale $X = \{X(t) : t \geq 0\}$ with a right continuous filtration and sample paths that are a.s. càdlàg. See [6] for full details. Choose and fix a continuous truncation function h on the real line; that is, $h : \mathbb{R} \to \mathbb{R}$ is a continuous function with compact support and such that $h(x) = x$ for all x in a neighbourhood N of zero. Define the process $\hat{X}$ for $t \geq 0$ by

$$\hat{X}(t) := X(t) - \sum_{s \leq t} \left[\Delta X(s) - h(\Delta X(s))\right], \tag{1.1}$$

where $\Delta X(s) := X(s) - X(s-)$ is the jump in X at s. The subtracted sum has only finitely many non-zero terms because X has càdlàg sample paths and in the interval $[0, t]$ a càdlàg function can have only a finite number of jumps outside the neighbourhood N of zero. In going from X to $\hat{X}$, each jump $\Delta X(s)$ that has a sufficiently large magnitude to differ from $h(\Delta X(s))$ has been replaced by $h(\Delta X(s))$. Therefore the jumps in $\hat{X}$ are uniformly bounded, which implies that $\hat{X}$ is a special semimartingale. Consequently $\hat{X}$ has a canonical decomposition

$$\hat{X}(t) = X(0) + M(t) + B(t), \quad (t \geq 0),$$

in which $M := \{M(t) : t \geq 0\}$ is a local martingale and $B := \{B(t) : t \geq 0\}$ is a predictable process with finite variation, the sample paths of M and B are almost surely càdlàg, $M(0) = 0$ and $B(0) = 0$. Moreover, M and B are uniquely determined up to indistinguishability. We also know that X has a continuous martingale part X^c, which is uniquely determined up to indistinguishability. Let us recall the following.

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Definition 1.1. The *characteristics* of the real-valued semimartingale X are the three predictable a.s. càdlàg processes B, C, ν, where:

B is the predictable finite variation process above;
$C := \langle X^c, X^c \rangle$ is the compensator of the process $(X^c)^2$, so C is the non-decreasing continuous process such that $(X^c)^2 - C$ is a local martingale;
ν is the compensator of the jump measure μ of X.

(B, C, ν) is called the *characteristic triple* of the semimartingale X.

Whereas the process B may depend on the choice of truncation function h, the processes C and ν do not. The characteristics B, C and ν play similar roles in semimartingale theory to those of the drift, the variance of the Gaussian part and the Lévy measure in the theory of processes with independent increments.

These ideas generalise in a natural way to semimartingales with values in $\mathbb{R}^d$ (for any positive integer d), and in the book of Jacod and Shiryaev [6] it is shown that, in suitable circumstances, conditions for the convergence in law of a sequence of $\mathbb{R}^d$-valued semimartingales to a limiting semimartingale can be given in terms of the characteristics of the semimartingales. It is natural to ask whether we can develop an analogous theory of semimartingales and their characteristics, together with a corresponding theory of convergence, for stochastic processes which take their values in a locally compact second countable abelian group G. Such a theory should be applicable, for example, to proving functional central limit theorems on G. This paper presents some first steps towards the development of such a theory by suggesting a way of defining G-valued semimartingales and their characteristics. It is proved that the third characteristic always exists and that the first two characteristics always exist if G is compact. In the case when $G = \mathbb{R}$ the definitions are consistent with the classical definitions for real-valued semimartingales, provided that an appropriate local inner product is used. For any locally compact second countable abelian group G that can support a Gaussian distribution, it is shown that any continuous additive Gaussian process on G is a semimartingale with respect to a suitable stochastic basis and that its characteristics exist and coincide with the canonical triple in the Lévy-Khinchine representation.

Except where stated otherwise, the stochastic processes considered in this paper will be defined on the same stochastic basis $\mathbf{B} = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$, in which the filtration is right continuous. Accordingly, when no other basis is mentioned, properties like being 'adapted' or 'predictable', and concepts like 'martingale', 'local martingale' and 'semimartingale' should be understood as being relative to this stochastic basis. As is customary in order to avoid tedious repetition, the qualification "a.s.(P)" is sometimes omitted and should therefore be mentally inserted wherever appropriate.

The following notation will be used throughout. G is a locally compact second countable abelian group and $\widehat{G}$ is its dual group. Therefore $\widehat{G}$ is the group of all continuous homomorphisms (characters) of G into the unit circle group $\mathbb{T}$ and is endowed with the topology of uniform convergence on compact subsets of G. It is well known that $\widehat{G}$ is also a locally compact second countable abelian group, and the Pontryagin duality theorem tells us that the dual group of $\widehat{G}$ can be identified with G. The value of the character $y \in \widehat{G}$ at the point $x \in G$ will be denoted by

$\langle x, y\rangle$ and the identity element of G will be denoted by e. The group operation will be denoted by $+$ both in G and in $\widehat{G}$. All neighbourhoods that appear will be assumed without loss of generality to be Borel sets.

General information about locally compact abelian groups can be found in the books by Hewitt and Ross [4] and Rudin [11]. The books by Heyer [5] and Parthasarathy [9] are recommended for probability theory on such groups.

A key role in this paper will be played by a local inner product, which we now define.

Definition 1.2. By a *local inner product* on $G\times\widehat{G}$ we mean a continuous function $g : G \times \widehat{G} \to \mathbb{R}$ with the following properties:

(i) $g(x, y_1 + y_2) = g(x, y_1) + g(x, y_2)$ for all $x \in G$, $y_1, y_2 \in \widehat{G}$;

(ii) $g(-x, y) = -g(x, y)$ for all $x \in G$, $y \in \widehat{G}$;

(iii) $\sup_{x\in G} \sup_{y\in K} |g(x, y)| < \infty$ for every compact set $K \subseteq \widehat{G}$;

(iv) for every compact $K \subseteq \widehat{G}$ there exists a neighbourhood N of the identity e in G such that $\langle x, y\rangle = \exp(ig(x, y))$ for all $x \in N$, $y \in K$;

(v) for every compact $K \subseteq \widehat{G}$, $\sup_{y\in K} |g(x, y)| \to 0$ as $x \to e$.

Parthasarathy, Ranga Rao and Varadhan proved in [10] that local inner products exist on $G \times \widehat{G}$ for every locally compact second countable abelian group G. Their proof can also be found in Parthasarathy [9]; see Lemma 5.3 in Chapter IV. See also Heyer [5], pages 340–343. Local inner products are not uniquely determined, as is illustrated by the following example.

Example 1.3. Consider the case in which $G = \mathbb{R}^d$, where $\mathbb{R}$ is the real line with its usual topology and additive group structure, and d is a positive integer. Then the dual group $\widehat{G}$ can be identified with $\mathbb{R}^d$, and $\langle x, y\rangle = \exp(ix \cdot y)$ holds for all $x, y \in \mathbb{R}^d$, where $x \cdot y$ is the usual inner product of x and y. Let g be any local inner product on $\mathbb{R}^d \times \mathbb{R}^d$. If $x \in \mathbb{R}^d$, then $g(x, \cdot)$ is continuous and satisfies the Cauchy functional equation:

$$g(x, y_1 + y_2) = g(x, y_1) + g(x, y_2) \quad \text{for all } y_1, y_2 \in \mathbb{R}^d.$$

Therefore $g(x, y) \equiv k(x) \cdot y$ for some $k(x) \in \mathbb{R}^d$. Using the properties of the local inner product g given in Definition 1.2, it is easy to see that k is a bounded continuous function from $\mathbb{R}^d$ into $\mathbb{R}^d$ and $k(-x) = -k(x)$ for all x. Also, the validity for all y of

$$\exp(k(x) \cdot y) = \exp(ig(x, y)) = \langle x, y\rangle = \exp(ix \cdot y)$$

whenever x is sufficiently near to the zero element $\mathbf{0}$ of $\mathbb{R}^d$ implies that $k(x) = x$ whenever x is in a sufficiently small neighbourhood N_1 of $\mathbf{0}$. Conversely, any bounded continuous function k from $\mathbb{R}^d$ into $\mathbb{R}^d$ such that $k(-x) = -k(x)$ for all x and $k(x) = x$ for all x in a neighbourhood of $\mathbf{0}$ leads to a local inner product on $\mathbb{R}^d \times \mathbb{R}^d$ given by $g(x, y) \equiv k(x) \cdot y$.

From the properties listed in Definition 1.2, it follows that any local inner product g also satisfies the following condition: given any compact set $K \subseteq \widehat{G}$, there is a neighbourhood M of the identity in G such that

$$g(x_1 + x_2, y) = g(x_1, y) + g(x_2, y) \quad \text{for all } x_1, x_2 \in M,\ y \in K. \tag{1.2}$$

To see this, let $K \subseteq \widehat{G}$ be compact and use properties (iv) and (v) of g in Definition 1.2 to deduce that there is a neighbourhood N of e in G such that $\langle x, y\rangle = \exp(ig(x, y))$ and $|g(x, y)| < \frac{1}{2}$ both hold for all $x \in N$ and $y \in K$. Then let M be a neighbourhood of e such that $M + M \subseteq N$. Using the fact that $x \mapsto \langle x, y\rangle$ is a homomorphism of G into $\mathbb{T}$, we then have for $x_1, x_2 \in M$ and $y \in K$

$$\begin{aligned}\exp(ig(x_1 + x_2, y)) &= \langle x_1 + x_2, y\rangle = \langle x_1, y\rangle\langle x_2, y\rangle \\ &= \exp(ig(x_1, y)) \exp(ig(x_2, y)) = \exp(i[g(x_1, y) + g(x_2, y)])\end{aligned}$$

As $|g(x_1 + x_2, y) - g(x_1, y) - g(x_2, y)| < \frac{3}{2}$, equation (1.2) follows.

Another property of g that will be used later is given by the following lemma, which is essentially the same as Lemma 2.4 in Bingham [1].

