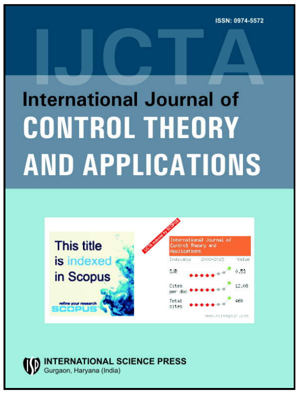

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Some I-convergent Triple Sequence Spaces of Fuzzy Numbers Defined by Orlicz Function (Paper Id: 59)

Sangita Saha[1] and Santanu Roy[2]

[1] *Research Scholar, Department of Mathematics National Institute of Technology Silchar; Assam, India, Emails: sangitasaha131@gmail.com*
[2] *Assistant Professor& Head, Department of Mathematics National Institute of Technology Silchar; Assam, India*

Abstract: In this research paper, using an Orlicz function the notion of some I-convergent triple sequence spaces of fuzzy numbers is introduced. We make an effort to investigate some basic algebraic and topological properties of the introduced sequence spaces and also derived some inclusion results between these spaces.

Keywords: Filter, Fuzzy numbers, Ideal, I-convergence, Normal, Orlicz function, Triple sequence.

1. INTRODUCTION

The basic mathematical concept of a set was extended by the introduction of the fuzzy set theory. Fuzzy set theory, compared to other mathematical theories, is perhaps the most easily adaptable theory to practice. The concepts of fuzzy sets and fuzzy set operations were first introduced by Lofti A. Zadeh [41] in 1965 and after his pioneering work done on fuzzy set theory, a huge number of research papers have been appeared on fuzzy theory and its applications as well as fuzzy analogues of the classical theories. Fuzzy set theory is a powerful hand set for modeling, uncertainty and vagueness in various problems arising in the field of science and engineering. Several mathematicians have discussed various aspects of the theory and applications of fuzzy sets such as fuzzy topological spaces, similarity relations and fuzzy orderings, fuzzy measures of fuzzy events, fuzzy mathematical programming and so on.

The theory of sequence of fuzzy numbers was first introduced by Matloka [18] and Matloka showed that every convergent sequence of fuzzy numbers is bounded. After a detailed study on the sequences of fuzzy numbers, it was proved by Nanda [19] that the set of all convergent sequences of fuzzy numbers forms a complete metric space. The notion of ideal convergence, as a generalization of statistical convergence was introduced by Kostyrko *et . al*. [14] in 2000-2001, which depends on the structure of the ideal I of the subset of the set of natural numbers N. Later, it was further developed by various researchers such as Šalát *et. al*. [27-28], Kumar and Kumar [16], Tripathy and Tripathy [40], Das *et. al*. [4], Tripathy and Sen [40], Tripathy and

Hazarika [37], Sen and Roy [32], Khan and Khan [13], Esi and Sharma [7], Gürdal and Huban [11], Nath and Roy [21-22] etc.

The summability theory of multiple sequences was first studied by Agnew [1] and certain theorems for double sequences was derived by him. Sahiner *et. al*. [25], Sahiner and Tripathy [26] introduced and investigated the different types of notions of triple sequences at the initial stage. Recently statistical convergence of triple sequences on probabilistic normed space was introduced by Savas and Esi [31]. Later, Esi [9] introduced statistical convergence of triple sequences in topological groups. Recently more works on triple sequences are done by Esi [8], Kumar *et. al.* [14], Dutta *et .al.* [6], Tripathy and Goswami [36], Nath and Roy [20] and many others.

Using the notion of Orlicz function, the scope for the studies on sequence spaces was enhanced by Lindenstrauss and Tzafriri [17], who used Orlicz function to construct the Banach space

$$\ell_M = \left\{ (x_k) : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty, \text{ for some } \rho > 0 \right\},$$

with the norm

$$\|x\| = \inf\left\{ \rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \le 1 \right\}.$$

In the later stage different classes of Orlicz sequence spaces were introduced and studied by Parashar and Choudhary [23] and discussed some properties of the sequence spaces defined by an Orlicz function M which generalized the Orlicz sequence space ℓ_M. More works on Orlicz sequence spaces can be found in ([2-3], [6], [10-11], [24], [29-30], [33-35], [38]).

2. PRELIMINARIES AND BACKGROUND

In this section, some fundamental notions and definitions are defined, which are closely related to the paper. Throughout w, c, c_0, ℓ_∞ denote the spaces of all, convergent, null and bounded sequences respectively and N, R and C denote the sets of natural and real numbers respectively.

A fuzzy number on R is a function $X : R \to L(=[0,1])$ associating each real number $t \in R$ having grade of membership $X(t)$. We can express every real number r as a fuzzy number $\bar{r}$ as:

$$\bar{r}(t) = \begin{cases} 1, & if \quad t = r \\ 0, & otherwise \end{cases}$$

The a-level set of a fuzzy number X, $0 < \alpha \le 1$, is defined and denoted by $[X]^\alpha = \{t \in R : X(t) \ge \alpha\}$.

