Communications on Stochastic Analysis
Vol. 14, No. 3-4 (2020)

THE INVERSE PROBLEM TO THE QUESTION OF THE EXISTENCE OF INTERPOLATING MARTINGALE MEASURES

IGOR PAVLOV, ANZHELIKA DANEKYANTS, NATALIA NEUMERZHITSKAIA

Abstract. In this paper, we formulate inverse problems that naturally arise when interpolating financial markets. Within the framework of this topic, one of the most important is the question of the existence of interpolation martingale measures, introduced into consideration by the first co-author in the 2000s. For static financial markets defined on finite or countable probability spaces, the inverse problem is formulated as follows: for a predetermined probability measure P and an initial condition $a > 0$, prove the existence of a stock whose price at the initial moment coincides with a, and the measure P for the price process is an interpolation martingale measure. It is shown that this is true for finite probability spaces. For countable probability spaces, sufficient conditions are found for this statement to hold.

1. Introduction

Let us consider on a finite or countable set Ω a static real-valued stochastic process (s. p.) $Z = (Z_0, Z_1)$, where $Z_0 = a = \text{const}$, and all the different values of the random variable (r. v.) Z_1 are denoted by b_k. Denote $B_k = \{\omega\colon Z_1(\omega) = b_k\}$, $1 \le k < r+1$ (here r can be both finite and infinite).

We say that b_k is of the order m_k, $1 \le m_k \le \infty$, if r. v. Z_1 takes this value m_k times. It means that for any k $B_k = \bigcup_{i=1}^{m_k} \omega_k^i$, where $\omega_k^i \in \Omega$ and $Z_1(\omega_k^i) := b_k^i = b_k$.

Denote by $\mathcal{F}_0$ the trivial σ-field $\{\Omega, \emptyset\}$, and by $\mathcal{F}_1$ the set of all subsets of Ω. It is clear that s. p. Z is adapted to the one-step filtration $\mathbf{F} = (\mathcal{F}_0, \mathcal{F}_1)$. Denote by $\mathcal{P}$ the set of all non degenerate probability measures on $\mathcal{F}_1$, i.e. $P \in \mathcal{P}$ iff $p_k^i := P(\omega_k^i) > 0$ for all $1 \le k < r+1$, $1 \le i < m_k + 1$. Denote $p_k = \sum_{i=1}^{m_k} p_k^i$ $(1 \le k < r+1)$ and introduce the set $\mathcal{P}(Z, \mathbf{F})$ of all non degenerate martingale measures (m. m.) P of the process Z. It is obvious that any $P = (p_k^i) \in \mathcal{P}(Z, \mathbf{F})$ satisfies the system:

Date: Date of Submission June 02, 2020 ; Date of Acceptance July 25, 2020 , Communicated by Yuri E. Gliklikh .

2010 *Mathematics Subject Classification.* Primary 60G42; Secondary 60H30.

Key words and phrases. Martingale measure, special interpolating martingale measures, non-coincidence barycenter condition, arbitrage-free markets, complete markets.

This research is supported by the RFBR (project 19-01-00451).

$$\begin{cases} \sum_{k=1}^{r} p_k = 1 \\ \sum_{k=1}^{r} |b_k| p_k < \infty \\ \sum_{k=1}^{r} b_k p_k = a \\ p_k > 0, \ 1 \le k < r+1. \end{cases} \tag{1.1}$$

Conversely, if $(p_k)_{k=1}^r$ is a solution of the system (1.1), then writing arbitrarily for any k a representation $p_k = \sum_{i=1}^{m_k} p_k^i$ $(1 \le k < r+1)$, where each term of the sum is strictly positive, we obtain a measure $P = (p_k^i) \in \mathcal{P}(Z, \mathbf{F})$. Since $r > 1$, then it is easy to see that the resolvability of the system (1.1) is equivalent to the fulfilment of the condition

$$\inf_k b_k < a < \sup_k b_k. \tag{1.2}$$

In what follows, we will deal with special martingale measures, such that the prices of contingent claims calculated with their help are the most fair. We denote by $|M|$ the cardinality of the set M.

