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Design Of LQR Based Stabilizer For Rotary Inverted Pendulum System

M. Siva Kumar* B. Dasu and G. Ramesh*****

Abstract : This paper presents the design of optimal controller for nonlinear Rotary Inverted Pendulum (RIP) dynamic system using Linear Quadratic Regulator (LQR). LQR, an optimal control technique is generally used for control of the linear dynamical systems, have been used in this paper to control the non linear dynamical system. The non linear system states are fed to LQR which is designed using linear state-space model. Inverted pendulum, a highly nonlinear unstable system is used as a benchmark for implementing the control methods. Here the controller objective is to control the system such that the arm reaches at a desired position and the inverted pendulum stabilizes in upright position. The MATLAB-SIMULINK model has been developed for implementation of control schemes. The same controllers have been tested on a test bed of Quanser QUBE-Servo hardware system and the results are compared in various aspects to verify the efficiency of the proposed controller.

Keywords : LQR, RIP system

1. INTRODUCTION

A typical unstable non-linear Rotary Inverted Pendulum (RIP) system is often used as a benchmark to study various control techniques in control engineering. Analysis of controllers on RIP system illustrates the analysis in cases such as control of a space booster rocket and a satellite, an automatic aircraft landing system, aircraft stabilization in the turbulent air-flow, stabilization of a cabin in a ship etc. RIP system is a test bed for the study of various controllers like PID controller LQR controller and fuzzy controller. A normal pendulum is stable when hanging downwards, an inverted pendulum is inherently unstable, and must be actively balanced in order to remain up right, this can be do neither by applying a torque at the pivot point, or by moving the pivot point horizontally as part of a feedback system.

Early studies of the inverted pendulum system was motivated by the need to design controllers to balance rockets during vertical take-off. At the instance of time during launch, the rocket is extremely unstable. Similar to the rocket launch, the inverted pendulum requires a continuous correction mechanism to stay upright, since the system with unstable in open loop configuration. This problem can be compared to the rocket during launch. Here, rocket boosters have to be fired in a controlled manner to maintain the rocket upright. The Linear Inverted Pendulum is widely in use and it has diversity of applications [1][2][3][4][5]. But in case of the Rotated Inverted Pendulum, there is ongoing research in this aspect and new applications are being invented and some of the old techniques are modified by the advent in the control of the RIP System. One such application is crane, which is used in the construction purposes. The rotary inverted pendulum (RIP) is a pendulum with its centre of gravity over its axis of rotation. The normal

* Gudlavalleru Engineering college, Gudlavalleru, A.P., India, 521356 Email: profsivakumar.m@gmail.com,
** Gudlavalleru Engineering college, Gudlavalleru, A.P., India, 521356 Email: dasu.geceee@gmail.com
*** Gudlavalleru Engineering college, Gudlavalleru, A.P., India, 521356 Email: ganta.ramesh25@gmail.com

pendulum has its centre of gravity under its axis of rotation and therefore, it is in stable state when it directs downwards. The rotary inverted pendulum is in unstable state because its centre of gravity is over its axis of rotation. A raised problem is how it is necessary to control the rotary inverted pendulum so that it can keep its equilibrium state when it directs upwards.

It is a challenge for a control engineer to design a controller for a RIP system, because the control of angular velocity is much more difficult than that of a linear moving object. It is further more difficult to control the pendulum such that it rotates to a certain degree and stop at that position and be balanced. A model of a Rotary Inverted Pendulum is shown in the following figure.In this paper LQR based optimal controller is developed that keep the pendulum upright without any oscillations. The model is simulated using the MATLAB software. The paper is organized as follows. Section 2 deals with the modeling of the system, Section 3 discusses the control technique LQR, Section 4 gives the test bed results, Section 5discusses the conclusion drawn from the analysis of these controllers in simulink and on test bed.

2. MODELING OF ROTARY INVERTED PENDULUM

The system, as shown in Fig. 1, consists of a vertical pendulum, a horizontal arm, a gear chain, and a servomotor which drives the pendulum through the gear transmission system. The rotating arm is mounted on the output gear of the gear chain. An encoder is attached to the arm shaft to measure the rotating angle of the arm. At the end of the rotating arm there is a hinge instrumented with an encoder. The pendulum is attached to the hinge.