Lemma 1.4. *Given a compact subset $K \subseteq \widehat{G}$ the following exist: a constant $c_K \geq 0$, a finite set $F_K \subseteq \widehat{G}$ and a neighbourhood N_K of the identity element in G such that*

$$\sup_{y\in K} |g(x, y)| \leq c_K \max_{y\in F_K} |g(x, y)| \quad \textit{for every } x \in N_K.$$

2. Unwrapping a càdlàg Function

Denote by $\mathbb{D}$ the Skorokhod space of G-valued càdlàg functions defined on the half-line $[0, \infty)$. Choose and fix a fixed local inner product g on $G \times \widehat{G}$. For later use we shall begin by seeing how we can use g to modify the jumps of an arbitrary function $\alpha \in \mathbb{D}$ such that $\alpha(0) = e$ to obtain a function $\beta \in \mathbb{D}$ such that, for each $y \in \widehat{G}$, the càdlàg function $t \mapsto \langle\beta(t), y\rangle$ with values in the unit circle $\mathbb{T}$ can be 'unwrapped' from the circle to the line to give a real-valued càdlàg function $t \mapsto w(\alpha, t, y)$ that is uniquely determined by α, g and y.

Lemma 2.1. *Let $\alpha \in \mathbb{D}$ with $\alpha(0) = e$ and let $\Delta\alpha(s) := \alpha(s) - \alpha(s-)$ for all $s > 0$, $\Delta\alpha(0) := e$. Then all of the following hold.*

(a) *There exists a uniquely determined $\gamma \in \mathbb{D}$ with $\gamma(0) = e$ such that*

$$\langle\gamma(t), y\rangle = \prod_{0\leq s\leq t} \langle\Delta\alpha(s), y\rangle \exp[-ig(\Delta\alpha(s), y)] \tag{2.1}$$

for all $t > 0$ and all $y \in \widehat{G}$.

(b) *If $\beta = \alpha - \gamma$, then, for each $y \in \widehat{G}$, there is a uniquely determined real-valued càdlàg function $t \mapsto w(\alpha, t, y)$ such that*

$$\langle\beta(t), y\rangle = \exp[iw(\alpha, t, y)] \quad \textit{for all } t \geq 0, \tag{2.2}$$

$$w(\alpha, s, y) - w(\alpha, s-, y) = g(\Delta\alpha(s), y) \quad \textit{for all } s > 0, \tag{2.3}$$

and

$$w(\alpha, 0, y) = 0. \tag{2.4}$$

(c) *For each $t \geq 0$, the mapping $y \mapsto w(\alpha, t, y)$ is continuous and*

$$w(\alpha, t, y_1 + y_2) = w(\alpha, t, y_1) + w(\alpha, t, y_2) \tag{2.5}$$

for all $y_1, y_2 \in \widehat{G}$.

(d) *If K is any compact subset of $\widehat{G}$, then*
  (i) *for every $t \geq 0$, $w(\alpha, s, y) \to w(\alpha, t, y)$ uniformly in $y \in K$ as $s \downarrow t$, and*
  (ii) *for every $t > 0$, $w(\alpha, s, y) \to w(\alpha, t-, y)$ uniformly in $y \in K$ as $s \uparrow t$.*

Remark 2.2. Given $y \in \widehat{G}$ there is, by property (iv) of g, a neighbourhood N of the identity in G such that $\langle x, y\rangle = \exp(ig(x, y))$ for all $x \in N$. As α is càdlàg there are only finitely many points $s \in [0, t]$ for which $\Delta\alpha(s) \notin N$. Consequently, only a finite number of the factors in the product on the right hand side of (2.1) can be different from 1. Therefore the product is well defined.

For each jump $\Delta\alpha(s)$ in α, property (i) of g and the continuity of g imply that the mapping $y \mapsto \exp(ig(\Delta\alpha(s), y))$ is a continuous character on $\widehat{G}$. Therefore, by the Pontryagin duality theorem, for each jump $\Delta\alpha(s)$ in α, there is a unique element $\Delta'\alpha(s)$ in G such that $\langle\Delta'\alpha(s), y\rangle = \exp(ig(\Delta\alpha(s), y))$ for all $y \in \widehat{G}$. Using this notation, (2.1) can be rewritten as

$$\langle\gamma(t), y\rangle = \prod_{0\leq s\leq t} \langle\Delta\alpha(s) - \Delta'\alpha(s), y\rangle.$$

From this it follows that, for $s > 0$, $\gamma(s) - \gamma(s-) = \Delta\alpha(s) - \Delta'\alpha(s)$. The function β can be obtained from the function α by subtracting γ from α, and the effect of this is to replace each jump $\Delta\alpha(s)$ in α by the corresponding element $\Delta'\alpha(s)$ of G.

Proof. Lemma 2.1 will be proved in a sequence of steps.

Step 1: First, let K be a compact subset of $\widehat{G}$. Using property (iv) of g and equation (1.2), let M be a neighbourhood of the identity element e in G such that

$$\langle x, y\rangle = \exp(ig(x, y)) \quad \text{for all } x \in M,\ y \in K \tag{2.6}$$

and

$$g(x_1 + x_2, y) = g(x_1, y) + g(x_2, y) \quad \text{for all } x_1, x_2 \in M,\ y \in K. \tag{2.7}$$

Let N be a symmetric neighbourhood of e in G such that $N + N + N + N \subseteq M$.

Let $\alpha \in \mathbb{D}$ with $\alpha(0) = e$. Define $w(\alpha, 0, y) = 0$ for all $y \in \widehat{G}$. For now fix a positive number T, and suppose that $\Delta\alpha(s) \in N$ whenever $0 < s \leq T$, where $\Delta\alpha(s) := \alpha(s) - \alpha(s-)$ is the jump in α at the point s. The right continuity of α implies that for every $s \in [0, T]$ there exists $\delta(s) > 0$ such that $\alpha(s') - \alpha(s) \in N$ whenever $s \leq s' < s + 2\delta(s)$. For $s > 0$ the existence of left limits and the assumption that $\Delta\alpha(s) \in N$ imply that $\delta(s)$ can be chosen so that, in addition, $\alpha(s') - \alpha(s) = \alpha(s') - \alpha(s-) + \Delta\alpha(s) \in N + N$ whenever $s - 2\delta(s) < s' < s$ and $s' \geq 0$. Hence $\alpha(s') - \alpha(s) \in N + N$ whenever $s - 2\delta(s) < s' < s + 2\delta(s)$ and $s' \in [0, T]$. The collection of intervals $\{(s - \delta(s), s + \delta(s)) : s \in [0, T]\}$ covers the compact set $[0, T]$, so there is a finite subcover, $\{(s_j - \delta(s_j), s_j + \delta(s_j)) : j = 1, 2, \ldots, J\}$ say, of $[0, T]$. Let $\delta = \min_{1\leq j\leq J} \delta(s_j)$.

If $t_1, t_2 \in [0, T]$ with $|t_1 - t_2| < \delta$, then $t_1 \in (s_j - \delta(s_j), s_j + \delta(s_j))$ for some j, so $\alpha(t_1) - \alpha(s_j) \in N + N$. Also, for the same j, $|s_j - t_2| \leq |s_j - t_1| + |t_1 - t_2|$

$< \delta(s_j) + \delta \leq 2\delta(s_j)$, so $\alpha(s_j) - \alpha(t_2) \in N + N$. Hence $\alpha(t_1) - \alpha(t_2) \in N + N + N + N \subseteq M$ whenever $t_1, t_2 \in [0, T]$ and $|t_1 - t_2| < \delta$.

Now suppose that $0 < t \leq T$ and let the partition

$$\mathcal{D} = (t_0 = 0 < t_1 < \cdots < t_r = t)$$

of $[0, t]$ have mesh less than δ; i.e., $\max_{1 \leq j \leq r}(t_j - t_{j-1}) < \delta$. Fix $y \in K$ and consider the sum

$$S(\mathcal{D}) := \sum_{j=1}^{r} g(\alpha(t_j) - \alpha(t_{j-1}), y).$$

Suppose that we create a refinement $\mathcal{D}'$ of $\mathcal{D}$ by inserting an extra point s. Then $t_{j-1} < s < t_j$ for some j and

$$S(\mathcal{D}') - S(\mathcal{D}) = g(\alpha(t_j) - \alpha(s), y) + g(\alpha(s) - \alpha(t_{j-1}), y) - g(\alpha(t_j) - \alpha(t_{j-1}), y).$$

By the choice of δ the elements $\alpha(t_j) - \alpha(s)$ and $\alpha(s) - \alpha(t_{j-1})$ are in M. Equation (2.7) then implies that $S(\mathcal{D}') - S(\mathcal{D}) = 0$. By a simple inductive argument, it therefore follows that $S(\mathcal{D}') = S(\mathcal{D})$ for every (finite) partition $\mathcal{D}'$ of $[0, t]$ that is a refinement of $\mathcal{D}$. By considering a common refinement of any two given partitions of mesh less then δ, it then follows also that $S(\mathcal{D})$ does not depend on the choice of $\mathcal{D}$ if the mesh of $\mathcal{D}$ is less than δ. Therefore, for $0 < t \leq T$, we can define

$$w_N(\alpha, t, y) := \sum_{j=1}^{r} g(\alpha(t_j) - \alpha(t_{j-1}), y)$$

for all (finite) partitions $\mathcal{D} = (0 = t_0 < t_1 < \cdots < t_r = t)$ such that $\max_j(t_j - t_{j-1}) < \delta$. Note that δ depends on T but not on the choice of $t \in [0, T]$.