Definition 1.1. We say that a probability measure $P = (p_k^i) \in \mathcal{P}(Z, \mathbf{F})$ satisfies noncoincidence barycenter condition ($P \in NBC$) if for any two subsets $I, J \subset \{(k,i), 1 \le k < r+1, 1 \le i < m_k + 1\}$ such that $I \cap J = \emptyset$ the following inequalities hold:

$$\frac{\sum_I b_k p_k^i}{\sum_I p_k^i} \ne \frac{\sum_J b_k p_k^i}{\sum_J p_k^i}. \tag{1.3}$$

If the inequality (1.3) holds only for such I that $|I| = 1$ and for J with finite complementation J^c (in the set of all duble indexes $\{(k,i), 1 \le k < r+1, 1 \le i < m_k + 1\}$), then we say that P satisfies weakened noncoincidence barycenter condition ($P \in WNBC$).

Remark 1.2. It is easy to see that: 1) $NBC \subset WNBC$; 2) if $|\Omega| < \infty$ and $WNBC \ne \emptyset$ or $|\Omega| = \infty$ and $NBC \ne \emptyset$, then $b_k \ne a$ and $m_k = 1$ $(1 \le k < r+1)$; 3) if $|\Omega| = \infty$ and $WNBC \ne \emptyset$, then $b_k \ne a$ $(1 \le k < r+1)$; 4) if $|\Omega| = \infty$, $WNBC \ne \emptyset$, and $r < \infty$, then among the numbers $b_1, \dots, b_r$ at least 2 have infinite order.

The sets NBC and $WNBC$ play an important role in the theory of Haar interpolations of financial markets. In arbitrage-free incomplete markets using measures $P \in \mathcal{SP}(Z, \mathbf{F})$ we obtain more fair prices of various contingent claims. That is why the question if whether NBC or $WNBC$ is not empty (direct problem) is significant.

2. Invers problem in the case of finite Ω

In this section we suppose that $|\Omega| < \infty$.

A filtration $\mathbf{H} = (\mathcal{H}_n)_{n=0}^{L}$ is called Haar filtration (HF) on Ω (c.f. [1]) if σ-field $\mathcal{H}_n$ is generated by a partition of Ω into exactly $n+1$ atoms. A HF $\mathbf{H}$ is called interpolating Haar filtration (IHF) of the filtration $\mathbf{F}$ (c.f. [2]–[3]) if $\mathcal{H}_0 = \mathcal{F}_0$ and $\mathcal{H}_L = \mathcal{F}_1$. It is obvious that $L = |\Omega| - 1$.

Let $P \in \mathcal{P}(Z, \mathbf{F})$ ($\Leftrightarrow$ $Z = (Z_k, \mathcal{F}_k, P)_{k=0}^{1}$ is a martingale) and consider $Y_n := E^P[Z_1|\mathcal{H}_n]$. Then the process $Y = (Y_n, \mathcal{H}_n)_{n=0}^{L}$ is called a martingale Haar interpolation of Z (c.f. [2]–[3]). We say that $P \in \mathcal{P}(Z, \mathbf{F})$ satisfies universal Haar uniqueness property (UHUP) if for every IHF $\mathbf{H} = (\mathcal{H}_n)_{n=0}^{L}$ of $\mathbf{F}$ $|\mathcal{P}(Y, \mathbf{H})| = 1$ (i.e. Y is a martingale only with respect to the initial measure P).

The following result clarifies the meaning of the relation (1.3).

Proposition 2.1 (c.f. [2]–[3])**.** *Measure $P \in \mathcal{P}(Z, \boldsymbol{F})$ satisfies UHUP if and only if it satisfies the condition* (1.3).

With the help of Proposition 2.1 the following result was proved.

Theorem 2.2 (c.f. [2]–[4])**.** *$NBC \neq \emptyset \Leftrightarrow b_k \neq a$ and $m_k = 1$ $(1 \leq k \leq |\Omega|)$.*

Consider the following question: is it possible for an arbitrary non-degenerate probability measure and an arbitrary number $a > 0$ to construct a stock whose price takes at the initial moment the value a and satisfies UHUP?

We will use the following lemma.