A fuzzy number X is said to be convex if $X(t) \ge X(s) \wedge X(r) = \min(X(s), X(r))$, where $s < t < r$.

A fuzzy number X is said to be normal if there exists $t_0 \in R$ such that $X(t_0) = 1$. A fuzzy number X is called upper semi-continuous if for each $\varepsilon > 0$, $X^{-1}[0, a+\varepsilon))$, for all $a \in L$ is open in the usual topology of R. The set of all upper semi continuous, normal, convex fuzzy number is denoted by $R(L)$, whose additive and multiplicative identities are denoted by $\bar{0}$ and $\bar{1}$ respectively.

Let D be the set of all closed bounded intervals $X=\left[X^L,X^R\right]$ on the real line R. Then $X \le Y$ if and only if $X^L \le Y^L$

and $X^R \le Y^R$. Let $d(X,Y) = \max(|X^L - X^R|, |Y^L - Y^R|)$. Then (D,d) is a complete metric space.

Also $\bar{d} : R(L)\times R(L) \to R$ defined by $\bar{d}(X,Y) = \sup\limits_{0\le\alpha\le 1} d([X]^\alpha,[Y]^\alpha)$, for $X,Y \in R(L)$ is a metric on $R(L)$.

A non-void class $I \subseteq 2^X$ (power set of non-empty set X) is said to be an ideal if I satisfies the following conditions:

(i) $A,B\in I \Rightarrow A\cup B \in I$ and (ii) $A\in I$ *and* $B\subseteq A \Rightarrow B\in I$.

An ideal $I\subseteq 2^X$ is said to be non-trivial if $I\neq\varnothing$ and $X\notin I$.

A non-trivial ideal $I\subseteq 2^X$ is said to be admissible if I contains each finite subset of X.

A non-trivial ideal I is said to be maximal if there does not exist any non-trivial ideal $J\neq I$ containing I as a subset.

Throughout the paper, the ideals of $2^{N\times N\times N}$ will be denoted by I_3 and ${}_3(w^F)$, ${}_3(\ell_\infty^F)$, ${}_3(c^F)$, ${}_3(c_0{}^F)$ denote the spaces of all, bounded, convergent in Pringsheim's sense and null in Pringsheim's sense fuzzy real-valued triple sequences respectively.

A non-empty family of sets $F\subseteq 2^X$ is called a filter on X if

(i) $\varnothing\notin F$ (ii) $A, B\in F \Rightarrow A\cap B\in F$ and (iii) $A\in F$ and $A\subseteq B \Rightarrow B\in F$.

For any ideal I, a filter $F(I)$ is defined as $F(I)=\{K\subseteq N : N\setminus K\in I\}$.

A subset E of $N\times N\times N$ is said to have asymptotic density $\delta(E)$ if

$\delta(E) = \lim\limits_{p,q,r\to\infty} \sum_{n=1}^{p}\sum_{l=1}^{q}\sum_{k=1}^{r} \chi_E(n,l,k)$ exists, where χ_E is the characteristic function of E.

A triple sequence is a mapping $x : N\times N\times N \to R(C)$.

A triple sequence $X=\langle X_{nlk}\rangle$, $X_{nlk}\in R(L)$ of fuzzy numbers is a triple infinite array of fuzzy numbers X_{nlk} for all n, $l,k\in N$.

A triple sequence $X=\langle X_{nlk}\rangle$ of fuzzy numbers is said to be convergent in Pringsheim's sense to the fuzzy number X, if for every $\varepsilon>0$, $\exists$ $n_0=n_0(\varepsilon), l_0=l_0(\varepsilon), k_0=k_0(\varepsilon)\in N$, such that $\bar{d}(X_{nlk},X)<\varepsilon$ for all $n\ge n_0, l\ge l_0, k\ge k_0$.

A triple sequence $X=\langle X_{nlk}\rangle$ of fuzzy numbers is said to be I_3-*convergent* to the fuzzy number X_0, if for all $\varepsilon>0$, $\{(n,l,k)\in N\times N\times N : \bar{d}(X_{nlk},X_0)\ge\varepsilon\}\in I_3$ and written as $I_3-\lim X_{nlk}=X_0$.

A triple sequence $X = \langle X_{nlk} \rangle$ of fuzzy numbers is said to be I_3-*bounded* if there exists a real number μ such that

$$\{(n,l,k) \in N \times N \times N : \bar{d},(X_{nlk}, \bar{0}) > \mu\} \in I_3.$$

A triple sequence space E^F of fuzzy numbers is said to be solid or normal if $\langle Y_{nlk} \rangle \in E^F$ whenever $\langle X_{nlk} \rangle \in E^F$ and $\bar{d}(Y_{nlk}, \bar{0}) \leq \bar{d}(X_{nlk}, \bar{0})$ for all n , $l,k \in N$.

A triple sequence space E^F of fuzzy numbers is said to be monotone if E^F contains the canonical pre-image of all its step spaces.

