

ONE DIMENSIONAL COMPLEX ORNSTEIN-UHLENBECK OPERATOR

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ABSTRACT. We show that for any fixed $\theta \in (-\frac{\pi}{2}, 0) \cup (0, \frac{\pi}{2})$, the complex Ornstein-Uhlenbeck operator

$$\tilde{L}_\theta = 4 \cos \theta \frac{\partial^2}{\partial z \partial \bar{z}} - e^{i\theta} z \frac{\partial}{\partial z} - e^{-i\theta} \bar{z} \frac{\partial}{\partial \bar{z}},$$

is a normal but nonsymmetric diffusion operator.

1. Introduction

In [16], the authors show that for a continuous-time stationary L^2 -exponential ergodic Markov process, if its infinitesimal generator \mathcal{A} is a normal but non-symmetric operator then all its power spectrums of real-valued observables are monotonic on $[0, \infty)$ if and only if the eigenvalues of \mathcal{A} satisfy the so-called IR-ratio rule:

$$\frac{|\operatorname{Im} z|}{|\operatorname{Re} z|} \leq \frac{1}{\sqrt{3}}, \quad \forall z \in \sigma(\mathcal{A}) \setminus \{0\}.$$

Shortly after that, the authors apply the above IR-ratio rule to several finite state normal but non-symmetric Markov operators in [7].

However, few normal but non-symmetric diffusion operators are well-studied in the literature. For example, the following 2-dimensional Ornstein-Uhlenbeck process

$$\begin{bmatrix} dX_1(t) \\ dX_2(t) \end{bmatrix} = \begin{bmatrix} -\lambda & -\omega \\ \omega & -\lambda \end{bmatrix} \begin{bmatrix} X_1(t) \\ X_2(t) \end{bmatrix} dt + \sqrt{2a} \begin{bmatrix} dB_t^1 \\ dB_t^2 \end{bmatrix} \quad (1.1)$$

is used to model the Chandler wobble, i.e., the variation of latitude concerning with the rotation of the earth, by M. Arató, A.N. Kolmogorov and Ya.G. Sinai [3] (see also [1, 2]) and to model the motion of a charged test particle in the presence of a constant magnetic field in [4, pp. 181–186]. But as far as we know, many mathematical properties such as the normality of the generator of (1.1) are still not written down in the previous literature.

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For simplicity, we can assume that $\lambda + i\omega = e^{i\theta}$, $a = \cos \theta$ and then rewrite the generator as a complex-valued operator:

$$\tilde{\mathcal{L}}_\theta = 4 \cos \theta \frac{\partial^2}{\partial z \partial \bar{z}} - e^{i\theta} z \frac{\partial}{\partial z} - e^{-i\theta} \bar{z} \frac{\partial}{\partial \bar{z}}, \tag{1.2}$$

where $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$ is fixed and $\frac{\partial f}{\partial z} = \frac{1}{2}(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y})$, $\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y})$ are the Wirtinger derivatives of f at point $z = x + iy$ with $x, y \in \mathbb{R}$. In [8], it is shown that the eigenfunctions are the complex Hermite polynomials and form an orthonormal basis of $L^2(\gamma)$ where $d\gamma = \frac{1}{2\pi} e^{-\frac{x^2+y^2}{2}} dx dy$ (see Proposition 2.2 below).

In this paper, we will firstly show that $\tilde{\mathcal{L}}_\theta$ can be realized as an unbounded normal operator (see [19, p. 368]) in $L^2(\gamma)$ but nonsymmetric when $\theta \neq 0$. Secondly, we extend the known fact about the 1-dimensional real symmetric diffusion operator [6, 22, 23] to the complex case. Precisely stated, we present the explicit expression of $\tilde{\mathcal{L}}_\theta$ in $L^2(\gamma)$ (see Theorem 4.3) and show that it is a normal diffusion operator (see Theorem 4.4).

This article is organized as follows. Section 2 provides necessary information of complex Hermite polynomials. Section 3 contains the proof of the normality of the complex Ornstein-Uhlenbeck semigroup. Section 4 contains the main results on the explicit expression of $\tilde{\mathcal{L}}_\theta$ and the property of the normal diffusion operator. Finally, some necessary approximation of identity and N-representation theorem are listed in Appendix.