Lemma 2.3. *If $p_1, p_2, \ldots, p_r$ are strictly positive numbers such that $\sum_{k=1}^{r} p_k = 1$ and $a > 0$, then there exists a strictly positive and strictly monotone sequence $b_1, b_2, \ldots, b_r$ such that $b_k \neq a$ $(1 \leq k \leq r)$ and $\sum_{k=1}^{r} b_k p_k = a$.*

Proof. The proof is trivial. □

Theorem 2.4. *Let $|\Omega| = r$, $P = (p_1, p_2, \ldots, p_r)$ be a non-degenerate probability measure on Ω and $a > 0$. Then there exists a stock with the price $Z = (Z_0, Z_1)$ such that $Z_0 = a$, the values $b_k \neq a$ $(1 \leq k \leq r)$ of Z_1 are strictly positive and strictly monotone, and in the obtained market the probability P satisfies UHUP.*

Proof. Define $B(b_1, b_2, \ldots, b_r)$ following Lemma 2.3. Therefore the point B lies on the hyperplane

$$\sum_{k=1}^{r} p_k x_k = a \tag{2.1}$$

of the space R^r. Taking into account inequalities (1.3) consider the following finite family of hyperplanes in R^r:

$$\frac{\sum\limits_{I} p_k x_k}{\sum\limits_{I} p_k} = \frac{\sum\limits_{J} p_k x_k}{\sum\limits_{J} p_k}, \tag{2.2}$$

where $I, J \subset \{1, 2, \ldots, r\}$ such that $I \cap J = \emptyset$. It is clear that the hyperplane (2.1) does not coinside with any of hyperplanes (2.2). Hence, if we denote by U the intersection of hyperplane (2.1) and the union of hyperplanes (2.2), then $\mu(U) = 0$, where μ is the Lebesgue measure on the of hyperplane (2.1).

Let us choose a number $\varepsilon > 0$ so small that the intervals $(b_k - \varepsilon, b_k + \varepsilon)$, $1 \leq k \leq r$, do not intersect in pairs. Denote by V the cartesian product of these intervals and $W := (\Pi \setminus U) \cap V$, where Π is the hyperplane (2.1). The set W is not empty since $\mu(W) > 0$. It is clear that any point of W satisfies the equation (2.1) and does not satisfy the equations (2.2). Q.E.D. □

3. Invers problem in the case of countable Ω

In this section we suppose that Ω is countable.

Instead of Haar filtration, we shall use in this section the notion of special Haar filtration. Recall that the basic property of Haar filtration is the following one: at every moment of time **only one atom** divides into two parts and other atoms do not change. Special Haar filtration is a particular case of Haar filtration. Namely, at every moment of time **only one of those two atoms** can be divided that were obtained by division at the previous moment [5].

Transform $P = \{p_k^i,\ 2 < k < r+1, 1 \le i < m_k + 1\}$ to a sequence $(q_1, q_2, \dots)$. For any permutation $\{k_1, \dots, k_n, \dots\}$ of $\{1, 2, \dots, n, \dots\}$ introduce interpolating special Haar filtration (SIHF) of $\mathbf{F} = (\mathcal{F}_{\mathbf{0}}, \mathcal{F}_{\mathbf{1}})$ in the manner:

$\mathcal{H}_0 = \mathcal{F}_0$,
$\mathcal{H}_1 = \sigma\{\omega_{k_1}\}$,
. . . .
$\mathcal{H}_n = \sigma\{\omega_{k_1}, \omega_{k_2}, \dots, \omega_{k_n}\}$,
. . . .
$\mathcal{H}_\infty = \mathcal{F}_1$.

Let $P \in \mathcal{P}(Z, \mathbf{F})$ and consider $Y_n := E^P[Z_1|\mathcal{H}_n]$. Then the process $Y = (Y_n, \mathcal{H}_n)_{n=0}^\infty$ is called a special martingale Haar interpolation of Z. We say that $P \in \mathcal{P}(Z, \mathbf{F})$ satisfies special Haar uniqueness property (SHUP) if for $\mathbf{F}$ and <u>*every*</u> SIHF $\mathbf{H} = (\mathcal{H}_n)_{n=0}^\infty$ $|\mathcal{P}(Y, \mathbf{H})| = \mathbf{1}$ (Y is a martingale <u>*only*</u> with respect to the initial measure P).

Proposition 3.1 (c.f. [6]–[7]). *Measure $P \in \mathcal{P}(Z, \boldsymbol{F})$ satisfies SHUP if and only if $P \in WNBC$.*

The proofs of various sufficient conditions for the inclusion $P \in WNBC$ to hold are presented in the articles [8]–[11].

As in the previous section consider the question: is it possible for an arbitrary non-degenerate probability measure and an arbitrary number $a > 0$ to construct a stock whose price takes at the initial moment the value a and satisfies SHUP? Taking into account points 3) and 4) of Remark 1.2, let us formulate the following theorem.