Step 2: Now let $\alpha \in \mathbb{D}$ be arbitrary with $\alpha(0) = e$, and let M, N, K, T, y be as above in Step 1. Define $\alpha_N \in \mathbb{D}$ by

$$\alpha_N(t) := \alpha(t) - \sum \{\Delta\alpha(s) : 0 \leq s \leq t,\ \Delta\alpha(s) \notin N\} \tag{2.8}$$

for all $t \geq 0$. Because $\alpha \in \mathbb{D}$, the set $\{s \in [0, t] : \Delta\alpha(s) \notin N\}$ is finite for each $t \geq 0$; consequently α_N is well defined. Moreover, for each $s \in [0, t]$, we have $\Delta\alpha_N(s) \in N$. By the argument in Step 1 applied to α_N (taking $0 < t \leq T$), there exists $\delta = \delta(T, N) > 0$ such that

$$w_N(\alpha_N, t, y) = \sum_{j=1}^{r} g(\alpha_N(t_j) - \alpha_N(t_{j-1}), y)$$

when the partition $0 = t_0 < t_1 < \cdots < t_r = t$ satisfies $\max_j(t_j - t_{j-1}) < \delta$.

Next define

$$s_N(\alpha, t, y) := w_N(\alpha_N, t, y) + \sum \{g(\Delta\alpha(s), y) : 0 \leq s \leq t, \Delta\alpha(s) \notin N\}.$$

Again this is well defined because the set $\{s \in [0, t] : \Delta\alpha(s) \notin N\}$ is finite.

It is now claimed that $s_N(\alpha, t, y)$ does not depend on the neighbourhood N of the identity, provided that N is sufficiently small. To prove this claim, it is sufficient to prove that $s_{N_2}(\alpha,t,y) = s_{N_1}(\alpha,t,y)$ whenever N_2 is a neighbourhood of e such that $N_2 \subset N_1 = N$. Given such N_1, N_2 there exists $\delta := \min(\delta(T,N_1),\delta(T,N_2)) > 0$ such that

$$w_{N_k}(\alpha_{N_k},t,y) = \sum_{j=1}^{r} g(\alpha_{N_k}(t_j) - \alpha_{N_k}(t_{j-1}),y) \quad \text{for } k=1,2 \tag{2.9}$$

whenever $0 = t_0 < t_1 < \cdots < t_r = t$, $r \in \mathbb{N}$ and $\max_j(t_j - t_{j-1}) < \delta$. As α is càdlàg, the set of points $F := \{s \in [0,t] : \Delta\alpha(s) \in N_1 \setminus N_2\}$ must be finite. Choose a finite partition $\mathcal{D} = (0 = t_0 < t_1 < \cdots < t_r = t)$, such that $\max_j(t_j - t_{j-1})$ is both less than δ and sufficiently small to ensure that no interval $(t_{j-1},t_j]$ in $\mathcal{D}$ contains more than one of the points that are in F. Consider one of the intervals $(t_{j-1},t_j]$ in $\mathcal{D}$. If this interval does not contain any point of F, then $\alpha_{N_1}(t_j) - \alpha_{N_1}(t_{j-1}) = \alpha_{N_2}(t_j) - \alpha_{N_2}(t_{j-1})$ and consequently

$$g(\alpha_{N_1}(t_j) - \alpha_{N_1}(t_{j-1}),y) = g(\alpha_{N_2}(t_j) - \alpha_{N_2}(t_{j-1}),y).$$

By the choice of $\mathcal{D}$, the only other possibility is that $(t_{j-1},t_j]$ contains exactly one point s that is in F. In this case $\alpha_{N_1}(t_j) - \alpha_{N_1}(t_{j-1}) = \alpha_{N_2}(t_j) - \alpha_{N_2}(t_{j-1}) + \Delta\alpha(s)$. Therefore, as $\alpha_{N_2}(t_j) - \alpha_{N_2}(t_{j-1})$ and $\Delta\alpha(s)$ are in M, (2.7) implies that

$$\begin{aligned} g(\alpha_{N_1}(t_j) - \alpha_{N_1}(t_{j-1}),y) &= g(\alpha_{N_2}(t_j) - \alpha_{N_2}(t_{j-1}) + \Delta\alpha(s),y) \\ &= g(\alpha_{N_2}(t_j) - \alpha_{N_2}(t_{j-1}),y) + g(\Delta\alpha(s),y). \end{aligned}$$

Summing over all the intervals in the partition $\mathcal{D}$ and using (2.9), we obtain

$$w_{N_1}(\alpha_{N_1},t,y) = w_{N_2}(\alpha_{N_2},t,y) + \sum\{g(\Delta\alpha(s),y) : 0 \le s \le t, \Delta\alpha(s) \in N_1 \setminus N_2\}.$$

Adding $\sum\{g(\Delta\alpha(s),y) : 0 \le s \le t, \Delta\alpha(s) \notin N_1\}$ to both sides of the last equation, we conclude that $s_{N_2}(\alpha,t,y) = s_{N_1}(\alpha,t,y)$ as required.

Step 3: Let K be a compact subset of $\widehat{G}$ and $t > 0$. By the previous steps, for any $\alpha \in \mathbb{D}$ with $\alpha(0) = e$ and all $y \in K$, we can define

$$\begin{aligned} w(\alpha,t,y) &:= s_N(\alpha,t,y) \\ &:= \sum\{g(\Delta\alpha(s),y) : 0 \le s \le t,\ \Delta\alpha(s) \notin N\} \\ &\qquad + \sum_{j=1}^{r} g(\alpha_N(t_j) - \alpha_N(t_{j-1}),y) \end{aligned} \tag{2.10}$$

for all sufficiently small symmetric neighbourhoods N of the identity e in G and all sufficiently fine partitions $0 = t_0 < t_1 < \cdots < t_r = t$ of $[0,t]$. In fact the quantity $w(\alpha,t,y)$ does not depend on the compact set K, provided that $y \in K$, the neighbourhood N is sufficiently small and the partition is sufficiently fine. For, if K_1, K_2 are compact subsets of $\widehat{G}$ such that $y \in K_1 \cap K_2$, let N_1, N_2 be neighbourhoods of e in G that correspond to K_1, K_2 respectively in the same way that N corresponds to K in the construction above of $w(\alpha,t,y)$. Then we can use $N_1 \cap N_2$ in place of N in (2.10), so the use of K_1 or K_2 leads to the same quantity $w(\alpha,t,y)$.

If $y_1, y_2 \in \widehat{G}$, we can choose the neighbourhood N small enough and the partition fine enough for (2.10) to hold for $y = y_1$, $y = y_2$ and $y = y_1 + y_2$. The validity of (2.5) therefore follows from property (i) of g in Definition 1.2.

Let $y_0 \in \widehat{G}$ and suppose that the compact set K is a neighbourhood of y_0 in $\widehat{G}$. Let the neighbourhood N and the number $\delta(T, N)$ correspond to K as in the construction of $w(\alpha, t, y)$ above. Fix a partition such that $\max_j(t_j - t_{j-1}) < \delta(T, N)$. Then (2.10) holds for all $y \in K$. As the two sums in (2.10) have finitely many terms, each of which is continuous in y, we deduce that $w(\alpha, t, y) \to w(\alpha, t, y_0)$ as $y \to y_0$. Thus, $y \mapsto w(\alpha, t, y)$ is continuous for each t and α.

The last two paragraphs prove that $w(\alpha, t, \cdot)$ has the properties stated in part (c) of Lemma 2.1. Therefore, the mapping $y \mapsto \exp[iw(\alpha, t, y)]$ is a continuous character on $\widehat{G}$. Hence, by the Pontryagin duality theorem, for each $t \geq 0$ there exists $\beta(t) \in G$ such that (2.2) holds for all $y \in \widehat{G}$. Define $\gamma(t) := \alpha(t) - \beta(t)$.

For any $y \in \widehat{G}$ choose a compact subset K of $\widehat{G}$ with $y \in K$, and choose a neighbourhood N of the identity in G that corresponds to K as before. Then, using (2.10) with a sufficiently fine partition of $[0, t]$, and also using (2.6),

$$\begin{aligned}
&\exp[iw(\alpha, t, y)] \\
&\quad = \left[\prod \{\exp[ig(\Delta\alpha(s), y)] : 0 \leq s \leq t,\ \Delta\alpha(s) \notin N\}\right] \\
&\qquad\qquad \times \prod_{j=1}^{r} \exp[ig(\alpha_N(t_j) - \alpha_N(t_{j-1}), y)] \\
&\quad = \left[\prod \{\exp[ig(\Delta\alpha(s), y)] : 0 \leq s \leq t,\ \Delta\alpha(s) \notin N\}\right] \\
&\qquad\qquad \times \prod_{j=1}^{r} \langle \alpha_N(t_j) - \alpha_N(t_{j-1}), y\rangle \\
&\quad = \left[\prod \{\exp[ig(\Delta\alpha(s), y)] : 0 \leq s \leq t,\ \Delta\alpha(s) \notin N\}\right] \\
&\qquad\qquad \times \left\langle \sum_{j=1}^{r} (\alpha_N(t_j) - \alpha_N(t_{j-1})), y \right\rangle \\
&\quad = \langle \alpha_N(t), y\rangle \left[\prod \{\exp[ig(\Delta\alpha(s), y)] : 0 \leq s \leq t,\ \Delta\alpha(s) \notin N\}\right]. \qquad (2.11)
\end{aligned}$$

In particular, (2.11) applied to α_N in place of α shows that

$$\langle \alpha_N(t), y\rangle = \exp[iw(\alpha_N, t, y)] \quad \text{for } y \in K. \qquad (2.12)$$

By (2.8) and (2.11),

$$\begin{aligned}
\langle \alpha(t), y\rangle &= \langle \alpha_N(t), y\rangle \prod \{\langle \Delta\alpha(s), y\rangle : 0 \leq s \leq t,\ \Delta\alpha(s) \notin N\} \\
&= \exp[iw(\alpha, t, y)] \prod \{\langle \Delta\alpha(s), y\rangle \exp[-ig(\Delta\alpha(s), y)] : 0 \leq s \leq t\} \\
&= \langle \beta(t), y\rangle \prod \{\langle \Delta\alpha(s), y\rangle \exp[-ig(\Delta\alpha(s), y)] : 0 \leq s \leq t\}.
\end{aligned}$$

Consequently, (2.1) holds for all $y \in \widehat{G}$. As the right hand side of (2.1) is uniquely determined, it follows that $\gamma(t)$ and $\beta(t)$ are also unique.