2. Preliminaries

Definition 2.1. (Definition of the complex Hermite polynomials [8, Def. 2.4]) We call $\partial := \frac{\partial}{\partial z}$ and $\bar{\partial} := \frac{\partial}{\partial \bar{z}}$ the complex annihilation operators. Let $m, n \in \mathbb{N}$. We define the sequence on \mathbb{C} (or say: \mathbb{R}^2)

$$J_{0,0}(z) = 1, \\ J_{m,n}(z) = \sqrt{\frac{2^{m+n}}{m!n!}} (\partial^*)^m (\bar{\partial}^*)^n 1,$$

where $(\partial^* \phi)(z) = -\frac{\partial}{\partial \bar{z}} \phi(z) + \frac{\bar{z}}{2} \phi(z)$, $(\bar{\partial}^* \phi)(z) = -\frac{\partial}{\partial z} \phi(z) + \frac{z}{2} \phi(z)$ for $\phi \in C^1_{\uparrow}(\mathbb{R}^2)$ (see Definition 5.1) are the adjoint of the operators $\partial, \bar{\partial}$ in $L^2(\gamma)$ respectively (the complex creation operator).

In [8, Theorem 2.7, Corollary 2.8], the authors show that $J_{m,n}(z)$ satisfies that:

Proposition 2.2. The complex Hermite polynomials $\{J_{m,n}(z) : m, n \in \mathbb{N}\}$ form an orthonormal basis of $L^2(\gamma)$ where $d\gamma = \frac{1}{2\pi} e^{-\frac{x^2+y^2}{2}} dx dy$. Thus, every function f in $L^2(\gamma)$ has a unique series expression

$$f = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} b_{m,n} J_{m,n}(z),$$

where the coefficients $b_{m,n}$ are given by

$$b_{m,n} = \langle f, J_{m,n} \rangle = \int_{\mathbb{R}^2} f \overline{J_{m,n}(z)} d\gamma.$$

Moreover, for any $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$ and each $m, n \in \mathbb{N}$,

$$\tilde{\mathcal{L}}_\theta J_{m,n}(z) = -[(m + n) \cos \theta + i(m - n) \sin \theta] J_{m,n}(z). \tag{2.1}$$

The real Hermite polynomials are defined by the formula¹

$$H_n(x) = \frac{(-1)^n}{\sqrt{n!}} e^{x^2/2} \frac{d^n}{dx^n} e^{-x^2/2}, \quad n = 1, 2, \dots$$

The following property gives the fundamental relation between the real and the complex Hermite polynomials [8, Corollary 2.8].

Proposition 2.3. *Let $z = x + iy$ with $x, y \in \mathbb{R}$. Both $\{J_{k,l}(z) : k + l = n\}$ and $\{H_k(x)H_l(y) : k + l = n\}$ generate the same linear subspace of $L^2(\gamma)$.*

3. The Normality of the Complex Ornstein-Uhlenbeck Semigroup

The Ornstein-Uhlenbeck process (1.1) can be rewritten as a complex-valued process:

$$\begin{cases} dZ_t = -e^{i\theta} Z_t dt + \sqrt{2 \cos \theta} d\zeta_t, & t \geq 0, \\ Z_0 = x \in \mathbb{C}, \end{cases} \tag{3.1}$$

where $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$, and $\zeta_t = B_1(t) + iB_2(t)$ is a complex Brownian motion. Solving for Z gives

$$Z_t = e^{-(\cos \theta + i \sin \theta)t} \left(x + \sqrt{2 \cos \theta} \int_0^t e^{(\cos \theta + i \sin \theta)s} d\zeta_s \right). \tag{3.2}$$