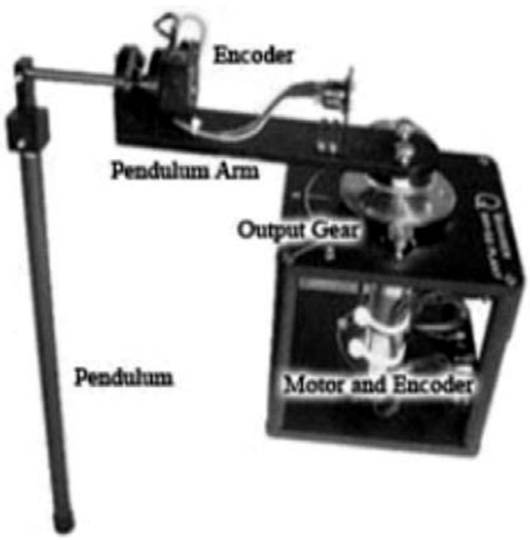

Fig. 1. Rotary Inverted Pendulum system.

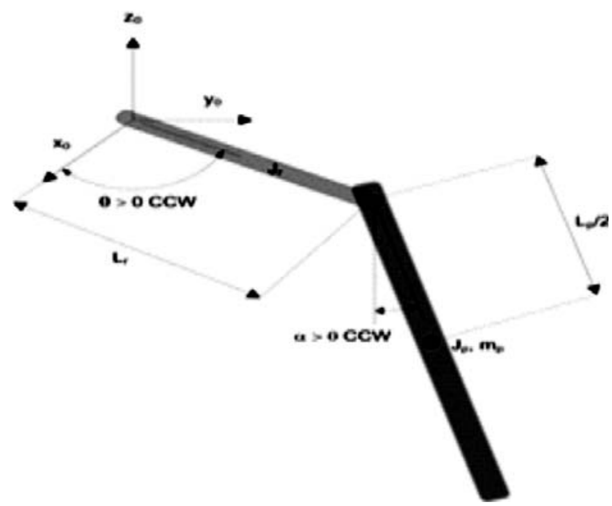

Fig. 2. Rotary inverted pendulum model.

The Rotary Inverted Pendulum (RIP) model is shown the below figure 2. The rotary arm pivot is attached to the QUBE-Servo base and is actuated. The arm has a length L_r, a moment of inertia of J_r, and its angle θ, increases positively when it is rotated in counter-clockwise direction (CCW). The servo and the arm should turn CCW direction when the control voltage is positive.

The pendulum link is connected to the end of the rotary arm. It has a total length of L_p and it centre of mass is $\frac{L_p}{2}$. The moment of inertia about its centre of mass is J_p. The inverted pendulum angle α is zero when it is perfectly upright in the vertical position and increases positively when rotated in CCW. The equations of motion (EOM) for the pendulum system were developed using the Euler-Lagrange method. This systematic method is often used to model complicated systems such as robot manipulators with multiple joints. The total kinetic and potential energy of the system is obtained, and then the Lagrangian can be found. A number of derivatives are then computed to yield the EOMs [13].

More specifically, the equations that describe the motion of the rotary arm and the pendulum with respect to the servo motor voltage will be obtained using Euler-Lagrange equation $\frac{\partial^2 L}{\partial t \partial \dot{q}_i} - \frac{\partial L}{\partial \dot{q}_i} = Q_i$

The variables q_i are called the *generalized coordinates.* For the system let $q(t)T = [\theta(t)\ a\ (t) =]$ (1.1)
Where $\theta(t)$ is the rotary arm angle and $\alpha(t)$ is the inverted pendulum angle.
The Euler-Lagrangian equations of the rotary pendulum are

$$\frac{\partial^2 L}{\partial t \partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = Q_1,$$

$$\frac{\partial^2 L}{\partial t \partial \dot{\alpha}} - \frac{\partial L}{\partial \alpha} = Q_2$$

The Lagrangian of the system is described by

$$L = T - V$$

Where, T is the total kinetic energy of the system and V is the total potential energy of the system.