A triple sequence space E^F of fuzzy numbers is said to be symmetric if $\langle X_{\pi(nlk)} \rangle \in E^F$, whenever $\langle X_{nlk} \rangle \in E^F$ where π is a permutation on $N \times N \times N$.

A triple sequence space E^F of fuzzy numbers is said to be sequence algebra if $\langle X_{nlk} \otimes Y_{nlk} \rangle \in E^F$, whenever $\langle X_{nlk} \rangle, \langle Y_{nlk} \rangle \in E^F$.

A triple sequence space E^F of fuzzy numbers is said to be convergence free if $\langle Y_{nlk} \rangle \in E^F$ whenever $\langle X_{nlk} \rangle \in E^F$ and $X_{nlk} = \bar{0}$ implies $Y_{nlk} = \bar{0}$.

Let M be an Orlicz function and $p = \langle p_{nlk} \rangle$ be a triple sequence of bounded strictly positive real numbers. In this paper, the following I-convergent fuzzy triple sequence spaces are introduced:

$$_3(c^{I(F)})(M,p) = \left\{ X = \langle X_{nlk} \rangle \in {}_3(w^F) : I_3\text{-lim} \left[M\left(\frac{\bar{d}\,(X_{nlk}\,,\,X_0)}{\rho} \right) \right]^{p_{nlk}} = 0, \text{ for some } \rho > 0 \text{ and } X_0 \in R(L) \right\},$$

$$_3(c_0{}^{I(F)})(M,p) = \left\{ X = \langle X_{nlk} \rangle \in {}_3(w^F) : I_3\text{-lim} \left[M\left(\frac{\bar{d}\,(X_{nlk}\,,\,\bar{0})}{\rho} \right) \right]^{p_{nlk}} = 0, \text{ for some } \rho > 0 \right\},$$

$$_3(\ell_\infty^{(F)})(M,p) = \left\{ X = \langle X_{nlk} \rangle \in {}_3(w^F) : \sup_{n,l,k} \left[M\left(\frac{\bar{d}\,(X_{nlk}\,,\,\bar{0})}{\rho} \right) \right]^{p_{nlk}} < \infty, \text{ for some } \rho > 0 \right\},$$

$$_3\ell_\infty^{I(F)}(M,p) = \left\{ \begin{array}{l} X = \langle X_{nlk} \rangle \in {}_3(w^F) : \text{ there exists a real number } \mu > 0 \text{ such that} \\ \{(n,l,k) \in N \times N \times N : \left[M\left(\frac{\bar{d}\,(X_{nlk}\,,\,\bar{0})}{\rho} \right) \right]^{p_{nlk}} > \mu\} \in I_3, \text{ for some } \rho > 0\} \end{array} \right\}.$$

Also we introduce

$$_3(m^{I(F)})(M,p) = \ _3(c^{I(F)})(M,p) \cap \ _3(\ell_\infty^{(F)})(M,p)$$

and $_3(c_0{}^{I(F)})(M,p) = \ _3(c_0{}^{I(F)})(M,p) \cap \ _3(\ell_\infty^{(F)})(M,p).$

To prove some results in the paper, the following Lemma's will be used.

Lemma 2.1- Every normal sequence space E^F *is monotone. .*

Lemma 2.2- For two triple sequences $p = \langle p_{nlk} \rangle$ and $q = \langle q_{nlk} \rangle$, $_3(c_0{}^{I(F)})^{BP}(p) \supseteq \ _3(c_0{}^{I(F)})^{BP}(q)$ *if and only if* $\lim\limits_{(n,l,k)\in K} \inf \left(\dfrac{p_{nlk}}{q_{nlk}} \right) > 0$, *where* $K \in F(I_3)$ *(Nath and Roy* [21]).

3. MAIN RESULTS

In this section, we examine some basic topological and algebraic properties of the introduced sequence spaces and obtain some inclusion relation related to these spaces.

Theorem 3.1- If M is an Orlicz function and $p = \langle p_{nlk} \rangle$ *is a triple sequence of bounded strictly positive numbers, then the sequence spaces* $_3(m^{I(F)})(M,p)$, $_3(m_0{}^{I(F)})(M,p)$ and $_3\ell_\infty^{I(F)}(M,p)$ *are closed with respect to addition and scalar multiplication operations.*

Proof. We prove the result for the space $_3(m_0^{I(F)})(M,p)$ and the result for the other spaces can be proved in a similar manner. Let $\langle X_{nlk} \rangle, \langle Y_{nlk} \rangle \in \ _3(m_0^{I(F)})(M,p).$

Then there exists positive numbers ρ_1 and ρ_2 such that the sets

$$A = \left\{ (n,l\ ,\ k) \in N \times N \times N : \left[M\left(\frac{\bar{d}\ (X_{nlk}\ ,\ \bar{0})}{\rho_1} \right) \right]^{p_{nlk}} \geq \frac{\varepsilon}{2} \right\} \in I_3 \text{ and}$$

$$B = \left\{ (n,l\ ,\ k) \in N \times N \times N : \left[M\left(\frac{\bar{d}\ (Y_{nlk}\ ,\ \bar{0})}{\rho_2} \right) \right]^{p_{nlk}} \geq \frac{\varepsilon}{2} \right\} \in I_3.$$