Thus, the associated Ornstein-Uhlenbeck semigroup of Eq.(3.1) has the following explicit representation, due to Kolmogorov, for each $\varphi \in C_b(\mathbb{R}^2)$ (the space of all continuous and bounded complex-valued functions on \mathbb{R}^2),

$$P_t \varphi(x) = E_x[\varphi(Z_t)] \tag{3.3}$$

$$= \frac{1}{2\pi(1 - e^{-2t \cos \theta})} \int_{\mathbb{R}^2} e^{-\frac{|y|^2}{2(1 - e^{-2t \cos \theta})}} \varphi(e^{-(\cos \theta + i \sin \theta)t} x - y) dy_1 dy_2, \tag{3.4}$$

where $y = y_1 + iy_2$ and $x, y \in \mathbb{C}$ and we write a function $\varphi(y_1, y_2)$ of the two real variables y_1 and y_2 as $\varphi(y)$ of the complex argument $y_1 + iy_2$ (i.e., we use the complex representation of \mathbb{R}^2 in (3.3-3.4)). The change of variable formula yields the following Mehler formula [8, p. 584].

Proposition 3.1. (Mehler formula) *For each $\varphi \in C_b(\mathbb{R}^2)$,*

$$P_t \varphi(x) = \int_{\mathbb{C}} \varphi(e^{-(\cos \theta + i \sin \theta)t} x + \sqrt{1 - e^{-2t \cos \theta}} y) d\gamma(y), \tag{3.5}$$

where

$$d\gamma(y) = \frac{1}{2\pi} \exp \left\{ -\frac{(y_1^2 + y_2^2)}{2} \right\} dy_1 dy_2. \tag{3.6}$$

¹Note that $H_n(x) = \frac{(-1)^n}{n!} e^{x^2/2} \frac{d^n}{dx^n} e^{-x^2/2}$ in [14, 20] and $H_n(x) = (-1)^n e^{x^2/2} \frac{d^n}{dx^n} e^{-x^2/2}$ in [8, 11], here we use the definition in [23].

Similar to the real Ornstein-Uhlenbeck semigroup [20, Proposition 2.3], using the rotation invariant of the measure γ and Lebesgue's dominated convergence theorem, it follows from Proposition 3.1 that γ is the unique invariant measure of P_t . In detail, for each $\varphi \in C_b(\mathbb{R}^2)$,

$$\int_{\mathbb{C}} P_t \varphi(x) \, d\gamma(x) = \int_{\mathbb{C}} \varphi(x) \, d\gamma(x) \tag{3.7}$$

and

$$\lim_{t \rightarrow \infty} P_t \varphi(x) = \int_{\mathbb{C}} \varphi(y) \, d\gamma(y), \quad \forall x \in \mathbb{C}. \tag{3.8}$$

Denote the associated transition probabilities on \mathbb{C} as $P_t(x, A) = P_t \mathbf{1}_A(x)$ for each $A \in \mathcal{B}(\mathbb{R}^2)$. Along the same line of the real case [6, 20, 23], for each $p \geq 1$, it follows from Jensen's inequality that for each $\varphi \in C_b(\mathbb{R}^2)$,

$$\begin{aligned} \|P_t \varphi\|_{L^p(\gamma)}^p &= \int_{\mathbb{C}} |P_t \varphi(x)|^p \, d\gamma(x) = \int_{\mathbb{C}} \left| \int_{\mathbb{C}} \varphi(y) P_t(x, dy) \right|^p \, d\gamma(x) \\ &\leq \int_{\mathbb{C}} d\gamma(x) \int_{\mathbb{C}} |\varphi(y)|^p P_t(x, dy) \\ &= \int_{\mathbb{C}} |\varphi|^p(x) \, d\gamma(x) \quad (\text{by (3.7)}) \\ &= \|\varphi\|_{L^p(\gamma)}^p. \end{aligned}$$

It follows from the B.L.T. theorem [17, p. 9] that $\{P_t\}_{t \geq 0}$ can be uniquely extended to a strong continuous contraction semigroup $\{T_t^p\}_{t \geq 0}$ on $L^p(\gamma)$ for each $p \geq 1$. Let \mathcal{A}_p be the (infinitesimal) generator, then \mathcal{A}_p is closed and $\overline{D(\mathcal{A}_p)} = L^p(\gamma)$ (i.e., densely defined) [15, 17].