The generalized forces Q_i are used to describe the non-conservative forces (*e.g.*, friction) applied on the system with respect to the generalized coordinates. In this case the generalized force acting on the rotary arm is

$$Q_1 = \tau - D_r \dot{\theta}$$

And acting on the pendulum is $Q_2 = -D_p \dot{\alpha}$

The total potential energy of the system is

$$V = mg \frac{L_p}{2} \cos(\alpha)$$

And the total kinetic energy of the system is $T = \frac{1}{2}(J_r + m_p L_r^2)\dot{\theta}^2 + \frac{2}{3} m_p L_p^2 \dot{\alpha}^2 - m_p L_p L_r \cos(\alpha)\dot{\theta}\dot{\alpha}$

Solving the above two equations for the Lagrangian and the derivatives, the EOM of the system are

$$\left(m_p L_r^2 + \frac{1}{4} m_p L_p^2 - \frac{1}{4} m_p L_p^2 \cos^2(\alpha) + J_r\right)\ddot{\theta} - \left(\frac{1}{2} m_p L_r \cos(\alpha)\right)\ddot{\alpha}$$

$$+\left(\frac{1}{2} m_p L_p^2 \sin(\alpha)\right)\ddot{\theta}\,\dot{\alpha} + \left(\frac{1}{2} m_p L_p L_r \sin(\alpha)\right)\dot{\alpha}^2 = \tau - D_r \dot{\theta} \tag{1.2}$$

$$\frac{1}{2} m_p L_r L_p \cos(\alpha)\ddot{\theta} + \left(J_p + \frac{1}{4} m_p L_p^2\right)\ddot{\alpha} - \frac{1}{4} m_p L_p^2 \cos(\alpha)\,\dot{\theta}^2 + \frac{1}{2} m_p L_p g \sin(\alpha)$$

$$= -D_p \dot{\alpha} \tag{1.3}$$

The torque applied at the base of the rotary arm is described as

$$\tau = \frac{\eta_g \eta_m k_g k_t k_m (V_m - k_m \dot{\theta})}{R_m} \tag{1.4}$$

When the nonlinear equations are linearized about the operating point $[\theta, \alpha]=[0,0]$, the resultant EOM of the inverted pendulum are defined as:

$$(m_p L_r^2 + J_r)\ddot{\theta} - \frac{1}{2} m_p L_p L_r \ddot{\alpha} = \tau - D_r \ddot{\theta}$$

And
$$\frac{1}{2} m_p L_r L_p \ddot{\theta} + \left(J_p + \frac{1}{4} m_p L_p^2\right)\ddot{\alpha} + \frac{1}{2} m_p L_p g \alpha = -D_p \dot{\alpha} \tag{1.5}$$

Solving the above equations for the acceleration terms yields

$$\ddot{\theta} = \frac{1}{J^T}\left\{-\left(J_p + \frac{1}{4} m_p L_p^2\right) D_r \ddot{\theta} + \frac{1}{2} m_p L_r D_p \ddot{\alpha} + \frac{1}{4} m_p^2 L_r g \alpha + \left(J_p + \frac{1}{4} m_p L_p^2\right)\tau\right\} \tag{1.6}$$

$$\ddot{\alpha} = \frac{1}{J^T}\left\{\frac{1}{2} m_p L_p L_r D_r \dot{\theta} - (J_r + m_p L_r^2) D_p \dot{\alpha} - \frac{1}{2} m_p L_p g (J_r + m_p L_r^2)\alpha - \frac{1}{2} m_p L_p L_T \tau\right\} \tag{1.7}$$

Where
$$J_T = J_p m_p L_r^2 + J_r J_p + \frac{1}{4} J_r m_p L_p^2 \tag{1.8}$$

The linear state-space equations are

$$\dot{x} = Ax + Bu \tag{1.9}$$

$$y = Cx + Du \tag{1.10}$$

Where x is the state, u is the control input. A, B, C, D are the state-space matrices. For the rotary inverted pendulum system, the state and output equations are defined as

$$x^T = [\theta\ \alpha\ \dot{\theta}\ \dot{\alpha}] \tag{1.11}$$

$$y^T = [x_1\ x_2\ ] \tag{1.12}$$

Substitute states into the equations of motion (in Equation 1.16 and 1.17).