Step 4: The construction described above defines $w(\alpha, t, y)$ for $\alpha \in \mathbb{D}$ with $\alpha(0) = e$, $t > 0$ and $y \in \widehat{G}$. Recall that we also defined $w(\alpha, 0, y) := 0$. It will now be demonstrated that the function $t \mapsto w(\alpha, t, y)$ is càdlàg. The construction of $w(\alpha, t, y)$ for $t > 0$ involved the selection of a neighbourhood $N = N(K)$ which depended on the compact set K, but did not depend on t. The choice of δ in that construction may, however, depend on the choice of a number T, with $T \geq t$, as well as on N; recall the meaning of the notation $\delta(T, N)$ in Step 2.

Let K be a compact subset of $\widehat{G}$ with $y \in K$ and let $N = N(K)$, as described above. Given $t > 0$, let $T > t$, $\delta := \delta(T, N)$ and let s satisfy $t < s < \min(T, t + \delta)$. Then consider any partition of $[0, t]$ given by $0 = t_0 < t_1 < \cdots < t_r = t$ with $\max_j(t_j - t_{j-1}) < \delta$ and the partition of $[0, s]$ obtained by adjoining to it the point s. Using these partitions gives for any $y \in K$

$$w_N(\alpha_N, t, y) = \sum_{j=1}^{r} g(\alpha_N(t_j) - \alpha_N(t_{j-1}), y)$$

and

$$w_N(\alpha_N, s, y) = \sum_{j=1}^{r} g(\alpha_N(t_j) - \alpha_N(t_{j-1}), y) + g(\alpha_N(s) - \alpha_N(t), y)$$

respectively. Therefore, by the right continuity of α_N,

$$w_N(\alpha_N, s, y) - w_N(\alpha_N, t, y) = g(\alpha_N(s) - \alpha_N(t), y) \to 0 \quad \text{as } s \downarrow t. \tag{2.13}$$

Hence, $s \mapsto w_N(\alpha_N, s, y)$ is right continuous at each $t > 0$. To prove the right continuity at $t = 0$, simply omit the partition of $[0, t]$.

With the same choice of $t_0, t_1, \ldots, t_r$ as above, let $t_{r-1} < s < t$. Then, as $s \uparrow t$,

$$\begin{aligned} & w_N(\alpha_N, s, y) \\ &= \sum_{j=1}^{r-1} g(\alpha_N(t_j) - \alpha_N(t_{j-1}), y) + g(\alpha_N(s) - \alpha_N(t_{r-1}), y) \\ &\to \sum_{j=1}^{r-1} g(\alpha_N(t_j) - \alpha_N(t_{j-1}), y) + g(\alpha_N(t-) - \alpha_N(t_{r-1}), y). \end{aligned} \tag{2.14}$$

In fact, if the neighbourhood N is sufficiently small, then the convergences in (2.13) and (2.14) occur uniformly in $y \in K$. To prove this we use Lemma 1.4. Choose the neighbourhoods M and N with the properties specified in Step 1 in such a way that M is closed and $M \subseteq N_K$, where N_K is as in Lemma 1.4. For $t < s < \min(T, t + \delta)$, (2.13) then gives

$$\begin{aligned} \sup_{y \in K} |w_N(\alpha_N, s, y) - w_N(\alpha_N, t, y)| &= \sup_{y \in K} |g(\alpha_N(s) - \alpha_N(t), y)| \\ &\leq c_K \max_{y \in F_K} |g(\alpha_N(s) - \alpha_N(t), y)| \\ &\to 0 \quad \text{as } s \downarrow t. \end{aligned}$$

For $t_{r-1} < s < t$, $\alpha_N(s) - \alpha_N(t-)$ and $\alpha_N(t-) - \alpha_N(t_{r-1})$ are in M because M is closed. Therefore, using (2.14), (2.7), $M \subseteq N_K$ and Lemma 1.4,

$$\begin{aligned}
&\sup_{y\in K} |w_N(\alpha_N, s, y) - w_N(\alpha_N, t-, y)| \\
&\quad = \sup_{y\in K} |\, g(\alpha_N(s) - \alpha_N(t_{r-1}), y) - g(\alpha_N(t-) - \alpha_N(t_{r-1}), y)| \\
&\quad = \sup_{y\in K} |\, g(\alpha_N(s) - \alpha_N(t-), y)| \\
&\quad \leq c_K \max_{y\in F_K} |\, g(\alpha_N(s) - \alpha_N(t-), y)| \\
&\quad \to 0 \quad \text{as } s \uparrow t.
\end{aligned}$$

Step 5: Continuing the development in Step 4, with K, y, N and α as before, now consider the quantity $w(\alpha, t, y) - w_N(\alpha_N, t, y)$ for $t \geq 0$. We have

$$\begin{aligned}
w(\alpha, t, y) - w_N(\alpha_N, t, y) &= \sum \{g(\Delta\alpha(s), y) : 0 \leq s \leq t,\ \Delta\alpha(s) \notin N\} \\
&= \int_{G\setminus N} g(x, y)\mu(\alpha, [0, t], dx) \qquad (2.15)
\end{aligned}$$

where $\mu(\alpha, \cdot, \cdot)$ is the jump measure associated with α: for each bounded interval $I \subset [0, \infty)$ and each Borel subset B of G such that $B \cap U = \emptyset$ for some neighbourhood U of the identity, $\mu(\alpha, I, B)$ is the (finite) number of points $s \in I$ such that $\Delta\alpha(s) \in B$. For $s > t$

$$\begin{aligned}
&\sup_{y\in K} \left| \int_{G\setminus N} g(x, y)\mu(\alpha, [0, s], dx) - \int_{G\setminus N} g(x, y)\mu(\alpha, [0, t], dx) \right| \\
&\quad \leq \sup_{x\in G} \sup_{y\in K} |g(x, y)|\, \mu(\alpha, (t, s], G \setminus N) \\
&\quad \to 0 \quad \text{as } s \downarrow t.
\end{aligned}$$

For $t > 0$ and $0 \leq s < t$,

$$\begin{aligned}
&\sup_{y\in K} \left| \int_{G\setminus N} g(x, y)\mu(\alpha, [0, t), dx) - \int_{G\setminus N} g(x, y)\mu(\alpha, [0, s], dx) \right| \\
&\quad \leq \sup_{x\in G} \sup_{y\in K} |g(x, y)|\, \mu(\alpha, (s, t), G \setminus N) \\
&\quad \to 0 \quad \text{as } s \uparrow t.
\end{aligned}$$

Using equation (2.15), these results imply that $t \mapsto w(\alpha, t, y) - w_N(\alpha_N, t, y)$ is right continuous uniformly in $y \in K$ at each point $t \geq 0$ and has the left limit $\int_{G\setminus N} g(x, y)\mu(\alpha, [0, t), dx)$ uniformly in $y \in K$ at each point $t > 0$. Taking this together with what was proved in Step 4, we see that part (d) of Lemma 2.1 is proved.

Step 6: The properties (d) of the unwrapping $w(\alpha, \cdot, \cdot)$, the validity of (2.2) for all $t \geq 0$ and $y \in \widehat{G}$, together with the Pontryagin duality theorem, imply that $\beta \in \mathbb{D}$ and $\beta(0) = e$. Therefore $\gamma \in \mathbb{D}$ and $\gamma(0) = e$. The uniqueness of β and γ were already noted in Step 3. To complete the proof of Lemma 2.1, it only remains to prove (2.3) and the uniqueness of $w(\alpha, \cdot, \cdot)$.

Taking the partition in (2.10) and $t_{r-1} < u < t$, we have

$$w(\alpha, u, y) = \sum\{g(\Delta\alpha(s), y) : 0 \le s \le u,\ \Delta\alpha(s) \notin N\} + \sum_{j=1}^{r-1} g(\alpha_N(t_j) - \alpha_N(t_{j-1}), y) + g(\alpha_N(u) - \alpha_N(t_{j-1}), y).$$

Subtracting this from (2.10) and using (2.7) yields

$$w(\alpha, t, y) - w(\alpha, u, y) = \sum\{g(\Delta\alpha(s), y) : u < s \le t,\ \Delta\alpha(s) \notin N\} + g(\alpha_N(t) - \alpha_N(u), y)$$

and then letting u increase to t yields (2.3).

Suppose that, for some $y \in \widehat{G}$, there are two càdlàg functions, $w(\alpha, \cdot, y)$ and $w'(\alpha, \cdot, y)$, satisfying the conditions (2.2), (2.3) and (2.4) in the lemma. Then, by (2.3), they have exactly the same discontinuities and therefore their difference, $d(\alpha, \cdot, y) := w(\alpha, \cdot, y) - w'(\alpha, \cdot, y)$, is continuous. But equations (2.2) and (2.4) imply that $d(\alpha, \cdot, y)/(2\pi)$ is integer-valued and $d(\alpha, 0, y) = 0$. It follows that $d(\alpha, t, y) = 0$ for all $t \ge 0$. This completes the proof of Lemma 2.1. □

Example 2.3. Using the same notation as in Example 1.3, let us return to the case in which $G = \mathbb{R}^d$. In this situation consider applying Lemma 2.1 to a càdlàg function $\alpha : [0, \infty) \to \mathbb{R}^d$ such that $\alpha(0) = \mathbf{0}$. Let $\mathbf{0} \ne y \in \widehat{G} = \mathbb{R}^d$ and $t > 0$. As in Step 1 of the proof of Lemma 2.1, choose neighbourhoods M, N of $\mathbf{0}$ in $\mathbb{R}^d$ with the extra requirement that $M \subseteq N_1$, where N_1 is as in Example 1.3. Then, by (2.10),

$$w(\alpha, t, y) := \sum\Big\{k(\Delta\alpha(s)) \cdot y\ : 0 \le s \le t,\ \Delta\alpha(s) \notin N\Big\} + \sum_{j=1}^{r} k(\alpha_N(t_j) - \alpha_N(t_{j-1})) \cdot y$$

for all sufficiently fine partitions $0 = t_0 < t_1 < \cdots < t_r = t$ of $[0, t]$. But, if the partition is fine enough, $\alpha_N(t_j) - \alpha_N(t_{j-1}) \in M \subseteq N_1$ and so $k(\alpha_N(t_j) - \alpha_N(t_{j-1})) = \alpha_N(t_j) - \alpha_N(t_{j-1})$ for all j. Therefore

$$\begin{aligned} w(\alpha, t, y) &= \Big[\sum\Big\{k(\Delta\alpha(s)) : 0 \le s \le t,\ \Delta\alpha(s) \notin N\Big\} + \alpha_N(t)\Big] \cdot y \\ &= \Big[\alpha(t) - \sum\Big\{\Delta\alpha(s) - k(\Delta\alpha(s)) : 0 \le s \le t\Big\}\Big] \cdot y. \qquad (2.16) \end{aligned}$$

The expression inside square brackets in (2.16) is $\beta(t)$ in this case and should be compared with the right hand side of (1.1).