Theorem 3.2. *Let Ω be countable, $a > 0$, and P be a non-degenerate probability measure on Ω. Then for any $r > 1$ there exists a stock with the price $Z = (Z_0, Z_1)$ such that $Z_0 = a$ and:*

1) among the values of the r.v. Z_1 exactly r values (which we denote $b_1, \dots, b_r$) are different and strictly positive;

2) $b_k \ne a$ $(1 \le k \le r)$;

3) the numbers $b_1, \dots, b_s$, $2 \le s \le r$, have infinite order and the numbers $b_{s+1}, \dots, b_r$ have a finite order;

4) in the obtained market the probability P satisfies SHUP.

Proof. Denote $m_1 = \infty, \dots, b_s = \infty$, $2 \le s \le r$, and if $s < r$, define in an arbitrary way integers $1 \le m_{s+1} < \infty, \dots, 1 \le m_r < \infty$. Let us compose in an arbitrary way from the components of the vector P the groups of numbers: (p_k^i), $1 \le k \le r$, $1 \le i < m_k + 1$. Denote $p_k := \sum_{i=1}^{m_k} p_k^i$ $(1 \le k \le r)$. Applying Lemma 2.3, we find numbers $b_1, b_2, \dots, b_r$. Taking into account that (1.3) is composed of countable many inequalities we can replicate the reasonings of the Theorem 2.4. Q.E.D. □

Now consider the case $r = \infty$. Sufficient conditions for the inclusion $P \in WNBC$ to hold are presented in the articles [11]–[15]. In the rest of this section, we assume that $m_k = 1$, $1 \le k < \infty$.

Let L be a linear subspace of the linear space R of real numbers. A nonzero sequence $(r_1, r_2, \dots)$ is called L-finite if its components belong to L and just a finite number of them are nonzero. Given a sequence of real numbers $d = (d_1, d_2, \dots)$, we denote by $\mathcal{L}(d)$ the set of numbers of the form $\sum r_i d_i$, where $(r_1, r_2, \dots)$ runs over all L-finite sequences. If $L = Q$ (Q is the set of rational numbers), we denote $\mathcal{Q}(d) = \mathcal{L}(d)$, and if $L = A$ (A is the set of algebraic numbers), we denote $\mathcal{A}(d) = \mathcal{L}(d)$. We suppose that the market under consideration satisfies the condition: $b_k \in L$, $1 \le k < \infty$.

Lemma 3.3. *If $P = (p_1, p_2, \dots) \in \mathcal{P}(Z, \boldsymbol{F})$ and*

$$a \notin \mathcal{L}(P) + L, \tag{3.1}$$

then $P \in WNBC$.

Proof. The proof can be found in [11]. □

We say that: $P = (p_1, p_2, \dots)$ is rational (resp., algebraic) if all its components $p_1, p_2, \dots$ are rational (resp., algebraic); a market under consideration is ratoinal (resp., algebraic, transcendental) if all the numbers $b_1, b_2, \dots$ are rational (resp., algebraic, transcendental).

Proposition 3.4. *Let $P = (p_1, p_2, \dots)$ be rational (resp., algebraic) and $a > 0$ be irrational (resp., transcendental). Then any rational (resp., algebraic) strictly positive solution of the equation $\sum_{k=1}^{\infty} p_k x_k = a$ generates a market in which $P \in WNBC$.*

Proof. Let $x_k = b_k$, $1 \le k < \infty$, be such a solution (that obviously exists). Put $L = Q$ (resp., $L = A$), $d = P$. It is clear that the relation (3.1) is fulfilled. Hence $P \in WNBC$. □

Proposition 3.5. *Let $P = (p_1, p_2, \dots)$ be rational (resp., algebraic) and $a > 0$ be also rational (resp., algebraic). Then there exists an algebraic (resp., transcendental) market in which $P \in WNBC$.*

Proof. Fix a not rational algebraic (resp., transcendental) number $c > 0$. Consider the converging series $\sum_{k=1}^{\infty} cp_k$. Let us find rational numbers r_k such that $\sum_{k=1}^{\infty} cr_k p_k = a$. Denote $b_k = cr_k$. It is clear that P is a martingale measure of obtained market. Put $L = cQ$ (resp., $L = cA$), $d = P$. Then (3.1) transforms to the inequalities $a \neq cu$, where u takes on nonzero rational (resp., algebraic) values. Q.E.D. □

Remark 3.6. In the case $r = \infty$ only one theorem of the existence of $P \in NBC$ is known (c.f. [16]). The inverse problem has not been resolved either.