Lemma 3.2. *Suppose $\vec{Y} = (y_1, \dots, y_n)$, $\vec{Z} = (z_1, \dots, z_n) \in \mathbb{C}^n$ and $\vec{Y} = M\vec{Z}$, where $M = (M_{ij})$ is an n -by- n unitary matrix over the field \mathbb{C} . If $z_i, i = 1 \dots, n$ are independent, each being centered complex normal such that $E|z_i|^2 = \sigma^2$, then $y_i, i = 1 \dots, n$ are also independent, each being centered complex normal such that $E|y_i|^2 = \sigma^2$.*

Proof. It follows from [9, Theorem 1.1] that $y_i, i = 1 \dots, n$ are centered complex normal. In addition, we have that

$$E[y_i \bar{y}_j] = \sum_{k,l} M_{ik} E[z_k \bar{z}_l] \bar{M}_{jl} = \sigma^2 \sum_k M_{ik} \bar{M}_{jk} = \sigma^2 \delta_{ij},$$

where δ_{ij} is the Kronecker delta. Thus, $y_i, i = 1 \dots, n$ have independent identical distributions with variance σ^2 . □

Proposition 3.3. *For each $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$, denote the semigroup P_t depending on θ in (3.3) by P_t^θ , then for each $\phi \in C_b(\mathbb{R}^2)$*

$$P_t^\theta (P_t^\theta)^* \phi(x) = (P_t^\theta)^* P_t^\theta \phi(x), \tag{3.9}$$

where $(P_t^\theta)^*$ is the adjoint operator of P_t^θ in $L^2(\gamma)$. Furthermore, when restricted on $C_b(\mathbb{R}^2)$, $(P_t^\theta)^* = P_t^{-\theta}$.

²Namely, T_t^p is the closure (see [17, p. 250]) in $L^p(\gamma)$ of the operator P_t .

Proof. Set $\alpha = e^{i\theta}$. For each $\phi, \psi \in C_b(\mathbb{R}^2)$ and $t \geq 0$, we have that

$$\begin{aligned} \langle P_t^\theta \phi, \psi \rangle &= \int_{\mathbb{C}} \bar{\psi}(z_1) d\gamma(z_1) \int_{\mathbb{C}} \phi(e^{-\alpha t} z_1 + \sqrt{1 - e^{-2t\operatorname{Re}\alpha}} z_2) d\gamma(z_2) \\ &= \int_{\mathbb{C}} \phi(y_1) d\gamma(y_1) \int_{\mathbb{C}} \overline{\psi(e^{-\bar{\alpha}t} y_1 - \sqrt{1 - e^{-2t\operatorname{Re}\alpha}} y_2)} d\gamma(y_2) \end{aligned} \quad (3.10)$$

$$\begin{aligned} &= \int_{\mathbb{C}} \phi(y_1) d\gamma(y_1) \int_{\mathbb{C}} \overline{\psi(e^{-\bar{\alpha}t} y_1 + \sqrt{1 - e^{-2t\operatorname{Re}\alpha}} y_2)} d\gamma(y_2) \quad (3.11) \\ &= \langle \phi, P_t^{-\theta} \psi \rangle, \end{aligned}$$

where (3.10) is deduced from Lemma 3.2 by taking $n = 2$ and

$$M = \begin{bmatrix} e^{-\alpha t} & \sqrt{1 - e^{-2t\operatorname{Re}\alpha}} \\ -\sqrt{1 - e^{-2t\operatorname{Re}\alpha}} & e^{-\bar{\alpha}t} \end{bmatrix},$$

and (3.11) is deduced from the rotation invariant of the measure γ . Therefore, the adjoint operator of P_t^θ in $L^2(\gamma)$ satisfies that $(P_t^\theta)^* = P_t^{-\theta}$ when restricted on $C_b(\mathbb{R}^2)$ for each $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$. Thus, for each $\phi \in C_b(\mathbb{R}^2)$,