Let α, β be two scalars and let $\rho = \max(2|\alpha|\rho_1,\ 2|\beta|\rho_2)$. Since M is continuous, the following inequality holds:

$$\left[M\left(\frac{\bar{d}\ (\alpha X_{nlk} + \beta Y_{nlk}\ \ ,\ \bar{0})}{\rho} \right) \right]^{p_{ijk}} \leq D \left\{ \left[M\left(\frac{\bar{d}\ (X_{nlk}\ ,\ \bar{0})}{\rho_1} \right) \right]^{p_{ijk}} + \left[M\left(\frac{\bar{d}\ (Y_{nlk}\ ,\ \bar{0})}{\rho_2} \right) \right]^{p_{ijk}} \right\},$$

where $D = \max\ (1, 2^{H-1}),\ \ H = \sup\limits_{i,j,k} p_{ijk} < \infty.$

From the above inequality, we obtained

$$\left\{(n,l\ ,\ k)\in N\times N\times N:\ \left[M\left(\frac{\bar{d}\ (\alpha X_{nlk}+\beta Y_{nlk}\ \ ,\ \bar{0})}{\rho}\right)\right]^{p_{nlk}}\geq\frac{\varepsilon}{2}\right\}\subseteq$$

$$\left\{(n,l\ ,\ k)\in N\times N\times N:D\left[M\left(\frac{\bar{d}\ (X_{nlk}\ ,\ \ \bar{0})}{\rho_1}\right)\right]^{p_{nlk}}\geq\frac{\varepsilon}{2}\right\}\cup\left\{(n,l\ ,\ k)\in N\times N\times N:D\left[M\left(\frac{\bar{d}\ (Y_{nlk}\ ,\ \ \bar{0})}{\rho_2}\right)\right]^{p_{nlk}}\geq\frac{\varepsilon}{2}\right\}.\in I_3.$$

$\therefore(\alpha X_{nlk}+\beta Y_{nlk})\in\ {}_3(m_0{}^{I(F)})(M,p).$

This completes the proof.

Theorem 3.2- Let the sequence $p=\langle p_{nlk}\rangle$ *be bounded. Then* ${}_3(c_0^{I(F)})(M,p)\subset{}_3(c^{I(F)})\ (M,p)\subset{}_3(\ell_\infty^{I(F)})(M,p)$ *and the inclusions are proper.*

Proof. From definition, the inclusion ${}_3(c_0^{I(F)})(M,p)\ \subset\ {}_3(c^{I(F)})(M,p)\subset\ {}_3(\ell_\infty^{I(F)})(M,p)$ follows.

To show that the inclusion ${}_3(c^{I(F)})(M,p)\ \ \subset\ {}_3(\ell_\infty^{I(F)})(M,p)$ is proper, we cite a counter example.

Example 3.1- Let $I_3(P)$ denote the class of all subsets of $N\times N\times N$ such that $A\in I_3(P)$

implies that $\exists\ n_0,l_0,k_0\in N$ such that

$A\subseteq N\times N\times N-\{(n\ ,\ l,k)\in N\times N\times N:n\geq n_0\ ,\ l\geq l_0,k\geq k_0\}.$

Let $M(x)=x^3$ and $n_0,l_0,k_0\in N$ be fixed such that

$$p_{nlk}=\begin{cases}\dfrac{1}{3},\ for\ \ 1\leq n\leq n_0\ ,\ \ 1\leq l\leq l_0\ ,\ \ 1\leq k\leq k_0\\ 3,\ \ otherwise\end{cases}$$

We define the sequence $\langle X_{nlk}\rangle$ as:

$$X_{nlk}=\bar{1}\ ,\ \ \text{for } 1\leq n\leq n_0,\ 1\leq l\leq l_0\ ,\ 1\leq k\leq k_0.$$

For $n>n_0,\ l>l_0,\ k>k_0$ and $(n+l+k)$ even,

$$X_{nlk}(t)=\begin{cases}\dfrac{nt-4n+1}{n+1},\ for\ \ \ 4-n^{-1}\leq t\leq 5\\ 6-t,\qquad\quad for\ \ \ 5<t\leq 6\\ 0,\qquad\qquad otherwise\end{cases}$$

otherwise
$$X_{nlk}(t) = \begin{cases} \dfrac{nt-1}{4n-1}, & \text{for} \quad n^{-1} \le t \le 4 \\ 5-t, & \text{for} \quad 4 < t \le 5 \\ 0, & \text{otherwise} \end{cases}$$

Then $\langle X_{nlk} \rangle \in {}_3(\ell_\infty^{I(F)})(M,p)$, but $\langle X_{nlk} \rangle \notin {}_3(c^{I(F)})(M,p)$. This implies ${}_3(c^{I(F)})(M,p) \subset {}_3(\ell_\infty^{I(F)})(M,p)$.