3. G-valued semimartingales

We begin by defining G-valued semimartingales that start at the identity e.

Definition 3.1. Let $X = \{X(t) : t \ge 0\}$ be a G-valued adapted stochastic process on the given stochastic basis $\mathbf{B}$. Suppose also that the sample paths of X are a.s. càdlàg and that $X(0) = e$, the identity of G. Then we call X a *G-valued* *semimartingale* (on the stochastic basis $\mathbf{B}$) if, for every $y \in \widehat{G}$, $\{\langle X(t), y\rangle : t \geq 0\}$ is (in the usual classical sense) a complex-valued semimartingale (on the stochastic basis $\mathbf{B}$).

Corollary 3.4 below shows that this definition is consistent with the classical notion of an $\mathbb{R}^d$-valued semimartingale starting at the origin $\mathbf{0}$. In this paper we consider only semimartingales that start at the identity; but, as in the classical situation, we could extend this to semimartingales such that $X(0) = X_0$, where X_0 is an $\mathcal{F}_0$-measurable G-valued random variable. This would mean that $\{X(t)-X_0 : t \geq 0\}$ is a semimartingale starting at e.

The next task is to define the characteristics of a G-valued semimartingale. Before we can give the definitions, however, we need some preparation. Let $X = \{X(t) : t \geq 0\}$ be a G-valued stochastic process which is a semimartingale in the sense of Definition 3.1. By discarding an ω-set of probability zero, we can and shall assume that all the sample paths of X are càdlàg. Then, for each $y \in \widehat{G}$, define a real-valued process $W(\cdot, y)$ as follows: for each sample path $\alpha = X(\cdot, \omega)$ of X the corresponding sample path of $W(\cdot, y)$ is given by

$$W(t, y, \omega) := w(\alpha, t, y) \quad \text{for all } t \geq 0, \tag{3.1}$$

where $w(\alpha, t, y)$ is as in Lemma 2.1. Thus, $W(\cdot, y) = w(X, \cdot, y)$. From what was proved in Lemma 2.1, it follows that the sample paths of $W(\cdot, y)$ are càdlàg.

From the following result we see that, for each $y \in \widehat{G}$, $W(\cdot\,, y)$ is an adapted stochastic process on the stochastic basis $\mathbf{B}$.

Lemma 3.2. *$W(t, y)$ is $\mathcal{F}_t$-measurable for each $y \in \widehat{G}$ and $t > 0$.*

Proof. Given $y \in \widehat{G}$, fix a closed (non-random) neighbourhood N of the identity e in G that satisfies the requirements in Step 1 in the proof of Lemma 2.1 with $K = \{y\}$. Then $W(t, y)$ is the limit (pointwise with respect to ω) of

$$\sum\{g(\Delta X(s), y) : 0 \leq s \leq t,\ \Delta X(s) \notin N\} + \sum_{j=1}^{r} g(X_N(t_j) - X_N(t_{j-1}), y) \tag{3.2}$$

as the partition $\mathcal{D} = (0 = t_0 < t_1 < \cdots < t_r = t)$ passes through a fixed sequence of partitions of $[0, t]$ whose meshes decrease to 0. (Here, as with the definition of $\alpha_N(t)$, $X_N(t) = X(t) - \sum\{\Delta X(s) : 0 \leq s \leq t,\ \Delta X(s) \notin N\}$.) Therefore, to prove the lemma, it is sufficient to prove that the expression (3.2) is $\mathcal{F}_t$-measurable for each fixed partition $\mathcal{D}$.

Define random times $(T_n)_{n=0,1,2,\ldots}$ inductively by $T_0 = 0$ and, for $n = 1, 2, \ldots$, $T_n = \inf\{s : s > T_{n-1},\ \Delta X(s) \notin N\}$. Because the complement of N is open and the filtration $(\mathcal{F}_t)_{t\geq 0}$ is right continuous, each T_n is an $(\mathcal{F}_t)$-stopping time. For T_1 this is proved for example in Sokol [12] for the case of $\mathbb{R}$-valued processes. It can be proved for G-valued processes in the same way as in [12] by replacing the Euclidean metric on $\mathbb{R}$ by a metric for the topology of G. The result for general n then follows by induction. Note that Sokol's proof does not require the filtration to be P-complete, whereas to obtain the same result by applying the Début Theorem (Dellacherie and Meyer [3] IV.50, or Meyer [8] IV.52) to the set $\{(t, \omega) : \Delta X(t, \omega) \notin N\}$ would require P-completeness.

For given $t > 0$ and all $n = 1, 2, \dots$, define $\Delta_n = \Delta X(T_n)$ when $T_n \le t$ and $\Delta_n = e$ when $T_n > t$. Then Δ_n is $\mathcal{F}_t$-measurable, because the process ΔX is progressively measurable and T_n is a stopping time. As X is càdlàg, we have $\Delta_n \ne e$ for only finitely many n. Therefore, in the identity

$$\sum\{g(\Delta X(s), y) : 0 \le s \le t,\ \Delta X(s) \notin N\} = \sum_{n=1}^{\infty} g(\Delta_n, y),$$

the sum on the right has only a finite number of non-zero terms, so it converges for every $\omega \in \Omega$. Its partial sums are all $\mathcal{F}_t$-measurable, and therefore the sum on the left is $\mathcal{F}_t$-measurable. By similar reasoning, the identity

$$\sum\{\Delta X(s) : 0 \le s \le t,\ \Delta X(s) \notin N\} = \sum_{n=1}^{\infty} \Delta_n$$

implies that the sum on its left side is $\mathcal{F}_t$-measurable. This in turn implies that $X_N(t)$ is $\mathcal{F}_t$-measurable for every $t > 0$, and hence that the expression (3.2) is $\mathcal{F}_t$-measurable. □

Lemma 3.3. *Let $y \in \widehat{G}$. Then $W(\cdot\,, y)$ is a semimartingale.*

Proof. Given $y \in \widehat{G}$, fix a closed neighbourhood N of the identity in G in the same way as in the proof of Lemma 3.2 and with the additional property that

$$|g(x, y)| \le \tfrac{1}{4} \quad \text{for all } x \in N. \tag{3.3}$$

This is possible by property (v) of g in Definition 1.2. With $w(\alpha, t, y)$ as in Lemma 2.1, let $W_N(\cdot\,, y)$ be the stochastic process whose sample path is $w(\alpha_N, t, y)$ whenever the sample path of X is α. As $W(\cdot\,, y)$ differs from $W_N(\cdot, y)$ by a finite variation process, which is given by the first sum in (3.2), the conclusion of the lemma will follow if we prove that $W_N(\cdot\,, y)$ is a semimartingale.

We have, by applying (2.12) and (2.8) to the sample paths of the processes $W_N(\cdot, y)$ and X_N,

$$\begin{aligned} \exp[iW_N(t, y)] &= \langle X_N(t), y\rangle \\ &= \langle X(t), y\rangle \Big\langle -\textstyle\sum\{\Delta X(s) : 0 \le s \le t,\ \Delta X(s) \notin N\},\ y\Big\rangle. \end{aligned}$$

Because $\langle X(\cdot), y\rangle$ is a semimartingale by hypothesis and the other factor in the last line of the above equation gives a finite variation process, which is therefore also a semimartingale, we conclude that $\exp[iW_N(\cdot\,, y)]$ is a semimartingale.