4. Applications

Let a hedger acting in the financial market believe that calculations of fair prices of contingent claims should be made using some probabilistic measure P. Suppose that he may describe all markets in which this measure is risk-neutral (martingale). Solving the inverse problem (that is, building the market according to such a predetermined measure) he can select those markets in which the martingale measure P allows to obtain the most fair prices. In view of the ability to interpolate these markets with complete ones, the hedger is able to form standard hedging portfolios of various contingent claims.

References

1. Neveu J.: *Discrete-Parameter Martingales*, North-Holland Publishing Company, New York, 1975.
2. Bogacheva M.N. and Pavlov I.V.: Haar extensions of arbitrage-free financial markets to markets that are complete and arbitrage-free, *Russian Math. Surveys* **57** (3) (2002) 581–583.
3. Bogacheva M.N. and Pavlov I.V.: Haar extensions of arbitrage-free financial markets to markets that are complete and arbitrage-free, *Izvestiya VUZov, Severo-Kavkaz. Region, Estestvenn. Nauki* (3) (2002) 16–24.
4. Pavlov I.V., Tsvetkova I.V. and Shamrayeva V.V.: Some results on martingale measures of static financial markets models relating noncoincidence barycenter condition, *Vestn. Rostov Gos. Univ. Putei Soobshcheniya* **45** (3) (2012) 177–181.
5. Pavlov I.V.: A conterexample to the conjecture that H^∞ is dense in BMO, *Theory Probab. Appl.* **25** (1) (1980) 152–154.
6. Danekyants A.G. and Pavlov, I.V.: On the weakened universal Haar uniqueness property, *Obozreniye prikladnoi i promishlennoi matematiki* **11** (3) (2004) 506–508.
7. Danekyants A.G.: On special Haar interpolations of martingales *Izvestiya VUZov, Severo-Kavkaz. Region, Estestvenn. Nauki, Addendum* (3) (2005) 3–20.
8. Pavlov I.V., Tsvetkova I.V. and Shamrayeva V.V.: On the existence of martingale measures satisfying weakened noncoincidence barycenter condition: constructivist approach, *Vestn. Rostov Gos. Univ. Putei Soobshcheniya* **47** (4) (2014) 132–138.
9. Shamraeva V.V.: New method for transforming systems of inequalities for finding interpolating martingale measures, *International Research Journal* **54** (12-5) (2016) 30–41.
10. Pavlov I.V., Tsvetkova I.V. and Shamrayeva V.V.: On the existence of martingale measures satisfying the weakened condition of noncoincidence of barycenters in the case of countable probability space, *Theory Probab. Appl.* **61** (1) (2017) 167–175.
11. Pavlov I.V., Danekyants A.G., Neumerzhitskaia N.V. and Tsvetkova I.V.: Interpolating martingale measures and interpolating deflators of one-step processes on countable probability spaces, *Global and Stochastic Analysis* **7** (1) (2020) 65–72.
12. Pavlov I.V. and Shamraeva V.V.: New results on the existence of interpolating and weakly interpolating martingale measures, *Russian Mathematical Surveys* **72** (4) (2017) 767-769.
13. Pavlov I.V.: New family of one-step processes admitting special interpolating martingale measures, *Global and Stochastic Analysis* **5** (2) (2018) 111–119.
14. Danekyants A.G. and Neumerzhitskaia N.V.: Generalization of a result on the existence of weakly interpolating martingale measures, *Theory Probab. Appl.* **64**, (1) (2019) 134–135.

15. Danekyants A.G. and Neumerzhitskaia N.V.: On rational and irrational interpolating martingale measures, *Theory Probab. Appl.* **65** (1) (2020) 127–128.
16. Shamraeva, V.V.: Some class ot the interpolating martingale measures on a countable probability space, *Global and Stochastic Analysis* **5**, N2 (2018) 119–124.

Igor V. Pavlov: Head of the Department of Higher Mathematics, Don State Technical University, Rostov-on-Don, 344001, Russia
E-mail address: pavloviv2005@mail.ru

Anzhelika G. Danekyants: Associate Professor of the Department of Higher Mathematics, Don State Technical University, Rostov-on-Don, 344001, Russia
E-mail address: dangegik@mail.ru

Natalia V. Neumerzhitskaia: Associate Professor of the Department of Higher Mathematics, Don State Technical University, Rostov-on-Don, 344001, Russia
E-mail address: neunata@yandex.ru