$$\dot{x}_3 = \frac{1}{J_T}\left\{-\left(J_p + \frac{1}{4} m_p L_p^2\right) D_r x_3 + \frac{1}{2} m_p L_p L_r D_p x_4 \frac{1}{4} m_p^2 L_p^2 L_r g x_2\right.$$

$$= \left. + \left(J_p + \frac{1}{4} m_p L_p^2\right) u\right\} \tag{1.13}$$

$$\dot{x}_4 = \frac{1}{J^T}\left\{\frac{1}{2} m_p L_p L_r D_r x_3 - (J_r + m_p L_r^2) D_p x_4 - \frac{1}{4} m_p L_p g (J_r + m_p L_r^2) x_2 - \frac{1}{2} m_p L_p L_r u\right\} \tag{1.14}$$

Considering actuator dynamics given in equation 1.4, A and B matrices will be modified as

$$A = \frac{1}{J_T}\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{1}{4} m_p^2 L_p^2 L_r g & -\left(J_p + \frac{1}{4} m_p L_p^2\right) D_r - \left(\frac{K_t K_t}{R_m B(3,1)}\right) & \frac{1}{2} m_p L_p L_r D_p \\ 0 & -\frac{1}{2} m_p L_p g (J^r + m_p L_r^2) & \frac{1}{2} m_p L_p L_r D_r - \left(\frac{K_t K_t}{R_m B(4,1)}\right) & -(J_r + m_p L_r^2) D_p \end{bmatrix}$$

$$B = K_t * B/R_m$$

In the output equation, only the positions of the servo and link angles are being measured. Based on this, the C and D matrices in the output equation are $C = \begin{bmatrix} 1\ 0\ 0\ 0 \\ 0\ 1\ 0\ 0 \end{bmatrix}$ And $D = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

The model parameters are shown in below table.

Table 1. Rotary Inverted pendulum Parameters

$R_m = 8.4$	Resistance
$k_t = 0.042$	Current-torque (N-m/A)
$k^m = 0.042$	Back-emf constant (V-s/rad)
$M_r = 0.095$	Mass (kg)
$L_r = 0.085$	Total length (m)
$J_r = 5.7198e\text{-}05$	Moment of inertia about pivot (kg-m^2)
$B_r = 0.0015$	Equivalent Viscous Damping Coefficient (N-m-s/rad)
$M_p = 0.024$	Mass (kg)
$L_p = 0.129$	Total length (m)
$J_p = 3.3282e\text{-}05$	Moment of inertia about pivot (kg-m^2)
$B_p = 0.0005$	Equivalent Viscous Damping Coefficient (N-m-s/rad)

3. PROPOSED CONTROLLER DESIGN

Assuming the pendulum is almost upright, a state feedback controller can be implemented that would maintain it upright (and handle disturbances up to a certain point). The state feedback controller is designed using the linear quadratic regulator and the linear model of the system. The Linear Quadratic Regulator (LQR) theory is a powerful method for the control of linear systems in the state-space domain. The LQR technique generates controllers with guaranteed closed loop stability robustness property even in the face of certain gain and phase variation at the plant input/output. In addition, the LQR-based controllers provide reliable closed-loop system performance despite of stochastic plant disturbance. The LQ control design framework is applicable to the class of stabilizable linear systems.

Briefly, the LQR theory says that, given a n^{th} order stabilizable system $\dot{x}(t) = \mathrm{A}x(t) + \mathrm{B}u(t)$, $t \geq 0, x(0) = x_0$ where $x(t) \in \mathrm{R}^n$ is the state vector and $u(t) \in \mathrm{R}^m$ is the input vector, determine the matrix gain $\mathrm{K} \in \mathrm{R}^{n \times m}$ such that the static, full-state feedback control law $u(t) = -\mathrm{K}x(t)$ satisfies the following criteria