Define a sequence of stopping times $(T_n)_{n=0,1,2,\dots}$ inductively by $T_0 = 0$ and, for $n = 1, 2, \dots$,

$$T_n := \inf\{t > T_{n-1} : |W_N(t, y) - W_N(T_{n-1}, y)| > \tfrac{1}{2}\},$$

where $\inf \emptyset = \infty$. Each path of $W_N(\cdot\,, y)$ is càdlàg and therefore, for any $\varepsilon > 0$, it can have only finitely many ε-oscillations in any bounded interval. Consequently $T_n \nearrow \infty$ as $n \to \infty$. For each $n = 1, 2, \dots$, the process with paths $t \mapsto \exp[iW_N(t \wedge T_n, y)]$ is a non-vanishing semimartingale, being the semimartingale $\exp[iW_N(\cdot\,, y)]$ stopped at T_n. Therefore the complex-valued process with paths

$$t \mapsto \exp[i\{W_N(t \wedge T_n, y) - W_N(t \wedge T_{n-1}, y)\}] = \frac{\exp[iW_N(t \wedge T_n, y)]}{\exp[iW_N(t \wedge T_{n-1}, y)]} \tag{3.4}$$

is also a semimartingale. Recalling the definition of the stopping times (T_n) and noting that, by (2.3) and (3.3), the jumps in the sample paths of $W_N(\cdot\,, y)$ will never exceed $\frac{1}{4}$ in magnitude, we conclude that we always have

$$|W_N(t \wedge T_n, y) - W_N(t \wedge T_{n-1}, y)| \leq \tfrac{3}{4}.$$

Therefore the process whose paths are given by (3.4) takes its values in the set S of all complex numbers z such that $|1 - z| < 1$. On the set S the principal value of the complex logarithm is a smooth function given by

$$\mathrm{Log}(z) = -\sum_{n=1}^{\infty} \frac{(1-z)^n}{n}.$$

Therefore the process V_n defined for all $t \geq 0$ by

$$V_n(t) := \frac{1}{i} \sum_{k=1}^{n} \mathrm{Log}\biggl(\exp\Bigl[i\Bigl\{W_N(t \wedge T_k, y) - W_N(t \wedge T_{k-1}, y)\Bigr\}\Bigr]\biggr)$$

is a semimartingale for each n. But $W_N(\cdot\,, y)$ is equal to V_n on the stochastic interval $[\![0, T_n[\![$ for each n. From 4.25 on page 44 of Jacod and Shiryaev [6], it follows that $W_N(\cdot . y)$ is a semimartingale. □

Corollary 3.4. *Let $X = \{X(t) : 0 \leq t < \infty\}$ be an $\mathbb{R}^d$-valued càdlàg stochastic process such that $X(0) = \mathbf{0}$. Then X is an $\mathbb{R}^d$-valued semimartingale (in the classical sense) if and only if* $\exp(iX \cdot y)$ *is a complex-valued semimartingale (in the classical sense) for every $y \in \mathbb{R}^d$.*

Proof. Because X is a classical semimartingale if and only if $X \cdot y$ is a classical semimartingale for every $y \in \mathbb{R}^d$, it is sufficient to prove the result for the case when $d = 1$. Accordingly, let X be a càdlàg real-valued process such that $X(0) = 0$.

If X is a classical semimartingale, then so is $\exp(iXy)$ for all $y \in \mathbb{R}$, because $x \mapsto \exp(ixy)$ is a smooth function. Conversely, suppose that $\exp(iXy)$ is a classical semimartingale for all $y \in \mathbb{R}$. It is enough to assume this for $y = 1$. Define the càdlàg process $\hat{X}$ by

$$\hat{X}(t) = X(t) - \sum\{\Delta X(s) :\ 0 \leq s \leq t,\ |\Delta X(s)| > \tfrac{1}{4}\}$$

for all $t \geq 0$, where $\Delta X(0) = 0$ and $\Delta X(s) = X(s) - X(s-)$ for $s > 0$. Because X is càdlàg, the sum has only a finite number of non-zero terms and therefore defines a finite variation process. To prove that X is a semimartingale, it is therefore sufficient to prove that $\hat{X}$ is a semimartingale.

On the right hand side of the identity

$$\exp(i\hat{X}(t)) = \exp(iX(t)).\exp\left[-i\sum\{\Delta X(s) : 0 \leq s \leq t,\ |\Delta X(s)| > \tfrac{1}{4}\}\right]$$

the first factor defines a semimartingale (by hypothesis) and the second factor defines a finite variation process. Therefore $\exp(i\hat{X}(\cdot))$ is a semimartingale. The same argument that was used in the proof of Lemma 3.3 to prove that $W_N(\cdot, y)$ is a semimartingale can now be used to conclude that $\hat{X}$ is a semimartingale. □

Lemma 3.3 shows that $W(\cdot\,, y)$ is a semimartingale for each $y \in \widehat{G}$. But we also have $|\Delta W(\cdot\,, y)| \le \sup_{x \in G} |g(x, y)| < \infty$, so the jumps of $W(\cdot\,, y)$ have uniformly bounded magnitudes. Therefore $W(\cdot\,, y)$ is a special semimartingale (Jacod and Shiryaev [6], 4.24, page 44). Consequently $W(\cdot\,, y)$ has a canonical decomposition

$$W(\cdot\,, y) = M(\cdot\,, y) + B(\cdot\,, y) \tag{3.5}$$

where $M(\cdot\,, y)$ is a local martingale with $M(0, y) = 0$, $B(\cdot\,, y)$ is a predictable process of locally integrable variation with $B(0, y) = 0$ and each of $M(\cdot\,, y)$ and $B(\cdot\,, y)$ almost surely has càdlàg paths (Jacod and Shiryaev [6], 4.21, page 43). Moreover, the processes $M(\cdot\,, y)$ and $B(\cdot\,, y)$ are uniquely determined up to evanescence.

We can now define the semimartingale characteristics for X, in analogy with the real line case, as follows.

Definition 3.5. Let G be a locally compact second countable abelian group and let X be a G-valued semimartingale in the sense of Definition 3.1. For each $y \in \widehat{G}$, let $W(\cdot, y) = w(X, \cdot, y)$ be the real-valued special semimartingale given by equation (3.1) above, and let $B(\cdot, y)$ be as in the canonical decomposition in (3.5). Then stochastic processes $\widetilde{B}$, Φ and ν with the following properties (1), (2) and (3) are called the *first, second, and third characteristics* of X respectively:

(1) $\widetilde{B}$ is a predictable a.s. càdlàg G-valued process such that, for each $y \in \widehat{G}$, the processes $\langle \widetilde{B}, y\rangle$ and $\exp[iB(\cdot\,, y)]$ are indistinguishable;
(2) Φ is a non-decreasing continuous process of random continuous nonnegative quadratic forms $\{\Phi(t, \cdot\,) : t \ge 0\}$ on $\widehat{G}$, such that, for each $y \in \widehat{G}$, $\Phi(\cdot\,, y)$ is the compensator of the square of the continuous martingale part of $W(\cdot\,, y)$;
(3) ν is the predictable compensator of the jump measure of X.

It is conjectured, but not yet proved, that all three characteristics always exist for semimartingales with values in locally compact second countable abelian groups. Theorem 3.6 below gives the partial result that ν always exists and that $\widetilde{B}$ and Φ always exist if G is compact. Proposition 3.7 establishes some uniqueness properties and Theorem 3.8 shows that all three characteristics exist when $G = \mathbb{R}$.

Theorem 3.6. *Let G be a locally compact second countable abelian group and let X be any G-valued semimartingale in the sense of Definition 3.1. Then the third characteristic of X exists. If G is compact, then the first and second characteristics of X exist.*

Proof. Let μ be the jump measure of X. By Proposition II 1.16 and Theorem II 1.8 in [6], replacing the Euclidean distance on $\mathbb{R}^d$ by a metric for the topology of G, we conclude that there is a predictable random measure ν, which is the compensator of μ and is unique up to a P-null set. Thus, the third characteristic of X exists and is unique (up to indistinguishability).

Now let $y_1, y_2 \in \widehat{G}$. It follows from (2.5) that $W(\cdot, y_1 + y_2) = W(\cdot, y_1) + W(\cdot, y_2)$. By the uniqueness of canonical decompositions (3.5) up to indistinguishability, it follows that

$$M(t, y_1 + y_2) = M(t, y_1) + M(t, y_2) \quad \text{for all } t \geq 0$$

$$\text{holds a.s.}(P) \text{ for all } y_1, y_2 \in \widehat{G} \tag{3.6}$$

and

$$B(t, y_1 + y_2) = B(t, y_1) + B(t, y_2) \quad \text{for all } t \geq 0$$

$$\text{holds a.s.}(P) \text{ for all } y_1, y_2 \in \widehat{G}. \tag{3.7}$$

Similarly, the continuous martingale part $M^c(\cdot\,, y)$ of $M(\cdot\,, y)$ (which is also the continuous martingale part of $W(\cdot\,, y)$) is also uniquely determined up to indistinguishability and therefore

$$M^c(t, y_1 + y_2) = M^c(t, y_1) + M^c(t, y_2) \quad \text{for all } t \geq 0$$

$$\text{holds a.s.}(P) \text{ for all } y_1, y_2 \in \widehat{G}. \tag{3.8}$$

The exceptional ω-sets of P-measure zero in equations (3.6), (3.7) and (3.8) may, in general, depend on y_1 and y_2. This fact causes technical difficulties that have long prevented the construction of a complete general proof of the existence of the first two characteristics for locally compact second countable abelian groups that are not compact. If, however, G is compact, these difficulties do not arise, because there is then an exceptional ω-set that is independent of y_1 and y_2.

For the rest of this proof assume that the locally compact second countable abelian group G is compact. Then the dual group $\widehat{G}$ is discrete and countable. We can therefore deduce from (3.7) that there is a set $\Omega_1 \in \mathcal{F}$ with $P(\Omega_1) = 1$ such that

$$B(t, y_1 + y_2, \omega) = B(t, y_1, \omega) + B(t, y_2, \omega) \quad \text{for all } t \geq 0$$

$$\text{holds for all } y_1, y_2 \in \widehat{G} \text{ for all } \omega \in \Omega_1. \tag{3.9}$$

Let $t \geq 0$ and $\omega \in \Omega_1$. Because every function on $\widehat{G}$ is continuous, we see from (3.9) that $y \mapsto \exp[iB(t, y, \omega)]$ is a continuous homomorphism from $\widehat{G}$ into $\mathbb{T}$. By the Pontryagin duality theorem there is a uniquely determined $\widetilde{B}(t, \omega) \in G$ such that

$$\left\langle \widetilde{B}(t, \omega), y \right\rangle = \exp[iB(t, y, \omega)] \quad \text{for all } y \in \widehat{G}. \tag{3.10}$$

Let $\widetilde{B}(t, \omega)$ be defined by condition (3.10) if $\omega \in \Omega_1$ and let $\widetilde{B}(t, \omega) = e$ if $\omega \in \Omega \backslash \Omega_1$. Because the set $\Omega_1 \times [0, \infty)$ is predictable, this defines a G-valued stochastic process $\widetilde{B} := \{\widetilde{B}(t) : t \geq 0\}$, such that the process $\langle \widetilde{B}, y \rangle$ is predictable for each $y \in \widehat{G}$. But the elements of $\widehat{G}$ generate the Borel σ-field of G, so this implies that the process $\tilde{B}$ is itself predictable; i.e., $\tilde{B}$ is measurable with respect to the predictable σ-field in $\Omega \times [0, \infty)$ and the Borel σ-field in G.