Theorem 3.3- The sequence spaces ${}_3(m^{I(F)})(M,p)$ *and* ${}_3(m_0^{I(F)})(M,p)$ are *complete with respect to the metric* ς *defined by*

$$\varsigma(X,Y) = \inf\left\{ \rho^{\frac{p_{nlk}}{J}} > 0 : \sup_{n,l,k} M\left(\frac{\bar{d}\,(X_{nlk}\,,\,Y_{nlk})}{\rho} \right) \le 1,\ \rho > 0 \right\},\ \textit{where } J = \max\,(1,H),\ \ H = \sup_{n,l,k} p_{nlk}.$$

Proof. We prove the result for the space ${}_3(m_0^{I(F)})(M,p)$.

Let $\langle X^{(i)} \rangle$ be a Cauchy sequence in ${}_3(m_0^{I(F)})(M,p)$ where $X^{(i)} = \langle X_{nlk}{}^{(i)} \rangle$.

For a fixed $x_0 > 0,\ r > 0$ is chosen such that $M\left(\dfrac{rx_0}{2}\right) > 1.$

For a given $\varepsilon > 0$, $\exists\ n_0 \in N$ such that

$$\tau\left(X^{(i)}\,,\,X^{(j)}\right) < \frac{\varepsilon}{rx_0},\ \text{ for all } i\,,j \ge n_0.$$

$$\Rightarrow\ \inf\left\{ \rho^{\frac{p_{nlk}}{J}} > 0 : \sup_{n,l,k} \left[M\left(\frac{\bar{d}(X_{nlk}{}^{(i)}\,,\,X_{nlk}{}^{(j)})}{\rho} \right) \right] \le 1,\ \rho > 0 \right\} < \frac{\varepsilon}{rx_0},\ \text{ for all } i\,,j \ge n_0.$$

$$\Rightarrow\ \sup_{n,l,k} \left[M\left(\frac{\bar{d}(X_{nlk}{}^{(i)}\,,\,X_{nlk}{}^{(j)})}{\rho} \right) \right] \le 1,\ \text{ for all } i\,,j \ge n_0.$$

$$\Rightarrow \sup_{n,l,k} \left[M\left(\frac{\bar{d}(X_{nlk}{}^{(i)}\,,\,X_{nlk}{}^{(j)})}{\tau\left(X^{(i)}\,,\,X^{(j)}\right)} \right) \right] \le 1,\ \text{for all } i\,,j \ge n_0. \tag{1}$$

$$\Rightarrow \left[M\left(\frac{\bar{d}(X_{nlk}{}^{(i)}\,,\,X_{nlk}{}^{(j)})}{\tau\left(X^{(i)}\,,\,X^{(j)}\right)} \right) \right] \le 1\ \le M\left(\frac{rx_0}{2}\right),\ \text{ for all } i\,,j \ge n_0.$$

$$\Rightarrow \ \bar{d}(X_{nlk}{}^{(i)} \ , X_{nlk}{}^{(j)}) < \frac{\varepsilon}{2}, \text{ for all } i \ , j \geq n_0 \ .$$

Hence $\langle X_{nlk}{}^{(j)} \rangle$ is a Cauchy sequence of fuzzy numbers. So $\exists$ a fuzzy number X_{nlk} such that

$\lim_{j \to \infty} X_{nlk}{}^{(j)} = X_{nlk}$, for each $n \ , \ l,k \in N$.

Since M is continuous, so taking limit as $j \to \infty$ in the equation (1),

$$\sup_{n,l,k} \left[M \left(\frac{\bar{d}(X_{nlk}{}^{(i)} \ , X_{nlk})}{\rho} \right) \right] \leq 1.$$

Now, taking infimum of such $\rho's$, we get

$$\inf \left\{ \rho^{\frac{p_{nlk}}{J}} > 0 : \ \sup_{n,l,k} \left[M \left(\frac{\bar{d}(X_{nlk}{}^{(i)} \ , X_{nlk})}{\rho} \right) \right] \leq 1, \right\} < \varepsilon, \text{ for all } i \geq n_0.$$

$$\Rightarrow \ \varsigma\left(X^{(i)} \ , X\right) < \varepsilon, \ \text{ for all } i \geq n_0.$$

Now for all $i \geq n_0$,

$$\varsigma\left(X,\bar{0}\right) \leq \varsigma\left(X,X^{(i)}\right) + \varsigma\left(X^{(i)},\bar{0}\right) < \varepsilon + K < \infty.$$

$\therefore X \in {}_3(m_0^{I(F)})(M,p).$

Hence ${}_3(m_0^{I(F)})(M,p)$ is complete.

Similarly the result can be established for the other space. ■

Theorem 3.4- Let M_1 and M_2 be two *Orlicz functions*, *then*

(*i*) $Z(M_1 \ , p) \cap Z(M_2 \ , p) \subseteq Z(M_1 + M_2 \ , p)$

(*ii*) $Z(M_2 \ , p) \subseteq Z(M_1 \circ M_2 \ , p)$, for Z $= {}_3(m_0^{I(F)}), {}_3(m^{I(F)}), {}_3(\ell_\infty^{I(F)})$.