1. The closed-loop system is asymptotically stable and
2. The quadratic performance functional

$$\mathrm{J} = \int_0^\infty (x(t))^{\mathrm{T}} \mathrm{Q}(x(t)) + u(t)^{\mathrm{T}} \mathrm{R} u(t) dt$$

is minimized. Q is a nonnegative-definite matrix that penalizes the departure of system states from the equilibrium and R is a positive-definite matrix that penalizes the control input. The solution of the LQR problem can be obtained via a Lagrange multiplier-based optimization technique and is given by

$$\mathrm{K} = \mathrm{R}^{-1} \mathrm{B}^{\mathrm{T}} \mathrm{P}$$

where $\mathrm{P} \in \mathrm{R}^{n \times n}$ is a nonnegative-definite matrix satisfying the matrix Riccati equation

$$\mathrm{A}^{\mathrm{T}} \mathrm{P} + \mathrm{PA} + \mathrm{Q} - \mathrm{PBR}^{-1} \mathrm{B}^{\mathrm{T}} \mathrm{P} = 0$$

Note that it follows that the LQR-based control design requires the availability of all state variables for feedback purpose. In our case the state vector x is defined as $x = [\theta\ \alpha\ \dot{\theta}\ \dot{\alpha}]^{\mathrm{T}}$ and quadratic performance functional as

$$J = \int_0^{\infty} (x_{ref} - x(t))^{T} Q(r_{ref} - x(t)) + u(t)^{T} R u(t) dt$$

The matrices Q and R hold the penalties on the deviations of the state variables from their set-point and the control actions, respectively. When an element of Q is increased, therefore, the cost function increases the penalty associated with any deviations from the desired set-point of that state variable, and thus the specific control gain will be larger. When the values of the R matrix are increased, a larger penalty is applied to the aggressiveness of the control action and the control gains are uniformly decreased.Since there is only one control variable, R is a scalar. The reference signal x_{ref} is set to $[\theta_r \ 0\ 0\ 0]$, and the control strategy used to minimize cost function J is thus given by

$$u = K(x_{ref} - x) = k_1(\theta_r - \theta) - k_2\alpha - k_3\dot{\theta} - k_4\dot{\alpha}$$

This control law is a state-feedback control and is illustrated in the above figure 3. For our system, the pivot arm angle θ and the pendulum angular position α are measured by two encoders. The pivot arm angular velocity $\dot{\theta}$ and pendulum angular velocity $\dot{\alpha}$ are not measured by any physical sensor, instead, we numerically compute $\dot{\theta}$ and $\dot{\alpha}$ by implementing a low-pass differentiator, $e.g. \frac{50s}{s+50}$, as a part of the overall control scheme.

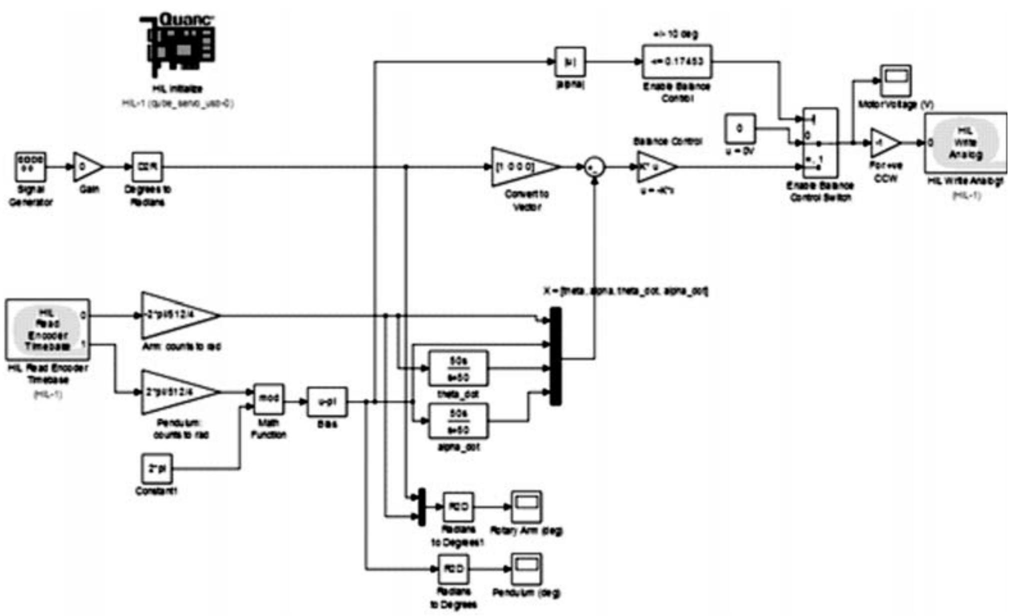

Fig. 3.