We also know that the sample paths of all of the processes $B(\cdot\,, y)$ for $y \in \widehat{G}$ are càdlàg with probability 1. Bearing in mind that uniform convergence on compact subsets of $\widehat{G}$ is the same as pointwise convergence, the Pontryagin duality theorem implies that the sample paths of $\widetilde{B}$ are also càdlàg with probability 1. Therefore $\widetilde{B}$ has all the properties of the first characteristic of X in Definition 3.5.

Next, for each $y \in \widehat{G}$, let $\Phi(\cdot\,,y) := \langle M^c(\cdot\,,y), M^c(\cdot\,,y)\rangle$ be the compensator of the process $(M^c(\cdot\,,y))^2$, so that $\Phi(\cdot\,,y)$ is the (unique up to indistinguishability) non-decreasing continuous process such that the process $(M^c(\cdot\,,y))^2 - \Phi(\cdot\,,y)$ is a local martingale. From (3.8) it follows that

$$\Phi(t, y_1 + y_2, \omega) + \Phi(t, y_1 - y_2, \omega) = 2\Phi(t, y_1, \omega) + 2\Phi(t, y_2, \omega)$$
$$\text{for all } t \geq 0 \text{ holds a.s.}(P) \text{ for all } y_1, y_2 \in \widehat{G}.$$

Again using the countability of $\widehat{G}$, we conclude that there exists $\Omega_2 \in \mathcal{F}$ with $P(\Omega_2) = 1$ such that

$$\Phi(t, y_1 + y_2, \omega) + \Phi(t, y_1 - y_2, \omega) = 2\Phi(t, y_1, \omega) + 2\Phi(t, y_2, \omega)$$
$$\text{for all } t \geq 0 \text{ holds for all } y_1, y_2 \in \widehat{G} \text{ for all } \omega \in \Omega_2. \quad (3.11)$$

If necessary, modify the random variables $\Phi(t, y)$ for all $t \geq 0$ and all $y \in \widehat{G}$ so that $\Phi(t, y, \omega) = 0$ for all $\omega \in \Omega \setminus \Omega_2$. Then (3.11) shows that $y \mapsto \Phi(t, y, \omega)$ is a quadratic form on $\widehat{G}$ for each $t \geq 0$ and each $\omega \in \Omega$. Thus, $\Phi := \{\Phi(t, \cdot\,) : t \geq 0\}$ is a process of random continuous nonnegative quadratic forms on $\widehat{G}$ such that $t \mapsto \Phi(t, y)$ is continuous and nondecreasing for each $y \in \widehat{G}$. Therefore Φ has all the properties required of the second characteristic in Definition 3.5 and Theorem 3.6 is proved. □

Proposition 3.7. *If it exists, the first characteristic $\widetilde{B}$ is unique (up to indistinguishability) for a given choice of g. If it exists, the second characteristic Φ is unique (up to indistinguishability) and does not depend on the choice of g. The third characteristic ν exists, is unique (up to indistinguishability) and does not depend on the choice of g.*

Proof. The existence and uniqueness (up to indistinguishability) of ν follow from the proof of Theorem 3.6. That ν does not depend on g is obvious from its definition.

For a given choice of g, the special semimartingale $W(\cdot, y)$ and its canonical decomposition are unique (up to indistinguishability) for each $y \in \widehat{G}$. Therefore, if $\widetilde{B}$ and $\widetilde{B}'$ are two possible candidates to be the first characteristic of X, then

$$\langle \widetilde{B}'(t), y\rangle = \exp\left[iB(t, , y)\right] = \langle \widetilde{B}(t), y\rangle \quad \text{for all } t \geq 0$$

holds a.s.(P) for each $y \in \widehat{G}$, where $B(t, y)$ is as in the canonical decomposition (3.5). Hence, if Y is a countable dense subset of $\widehat{G}$, there is a set $\Omega_1 \in \mathcal{F}$ with $P(\Omega_1) = 1$, such that $\langle \widetilde{B}'(t, \omega), y\rangle = \langle \widetilde{B}(t, \omega), y\rangle$ for all $t \geq 0$, all $y \in Y$ and all $\omega \in \Omega_1$. By continuity of $y \mapsto \langle x, y\rangle$, this implies that, when $\omega \in \Omega_1$, $\langle B'(t, \omega), y\rangle = \langle \widetilde{B}(t, \omega), y\rangle$ for all $y \in \widehat{G}$ and all $t \geq 0$. Hence, $\widetilde{B}'(t, \omega) = \widetilde{B}(t, \omega)$ for all $t \geq 0$ whenever $\omega \in \Omega_1$. Therefore $\widetilde{B}$ and $\widetilde{B}'$ are indistinguishable.

In order to prove the statement about the second characteristic, we first prove that, for each $y \in \widehat{G}$, the continuous martingale part $M^c(\cdot, y)$ of $W(\cdot, y)$ is unique and does not depend on g. The uniqueness of $W(\cdot, y)$ for a given choice of g implies the uniqueness of $M^c(\cdot, y)$ for that choice of g. Therefore it is sufficient to prove that $M^c(\cdot, y)$ does not depend on g. Let $W'(\cdot, y)$ be the process obtained instead of $W(\cdot,y)$ when a different local inner product g' is used instead of g. From properties (iv) and (v) in Definition 1.2, it follows that there is a neighbourhood M_0 of e in G such that $g'(x,y) = g(x,y)$ for all $x \in M_0$. Let $K = \{y\}$ and fix neighbourhoods M and N of e in G as specified in Step 1 of the proof of Lemma 2.1, taking $M \subseteq M_0$ and N closed. Consider the expression (3.2) for a given ω. Then, for all sufficiently fine $\mathcal{D}$ (depending on ω), the second sum in (3.2) has the same value whether we use g or g' as the inner product, because $X_N(t_j,\omega) - X_N(t_{j-1},\omega) \in M \subseteq M_0$ for all j. Therefore the processes $W(\cdot,y)$ and $W'(\cdot,y)$ differ only by a finite number of jumps over each finite interval $[0,t]$, and consequently they have the same continuous martingale part $M^c(\cdot.y)$.

From what was proved in the previous paragraph, it follows that the process $\Phi(\cdot,y)$ is unique (up to indistinguishability) for each $y \in \widehat{G}$ and does not depend on g. Suppose that there are two possible candidates Φ, Φ' to be the second characteristic of X. Then $\Phi'(t,y) = \Phi(t,y)$ holds for all $t \geq 0$ a.s.(P) for each $y \in \widehat{G}$. Therefore, if Y is a countable dense subset of $\widehat{G}$, we conclude that there exists $\Omega_2 \in \mathcal{F}$ with $P(\Omega_2) = 1$ such that $\Phi'(t,y,\omega) = \Phi(t,y,\omega)$ for all $t \geq 0$ and all $y \in Y$ whenever $\omega \in \Omega_2$. The continuity of $\Phi'(t,y,\omega)$ and $\Phi(t,y,\omega)$ with respect to y then implies that $\Phi'(t,y,\omega) = \Phi(t,y,\omega)$ holds for all $t \geq 0$ and all $y \in \widehat{G}$ whenever $\omega \in \Omega_2$. Hence Φ and Φ' are indistinguishable. □

Theorem 3.8. *Let X be an $\mathbb{R}$-valued semimartingale. Then all three of the semimartingale characteristics of X described in Definition 3.5 exist. The second and third characteristics described in Definition 3.5 are indistinguishable from those described in Definition 1.1, if each nonnegative quadratic form $y \mapsto cy^2$ on $\mathbb{R}$ is identified with its coefficient c. If the truncation function h and the local inner product g are chosen suitably, then the first characteristics described in Definitions 1.1 and 3.5 are also indistinguishable.*

Proof. Let X be a real-valued semimartingale. The third characteristic ν is defined in the same way in the two definitions and we already know that it exists, so only the statements about the first two characteristics have to be proved. Equation (2.16) shows that, when we apply Lemma 2.1 to the sample paths of X, we obtain

$$W(t,y) = \Big[X(t) - \sum\Big\{\Delta X(s) - k(\Delta X(s)) : 0 \leq s \leq t\Big\}\Big]\, y \tag{3.12}$$

for each $y \in \mathbb{R}$. Consider the process $B(\cdot,y)$ in the canonical decomposition (3.5). In the special situation under consideration, $B(\cdot,y)$ is indistinguishable from $B(\cdot,1)y$. Therefore, for each y, $\langle B(\cdot,1),y\rangle \equiv \exp[iB(\cdot,1)y]$ is indistinguishable from $\exp[iB(\cdot,y)]$. Hence, $B(\cdot,1)$ qualifies to be the first characteristic of X in the sense of Definition 3.5.

Similarly, the continuous martingale part $M^c(\cdot,y)$ of $W(\cdot,y)$ is indistinguishable from $M^c(\cdot,1)y$, and therefore the compensator $\Psi(\cdot,y)$ of its square is indistinguishable from $\Psi(\cdot,1)y^2$ for each $y \in \mathbb{R}$. Therefore we can take Φ as the second characteristic in Definition 3.5, where $\Phi(\cdot,y) = \Psi(\cdot,1)y^2$ for each y. If we replace k by any continuous truncation function h on $\mathbb{R}$, then the continuous martingale part of the process defined by the right hand side of (3.12) is unchanged , so $\Psi(\cdot,1)$ is the second characteristic of X in the sense of Definition 1.1.