Proof. We prove both the results (*i*) and (*ii*) for $Z = {}_3(m_0^{I(F)})$. Similarly, the other cases can be derived.

(*i*) Let $\langle X_{nlk} \rangle \in {}_3(m_0^{I(F)})(M_1,p) \cap {}_3(m_0^{I(F)})(M_2,p)$. Then $\exists$ $\rho_1 \ , \rho_2 > 0$ such that such that the sets

$$A = \left\{ (n,l \ , \ k) \in N \times N \times N : \ \left[M_1 \left(\frac{\bar{d} \ (X_{nlk} \ , \ \bar{0})}{\rho_1} \right) \right]^{p_{nlk}} \geq \frac{\varepsilon}{2} \right\} \in I_3 \text{ and}$$

$$B = \left\{ (n,l\ ,\ k) \in N \times N \times N :\ \left[M_2 \left(\frac{\bar{d}\ (X_{nlk}\ ,\ \bar{0})}{\rho_2} \right) \right]^{p_{nlk}} \geq \frac{\varepsilon}{2} \right\} \in I_3.$$

Let $\rho = \rho_1 + \rho_2$. Since M is continuous, we have the following inequality:

$$\left[(M_1 + M_2) \left(\frac{\bar{d}\ (X_{nlk}\ ,\ \bar{0})}{\rho} \right) \right]^{p_{nlk}} \leq D \left[\frac{\rho_1}{\rho_1 + \rho_2} M_1 \left(\frac{\bar{d}\ (X_{nlk}\ ,\ \bar{0})}{\rho_1} \right) \right]^{p_{nlk}} + D \left[\frac{\rho_2}{\rho_1 + \rho_2} M_2 \left(\frac{\bar{d}\ (X_{nlk}\ ,\ \bar{0})}{\rho_2} \right) \right]^{p_{nlk}},$$

where $D = \max\ (1\ ,\ 2^{H-1})$, $H = \sup_{n,l,k} p_{nlk}$.

From the above relation, we obtained

$$\left\{ (n,l\ ,\ k) \in N \times N \times N :\ \left[(M_1 + M_2) \left(\frac{\bar{d}\ (X_{nlk}, \bar{0})}{\rho} \right) \right]^{p_{nlk}} \geq \frac{\varepsilon}{2} \right\}$$

$$\subseteq \left\{ (n,l\ ,\ k) \in N \times N \times N : D \left[\rho M_1 \left(\frac{\bar{d}\ (X_{nlk}\ ,\ \bar{0})}{\rho_1} \right) \right]^{p_{nlk}} \geq \frac{\varepsilon}{2} \right\} \cup \left\{ (n,l\ ,\ k) \in N \times N \times N : D \left[\rho M_2 \left(\frac{\bar{d}\ (X_{nlk}\ ,\ \bar{0})}{\rho_2} \right) \right]^{p_{nlk}} \geq \frac{\varepsilon}{2} \right\}. \in I_3.$$

Thus $\langle X_{nlk} \rangle \in {}_3(m_0^{I(F)})(M_1 + M_2, p)$.

(ii) Let $\varepsilon > 0$ be given. Since M_1 is continuous, so $\exists\ \eta > 0$ such that $M_1(\eta) = \varepsilon$.

Let $\langle X_{nlk} \rangle \in {}_3(m_0^{I(F)})(M_2, p)$.

So $\exists\ \rho > 0$ such that $I_3\text{-}\lim \left[M_2 \left(\frac{\bar{d}\ (X_{nlk}\ ,\ \bar{0})}{\rho} \right) \right]^{p_{nlk}} = 0.$

Then $\exists\ n_0,\ l_0, k_0 \in N$ such that

$$\left[M_2 \left(\frac{\bar{d}\ (X_{nlk}\ ,\ \bar{0})}{\rho} \right) \right]^{p_{nlk}} < \eta, \text{ for all } n \geq n_0,\ l \geq l_0, k \geq k_0.$$

$$\Rightarrow \left[(M_1 \circ M_2) \left(\frac{\bar{d}\ (X_{nlk}\ ,\ \bar{0})}{\rho} \right) \right]^{p_{nlk}} < \varepsilon, \text{ for all } n \geq n_0,\ l \geq l_0, k \geq k_0.$$

$$\therefore I_3\text{-}\lim \left[(M_1 \circ M_2) \left(\frac{\bar{d}\ (X_{nlk}\ ,\ \bar{0})}{\rho} \right) \right]^{p_{nlk}} = 0.$$

$\Rightarrow \langle X_{nlk} \rangle \in {}_3(m_0^{I(F)})(M_1 \circ M_2, p)$. ■

Using the standard techniques, the following result can be easily proved.