4. TEST BED RESULTS AND DISCUSSIONS

LQR is a method in modern control theory that uses state-space approach to analyze a system. Using state-space methods it is relatively simple to work with a multi-output system. LQR is a control scheme that provides the best possible performance with respect to some given measure of performance. The LQR controller is designed using MATLAB. First the value of the vector K that provides the feedback control law is determined using R = 1 and Q = diag ([1 1 1 1]). Consequently matrix K is obtained as K = [–1.0, –34.2418, –1.2254, 3.0770]. The steady-state values of the states are first computed and then multiplied by a chosen gain K to provide a new value as the reference for computing the input. By using these new values of states, controlled response of Rotary Inverted pendulum system is obtained. We give e square waveform reference of 20 degrees and we obtain the behavior presented in Figures.

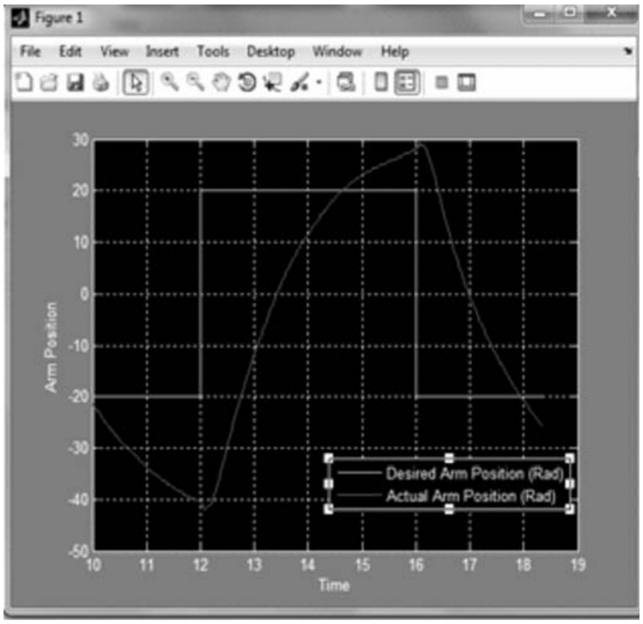

Fig. 4. Rotary Arm Position.

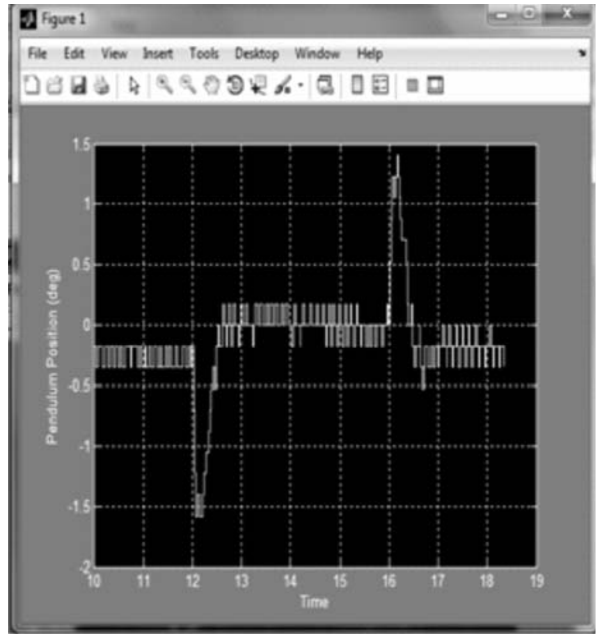

Fig. 5. Pendulum Position.