$$\begin{aligned} &P_t^\theta (P_t^\theta)^* \phi(x) \\ &= \int_{\mathbb{C}^2} \phi(e^{-\bar{\alpha}t}(e^{-\alpha t} x + \sqrt{1 - e^{-2t\operatorname{Re}\alpha}} z_1) + \sqrt{1 - e^{-2t\operatorname{Re}\alpha}} z_2) d\gamma(z_1) d\gamma(z_2) \\ &= \int_{\mathbb{C}} \phi(e^{-2t\operatorname{Re}\alpha} x + \sqrt{1 - e^{-4t\operatorname{Re}\alpha}} z) d\gamma(z) \quad (3.12) \\ &= \int_{\mathbb{C}^2} \phi(e^{-\alpha t}(e^{-\bar{\alpha}t} x + \sqrt{1 - e^{-2t\operatorname{Re}\alpha}} z_1) + \sqrt{1 - e^{-2t\operatorname{Re}\alpha}} z_2) d\gamma(z_1) d\gamma(z_2) \\ &= (P_t^\theta)^* P_t^\theta \phi(x), \end{aligned}$$

where (3.12) is deduced from the well-known fact that if Z_1, Z_2 are two independent standard complex normal random variables, then $e^{-\bar{\alpha}t} \sqrt{1 - e^{-2t\operatorname{Re}\alpha}} Z_1 + \sqrt{1 - e^{-2t\operatorname{Re}\alpha}} Z_2$ and $\sqrt{1 - e^{-4t\operatorname{Re}\alpha}} Z_1$ have the same law [9, Theorem 1.1]. \square

Theorem 3.4. $\{T_t^2\}_{t \geq 0}$ is a semigroup of normal operators (see [19, p. 382]) in $L^2(\gamma)$ and thus the generator \mathcal{A}_2 is a normal operator in $L^2(\gamma)$.

Proof. Since T_t^2 is the closure of the contraction operator P_t in $L^2(\gamma)$, it follows from (c) of Theorem VIII.1 in [17, p. 253] that the adjoint operator of T_t^2 equals to that of P_t . It follows from the density argument that (3.9) can be extended to each $\phi \in L^2(\gamma)$, i.e.,

$$T_t^2 (T_t^2)^* \phi = (T_t^2)^* T_t^2 \phi, \quad \forall \phi \in L^2(\gamma).$$

Thus $\{T_t^2\}_{t \geq 0}$ is a semigroup of normal operators. It follows from [19, Theorem 13.38] that the generator \mathcal{A}_2 is a normal operator in $L^2(\gamma)$. \square

4. The Normal Diffusion Operators in \mathbb{C}

The first aim of this section is to show the explicit expression of the generator \mathcal{A}_2 .

Definition 4.1. For any $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$, define

$$\mathcal{D}(\mathcal{L}_\theta) = \left\{ f \in L^2(\gamma), \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (m^2 + n^2 + 2mn \cos 2\theta) |\langle f, J_{m,n} \rangle|^2 < \infty \right\} \quad (4.1)$$

and

$$\mathcal{L}_\theta f = - \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} [(m+n) \cos \theta + i(m-n) \sin \theta] \langle f, J_{m,n} \rangle J_{m,n}(z). \quad (4.2)$$

Theorem 4.2. If $\phi \in C^2_+(\mathbb{R}^2)$, then $\phi \in \mathcal{D}(\mathcal{L}_\theta)$ and

$$\mathcal{L}_\theta \phi = [4 \cos \theta \frac{\partial^2}{\partial z \partial \bar{z}} - e^{i\theta} z \frac{\partial}{\partial z} - e^{-i\theta} \bar{z} \frac{\partial}{\partial \bar{z}}] \phi. \quad (4.3)$$

Theorem 4.3. Let \mathcal{A}_2 be as in Theorem 3.4 and \mathcal{L}_θ be as in Definition 4.1. For any $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$, $\mathcal{A}_2 = \mathcal{L}_\theta$, i.e., $D(\mathcal{A}_2) = \mathcal{D}(\mathcal{L}_\theta)$ and $\mathcal{A}_2 \varphi = \mathcal{L}_\theta \varphi$ on $\mathcal{D}(\mathcal{L}_\theta)$.