Theorem 3.5- If $M_1(x) \le M_2(x)$ *for all* $x \in [0,\infty)$, *then* $Z(M_2\ , p) \subseteq Z(M_1\ , p)$ *for*

$Z\ =\ {}_3(c_0^{I(F)}),\ {}_3(c^{I(F)}),\ {}_3(\ell_\infty^{I(F)})$.

Theorem 3.6- For two triple sequences $p = \langle p_{nkl} \rangle$ and $q = \langle q_{nkl} \rangle$,

${}_3(m_0{}^{I(F)})(M,p)\ \supseteq\ {}_3(m_0{}^{I(F)})(M,q)$ *if and only if* $\liminf\limits_{(n,l,k)\in K} \left(\dfrac{p_{nlk}}{q_{nlk}} \right) > 0,$ *where* $K \in F(I_3)$.

Proof. The result follows immediately from *Lemma 2.2.* ■

Theorem 3.7- The sequence spaces ${}_3(m_0^{I(F)})(M,p)$ *is both normal and monotone.*

Proof. Let $\langle X_{nlk} \rangle \in {}_3(m_0^{I(F)})(M,p)$ and $\langle Y_{nlk} \rangle$ be such that $\bar{d}(Y_{nlk}\ ,\ \bar{0}) \le \bar{d}(X_{nlk}\ ,\ \bar{0})$, for all $n\ ,\ l,k \in N$.

Let $\varepsilon > 0$ be given.

Then from the following inclusion relation:

$$\left\{ (n\ ,l,k) \in N \times N \times N : \left[M\ \left(\frac{\bar{d}\ (X_{nlk}\ ,\ \bar{0})}{\rho} \right) \right]^{p_{nlk}} \ge \varepsilon \right\} \supseteq \left\{ (n\ ,l,k) \in N \times N \times N :: \left[M\ \left(\frac{\bar{d}\ (Y_{nlk}\ ,\ \bar{0})}{\rho} \right) \right]^{p_{nlk}} \ge \varepsilon \right\}$$

it follows that ${}_3(m_0^{I(F)})(M,p)$ is normal.

Also by *Lemma 2.1*, the space ${}_3(m_0^{I(F)})(M,p)$ is monotone. ■

Proposition 3.8- The sequence spaces ${}_3(m^{I(F)})(M,p)$ *is neither monotone nor normal.*

Proof. The result follows from the following example.

Example 3.2- Let $I_3(\rho) \subset 2^{N\times N\times N}$ denote the class of all subsets of $N \times N \times N$ of zero natural density.

Let $I_3 = I_3(\rho)$, $A \in I_3$, $p_{nlk} = 1$ for all $n\ ,\ l,k \in N$ and $M(x) = x^2$.

We define the sequence $\langle X_{nlk} \rangle$ by:

For all $(n,l,k) \notin A$,

$$X_{nlk}(t) = \begin{cases} 1 + \sqrt{n+l+k}(t-1), & \textit{for} \quad 1 - \dfrac{1}{\sqrt{n+l+k}} \le t \le 1 \\ 1 - \sqrt{n+l+k}(t-1), & \textit{for} \quad 1 < t \le 1 + \dfrac{1}{\sqrt{n+l+k}} \\ 0, & \textit{otherwise} \end{cases}$$

otherwise $X_{nlk} = \bar{1}.$

Then $\langle X_{nlk} \rangle \in {}_3(m^{I(F)})(M,p).$

Let $K = \{2i : i \in N\}$.

Consider the sequence $\langle Y_{nlk} \rangle$ defined as:

$$Y_{nlk} = \begin{cases} X_{nlk}, & if \ \ (n, l, k) \in K \\ \bar{0}, & otherwise \end{cases}$$

Then $\langle Y_{nlk} \rangle$ belongs to the canonical pre-image of K step space of ${}_3(m^{I(F)})(M,p)$. But $\langle Y_{nlk} \rangle \notin {}_3(m^{I(F)})(M,p).$

Therefore ${}_3(m^{I(F)})(M,p)$ is not monotone and hence it is not normal. ■

Proposition 3.9- The sequence spaces ${}_3(m^{I(F)})(M,p)$ *and* ${}_3(m_0^{I(F)})(M,p)$ are *not symmetric.*

Proof. To prove the result, we cite a counter example.

Example.3.3- Let $I_3(\rho) \subset 2^{N \times N \times N}$ denote the class of all subsets of $N \times N \times N$ of zero natural density.