We have a great steady-state error and dynamic response is not good. Increasing the weight for the rotary arm error, we obtain the results from Fig. 5. : Q = diag([20 10 1 5]), R = 1, K = [–4.4721, 50.4191, –2.1203, 4.9542]

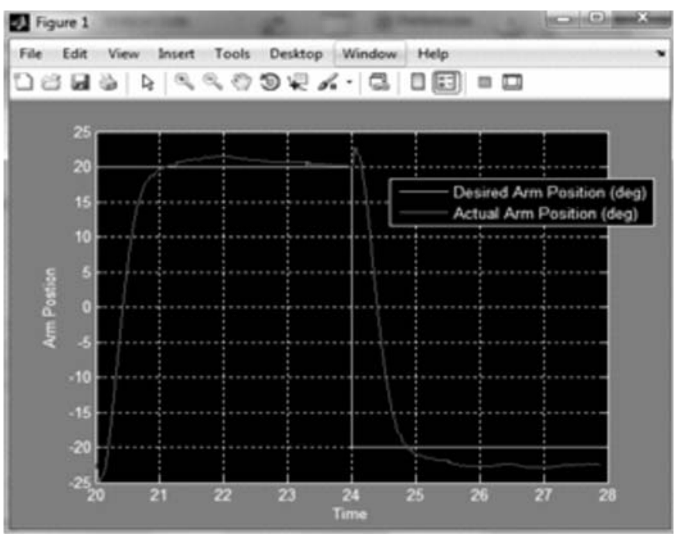

Fig. 6. Rotary Arm Position.

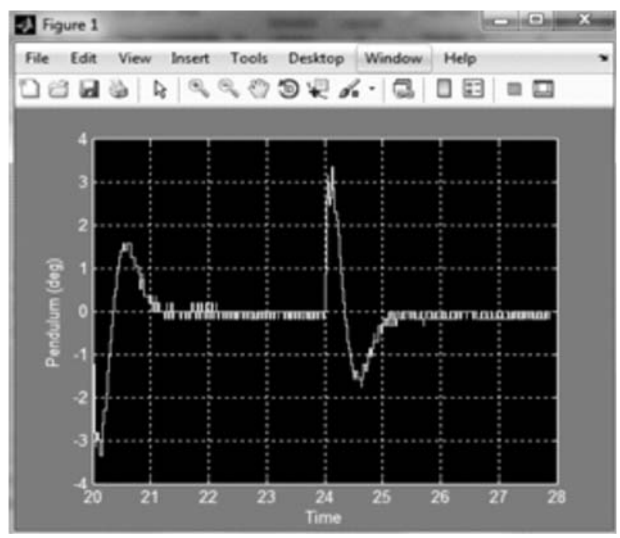

Fig. 7. Pendulum Position.

The steady-state error is eliminated. It is very important to obtain an accurate model of the rotary inverted pendulum system. With this model we are able to simulate the evolution of the control architecture and to tune the feedback parameters in order to obtain better performances. Another testing signal has been applied with sinusoidal references. The following results are obtained to a sinusoidal input of 0.04 Hz with amplitude of 20 degrees.

Here, controller is activated as soon as pendulum reaches to the reference position and pendulum is stabilized. The Quanser QUBE-Servo hardware system of rotary inverted pendulum consists of a DC servo motor, an angular sensor, a pendulum bar and a rotary arm. The Hardware set up of rotary inverted pendulum shown here.

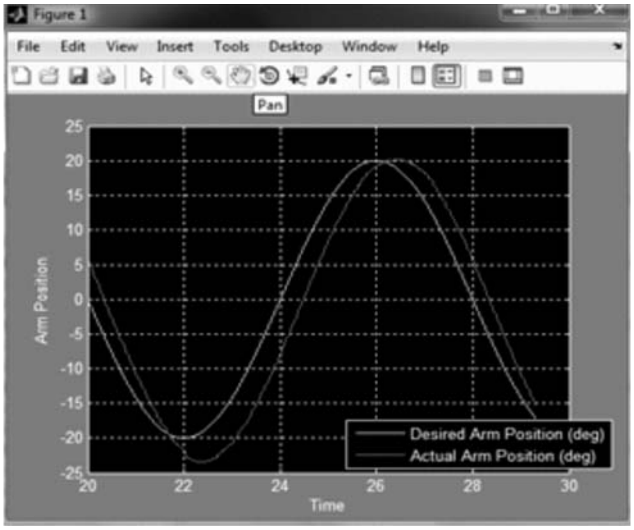

Fig. 8. Rotary Arm Position.