Finally, let h be a continuous truncation function on $\mathbb{R}$ such that $h(-x) = -h(x)$ for all x, and let g be the local inner product on $G \times \widehat{G}$ defined by $g(x, y) = h(x)y$ for all $x, y \in \mathbb{R}$. If h and g are used in Definitions 1.1 and 3.5, then the process $B(\cdot, 1)$ above is the first characteristic of X in the sense of both definitions. □

Remark 3.9. It is clear that Theorem 3.8 can be extended to $\mathbb{R}^d$-valued semimartingales for any positive integer d, provided that each nonnegative quadratic form on $\mathbb{R}^d$ is identified with its nonnegative-definite coefficient matrix.

4. Continuous Additive Gaussian Processes on G are Semimartingales

Suppose that the locally compact second countable abelian group G can support Gaussian distributions. This implies the existence of a non-trivial collection $\{\phi(t, \cdot) : t \geq 0\}$ of continuous nonnegative quadratic forms on $\widehat{G}$ such that $\phi(0, y) = 0$ for all $y \in \widehat{G}$, $t \mapsto \phi(t, y)$ is continuous for each $y \in \widehat{G}$ and $\phi(s, y) \leq \phi(t, y)$ whenever $0 \leq s \leq t$. From Bingham [2] there exists on some underlying probability space $(\Omega, \mathcal{F}, P)$ a G-valued Gaussian process $\{X(t) : t \geq 0\}$ with continuous sample paths and independent increments such that

$$E\,\langle X(t), y\rangle = \exp\left[-\tfrac{1}{2}\phi(t, y)\right] \quad \text{for all } t \geq 0,\ y \in \widehat{G}. \tag{4.1}$$

In fact the existence of such a process is proved in Bingham [2] only for $0 \leq t \leq 1$. But the result is easily extended to the case $0 \leq t < \infty$ by concatenating the sample paths of independent processes $\{Y_n : n = 0, 1, 2, \dots\}$, where each Y_n is a G-valued process $\{Y_n(s) : 0 \leq s \leq 1\}$ with continuous sample paths and independent increments such that

$$E\,\langle Y_n(s), y\rangle = \exp\left[-\tfrac{1}{2}\{\phi(s + n, y) - \phi(n, y)\}\right] \quad \text{for } s \in [0, 1],\ y \in \widehat{G}.$$

The process X given by

$$X(t) = \left[\sum_{k=0}^{n-1} Y_k(1)\right] + Y_n(t - n) \quad \text{for } n \leq t \leq n + 1$$

has the required properties. We shall show that any G-valued process X with continuous sample paths and independent increments that satisfies (4.1) is a G-valued semimartingale with respect to an appropriate right continuous filtration.

Let $\{\mathcal{G}_t : t \geq 0\}$ be the filtration in $\mathcal{F}$ generated by X; i.e., for each t, $\mathcal{G}_t$ is the σ-algebra generated by $\{X(s) : 0 \leq s \leq t\}$. Then X is adapted to the right-continuous filtration $\{\mathcal{G}_{t+} : t \geq 0\}$, where $\mathcal{G}_{t+} = \bigcap\{\mathcal{G}_u : u > t\}$. For each $y \in \widehat{G}$, define the process $N(\cdot\,, y)$ by

$$N(t, y) := \langle X(t), y\rangle \exp\left[\tfrac{1}{2}\phi(t, y)\right].$$

Then each $N(\cdot\,, y)$ is a $\{\mathcal{G}_t\}$-martingale. By the optional sampling theorem (see for example Theorem 3.22 on page 19 in Karatzas and Shreve [7])

$$E\left[N(t, y) \mid \mathcal{G}_{s+}\right] = N(s, y) \quad \text{a.s. for } 0 \leq s \leq t,$$

so $N(\cdot\,, y)$ is a $\{\mathcal{G}_{t+}\}$-martingale. But, from the definition of $N(t, y)$, this implies that $\langle X(\cdot), y\rangle$ is the product of a $\{\mathcal{G}_{t+}\}$-martingale and a (deterministic) finite variation process, so $\langle X(\cdot), y\rangle$ is a $\{\mathcal{G}_{t+}\}$-semimartingale. Hence X is a G-valued semimartingale with respect to the filtration $\{\mathcal{G}_{t+}\}$.

If we apply Lemma 2.1 to the sample paths of X, we obtain, for each $y \in \widehat{G}$, a real-valued process $W(\cdot\,, y)$ such that

$$\langle X(t), y\rangle = \exp\left[i\, W(t,y)\right] \quad \text{for all } t \geq 0.$$

Furthermore, $W(\cdot\,, y)$ has continuous paths, independent increments and $W(0, y) = 0$. Therefore $W(\cdot\,, y)$ is Gaussian. Because $X(s)$ has a symmetric distribution for every s and $g(-x, y) = -g(x, y)$ for all $x \in G$, $y \in \widehat{G}$, the distribution of $W(t, y)$ is also symmetric. Therefore $W(t, y)$ has expectation zero. Also,

$$E \exp[i\xi W(t,y)]\Big|_{\xi=1} = E\langle X(t), y\rangle = \exp[-\tfrac{1}{2}\phi(t,y)],$$

whence $W(t, y)$ has variance $\phi(t, y)$.

For $\xi \in \mathbb{R}$ and $y \in \widehat{G}$ define the process $N(\cdot\,, y, \xi)$ by

$$N(t,y,\xi) := \exp\left[i\xi W(t,y) + \tfrac{1}{2}\xi^2\phi(t,y)\right].$$

Then each $N(\cdot\,, y, \xi)$ is a $\{\mathcal{G}_t\}$-martingale. By the optional sampling theorem,

$$E\left[N(t,y,\xi) \mid \mathcal{G}_{s+}\right] = N(s,y,\xi) \quad \text{a.s. for } 0 \leq s \leq t$$

and therefore

$$E\Big(\exp\left[i\xi W(t,y)\right] \mid \mathcal{G}_{s+}\Big) = \exp\left[i\xi W(s,y) - \tfrac{1}{2}\xi^2\{\phi(t,y) - \phi(s,y)\}\right] \qquad (4.2)$$

holds a.s. for $0 \leq s \leq t$. Generally, the exceptional set of P-measure zero in (4.2) may depend on ξ, but there exist versions of the conditional expectations such that the exceptional set of P-measure zero does not depend on ξ. To see this, fix a version of the conditional distribution of $W(t, y)$ given $\mathcal{G}_{s+}$ and use this version of the conditional distribution to evaluate $E\big(\exp\left[i\xi W(t,y)\right] \mid \mathcal{G}_{s+}\big)$ for all $\xi \in \mathbb{R}$. With these versions of the conditional expectations, both sides of (4.2) are continuous in ξ. Because (4.2) holds a.s for each $\xi \in \mathbb{R}$, there exists $\Omega' \in \mathcal{F}$ with $P(\Omega') = 1$ such that (4.2) holds for all rational $\xi \in \mathbb{R}$ when $\omega \in \Omega'$. The continuity in ξ of both sides of (4.2) then implies that (4.2) holds for all $\xi \in \mathbb{R}$ when $\omega \in \Omega'$. Hence, the conditional distribution of $W(t, y)$ given $\mathcal{G}_{s+}$ is a.s. normal with conditional expectation $W(s, y)$ and conditional variance $\phi(t, y) - \phi(s, y)$. This implies that the continuous process $W(\cdot\,, y)$ is a $\{\mathcal{G}_{t+}\}$-martingale and that the compensator of its square is given by $\phi(\cdot\,, y)$.

It follows that X is a G-valued semimartingale with characteristics $\widetilde{B} \equiv e$, $\Phi \equiv \phi$ and $\nu \equiv 0$. Thus, in this example, the semimartingale characteristics exist and are the same as the canonical triple in the Lévy-Khinchine representation.

References

1. Bingham, M. S.: Stochastic processes with independent increments taking values in an abelian group, *Proc. London Math. Soc.* **22**, (1971) 507–530.
2. Bingham, M. S.: Functional central limit theorems for approximate martingale arrays in a locally compact abelian group, *J. Theoret. Prob.* **1**, No. 1, (1988) 3–26.
3. Dellacherie, C. and Meyer, P.-A.: *Probabilities and Potential*, Mathematics Studies 29, North Holland Publishing company, Amsterdam – New York – Oxford, 1978.
4. Hewitt, E. and Ross, K. A.: *Abstract Harmonic Analysis, Volumes 1 and 2*, Springer Verlag, Berlin – Heidelberg – New York, 1963, 1970.

5. Heyer, H.: *Probability Measures on Locally Compact Groups*, Springer Verlag, Berlin – Heidelberg – New York, 1977.
6. Jacod, J. and Shiryaev, A. N.: *Limit Theorems for Stochastic Processes*, Springer Verlag, Berlin – Heidelberg – New York, 1987.
7. Karatzas, I. and Shreve, S. E.: *Brownian Motion and Stochastic Calculus, 2nd edition*, Springer Verlag, Berlin – Heidelberg – New York, 1991 (corrected printing 2005).
8. Meyer, P.-A.: *Probability and Potentials*, Blaisdell Publishing Company, Waltham – Toronto – London, 1966.
9. Parthasarathy, K. R.: *Probability Measures on Metric Spaces*, Academic Press, New York, 1967 (republished by AMS Chelsea Publishing, 2005).
10. Parthasarathy, K. R., Ranga Rao, R., and Varadhan, S. R. S.: Probability distributions on locally compact abelian groups, *Illinois J. Math.* **7**, 1963, 337–369.
11. Rudin, W.: *Fourier Analysis on Groups*, Wiley (Interscience), New York, 1962.
12. Sokol, A.: An elementary proof that the first hitting time of an open set by a jump process is a stopping time, in *Séminaire de Probabilités, XLV*, (2013) 301–304, Lecture Notes in Math., Vol. 2078, Springer, Berlin.

M. S. Bingham: School of Mathematics & Physical Sciences, University of Hull, Hull, HU6 7RX, United Kingdom

E-mail address: `m.s.bingham@hull.ac.uk`