Let $I_3 = I_3(\rho),\ M(x) = x^2$ and $p_{nlk} = \begin{cases} 1, \ for \ n \ \ even \ \ and \ \ all \ \ l,k \in N \\ 2, \ otherwise \end{cases}$

Consider the sequence $\langle X_{nlk} \rangle$ defined by:

For $n = i^2,\ i \in N$ and for all $l,k \in N,$

$$X_{nlk}(t) = \begin{cases} 1 + \dfrac{t}{3\sqrt{n} - 2}, & for \quad 2 - 3\sqrt{n} \le t \le 0 \\ 1 - \dfrac{t}{3\sqrt{n} - 2}, & for \quad 0 < t \le 3\sqrt{n} - 2 \\ 0, & otherwise \end{cases}$$

otherwise $X_{nlk} = \bar{0}.$

Then $\langle X_{nlk} \rangle \in Z(M,p)$, for Z = ${}_3(m^{I(F)}),\ {}_3(m_0^{I(F)}).$

The rearrangement $\langle Y_{nlk} \rangle$ of $\langle X_{nlk} \rangle$ is defined as:

For k odd and for all $n, l \in N,$

$$Y_{nlk}(t) = \begin{cases} 1 + \dfrac{t}{3n-2}, & for \quad 2-3n \le t \le 0 \\ 1 - \dfrac{t}{3n-2}, & for \quad 0 < t \le 3n-2 \\ 0, & otherwise \end{cases}$$

otherwise $Y_{nlk} = \bar{0}.$

Then $\langle Y_{nlk} \rangle \notin Z(M,p),$ for Z = ${}_3(m^{I(F)}),\ {}_3(m_0^{I(F)}).$

Hence the sequence spaces ${}_3(m^{I(F)})(M,p)$ *and* ${}_3(m_0^{I(F)})(M,p)$ are not symmetric. ■

Proposition 3.10- The sequence spaces ${}_3(m^{I(F)})(M,p)$ *and* ${}_3(m_0^{I(F)})(M,p)$ are not convergence free.

Proof. The result follows from the following example.

Example 3.4- Let $I_3(\rho) \subset 2^{N \times N \times N}$ denote the class of all subsets of $N \times N \times N$ of zero natural density.

Let $I_3 = I_3(\rho),\ A \in I_3,\ M(x) = x$ and $p_{nlk} = \dfrac{1}{3}$ for all $n\ ,\ l,k \in N$

Define the sequence $\langle X_{nlk} \rangle$ by:

For all $(n,l,k) \notin A,$

$$X_{nlk}(t) = \begin{cases} 1 + 3(n+l+k)t, & for \quad -\dfrac{1}{3(n+l+k)} \le t \le 0 \\ 1 - 3(n+l+k)t, & for \quad 0 < t \le \dfrac{1}{3(n+l+k)} \\ 0, & otherwise \end{cases}$$

otherwise $X_{nlk} = \bar{0}.$

Then $\langle X_{nlk} \rangle \in Z(M,p),$ for Z = ${}_3(m^{I(F)}),\ {}_3(m_0^{I(F)}).$

Next the sequence $\langle Y_{nlk} \rangle$ is defined as:

For all $(n,l,k) \notin A,$

$$Y_{nlk}(t)=\begin{cases} 1+\dfrac{3t}{n+l+k}, & for \quad -\dfrac{(n+l+k)}{3}\le t\le 0 \\ 1-\dfrac{3t}{n+l+k}, & for \quad 0<t\le\dfrac{(n+l+k)}{3} \\ 0, & otherwise \end{cases}$$

otherwise $X_{nlk}=\overline{0}.$

Then $\langle Y_{nlk}\rangle \notin Z(M,p)$, for Z = ${}_3(m^{I(F)})$, ${}_3(m_0^{I(F)})$.

Hence ${}_3m^{I(F)}(M,p)$ *and* ${}_3m_0^{I(F)}(M,p)$ are not convergence free. ■

Theorem 3.11- The sequence spaces ${}_3m^{I(F)}(M,p)$ *and* ${}_3m_0^{I(F)}(M,p)$ *are sequence algebras.*

Proof. We prove the result for the space ${}_3(m_0^{I(F)})(M,p)$.

Let $\langle X_{nlk}\rangle,\langle Y_{nlk}\rangle \in {}_3(m_0^{I(F)})(M,p)$ and $0<\varepsilon<1$.

Then from the following inclusion relation, the result follows:

$$\left\{(n\ ,\ l,k)\in N\times N\times N : M\left[\frac{\overline{d}(X_{nlk}\otimes Y_{nkl}\ ,\ \overline{0})}{\rho}\right]^{p_{nlk}}<\varepsilon\right\}$$

$$\supset\ \left\{(n\ ,\ l,k)\in N\times N\times N : M\left[\frac{\overline{d}(X_{nlk}\ ,\ \overline{0})}{\rho}\right]^{p_{nlk}}<\varepsilon\right\}\cap\left\{(n\ ,\ l,k)\in N\times N\times N : \left[\frac{\overline{d}(Y_{nlk}\ ,\ \overline{0})}{\rho}\right]^{p_{nlk}}<\varepsilon\right\}.$$

Similarly the result for the space ${}_3(m^{I(F)})(M,p)$ can be established. ■

4. CONCLUSION

Convergence theory can be applied as a basic tool in measure spaces, sequences of random variables, information theory etc. In this research paper, the notion of ideal convergent triple sequence spaces of fuzzy numbers defined by an Orlicz function is introduced and studied. Few basic algebraic and topological properties of the introduced sequence spaces are studied and also some inclusion relations between these spaces are established. The introduced notion can be applied for further investigations from different aspects.

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