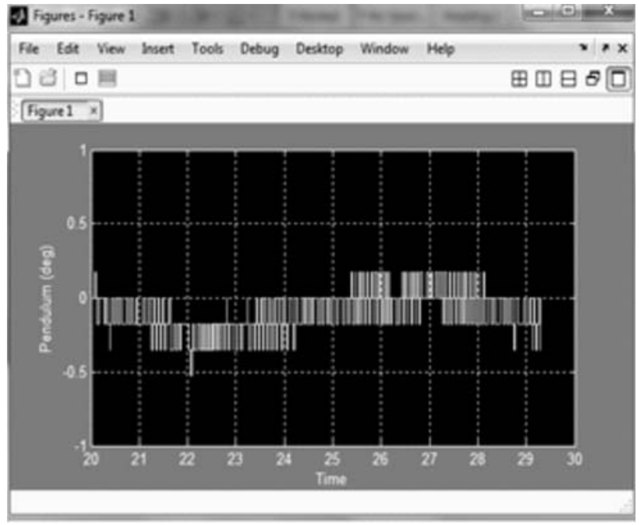

Fig. 9. Pendulum Position.

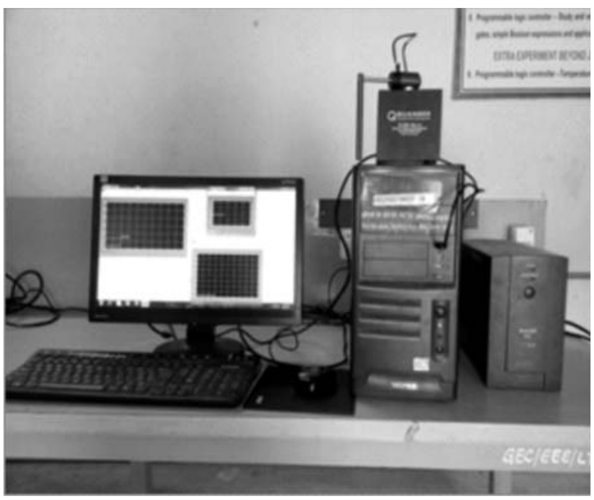

Fig. 10. Quanser QUBE-Servo hardware system

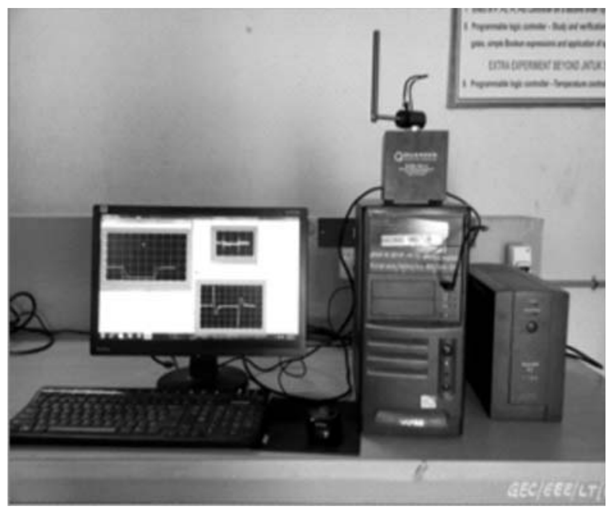

Fig. 11. Quanser QUBE-Servo hardware system with LQR controller

Fig.9 shows an unexcited (without controller) rotary inverted pendulum. As system is unexcited, pendulum is in downward position. Fig. 10 describes the response of pendulum angle after applying the LQR controller.

5. CONCLUSION

Design of an optimal control technique scheme (LQR) has been implemented to control the non linear rotary inverted pendulum system. In the optimal control of non linear inverted pendulum dynamical system using LQR approach, all the instantaneous states of the nonlinear system, are considered to be available for measurement, which are directly fed to the LQR. The LQR is designed using the linear state-space model of the system. The MATLAB-SIMULINK models have been developed for simulation of the control schemes. And the same are tested on the test bed of Quanser QUBE-Servo hardware system and the results are presented. The pendulum stabilizesing upright position justify that the control scheme is effective.

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