The second aim of this section is to show that the operator \mathcal{L}_θ defined above satisfies the following theorem, which is named as the normal diffusion operator analogous to the symmetric diffusion operator given by Stroock [6, 22, 23].

Theorem 4.4. The densely defined linear closed operator \mathcal{L}_θ defined in Proposition 4.1 is a normal diffusion operator. Namely, it satisfies that:

- 1) \mathcal{L}_θ is a normal operator on $\mathcal{D}(\mathcal{L}_\theta)$.
- 2) $1 \in \mathcal{D}(\mathcal{L}_\theta)$ and $\mathcal{L}_\theta 1 = 0$.
- 3) There exists a linear subspace

$$\mathcal{D} \subset \left\{ \phi \in \mathcal{D}(\mathcal{L}_\theta) \cap L^4(\gamma) : \mathcal{L}_\theta \phi \in L^4(\gamma), |\phi|^2 \in \mathcal{D}(\mathcal{L}_\theta) \right\}$$

such that $\text{graph}(\mathcal{L}_\theta|_{\mathcal{D}})$ is dense in $\text{graph}(\mathcal{L}_\theta)$.

- 4) For any $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$, define

$$(\phi, \psi)_\theta = \frac{1}{2 \cos \theta} [\mathcal{L}_\theta(\phi \bar{\psi}) - \phi \mathcal{L}_\theta(\bar{\psi}) - \bar{\psi} \mathcal{L}_\theta(\phi)]$$

for $\phi, \psi \in \mathcal{D}$. Then $(\cdot, \cdot)_\theta : \mathcal{D} \times \mathcal{D} \rightarrow L^2(\gamma)$ is a non-negative definite bilinear form on the field \mathbb{C} .

- 5) (Diffusion property) If $\vec{\phi} = (\phi_1, \dots, \phi_n) \in \mathcal{D}^n$ and $F \in C^2_+(\mathbb{C}^n)$, then $F \circ \vec{\phi} \in \mathcal{D}(\mathcal{L}_\theta)$ and

$$\begin{aligned} & \mathcal{L}_\theta(F \circ \vec{\phi}) \\ &= \cos \theta \sum_{i,j=1}^n (\phi_i, \bar{\phi}_j)_\theta \frac{\partial^2 F}{\partial z_i \partial z_j} \circ \vec{\phi} + (\bar{\phi}_i, \phi_j)_\theta \frac{\partial^2 F}{\partial \bar{z}_i \partial \bar{z}_j} \circ \vec{\phi} + 2(\phi_i, \phi_j)_\theta \frac{\partial^2 F}{\partial z_i \partial \bar{z}_j} \circ \vec{\phi} \\ &+ \sum_{i=1}^n \mathcal{L}_\theta \phi_i \frac{\partial F}{\partial z_i} \circ \vec{\phi} + \mathcal{L}_\theta \bar{\phi}_i \frac{\partial F}{\partial \bar{z}_i} \circ \vec{\phi}. \end{aligned} \quad (4.4)$$

- 6) \mathcal{L}_θ has an extension \mathcal{A}_1 to $L^1(\gamma)$ with domain $\mathcal{D}(\mathcal{A}_1)$ such that

$$\mathcal{D}(\mathcal{L}_\theta) = \left\{ \phi \in \mathcal{D}(\mathcal{A}_1) \cap L^2(\gamma) : \mathcal{A}_1 \phi \in L^2(\gamma) \right\},$$

i.e., the closure of \mathcal{L}_θ in $L^1(\gamma)$ is \mathcal{A}_1 .

Proofs of Theorems 4.2-4.4 are presented in Section 4.1.

4.1. Proofs of Theorems.

Proposition 4.5. *Let \mathcal{L}_θ be as in Definition 4.1. Then \mathcal{L}_θ is closed on $L^2(\gamma)$.*

Proof. Suppose that $f_k \in \mathcal{D}(\mathcal{L}_\theta)$ such that $f_k \rightarrow f$, $\mathcal{L}_\theta f_k \rightarrow g$ in $L^2(\gamma)$, we will show that $f \in \mathcal{D}(\mathcal{L}_\theta)$ and $\mathcal{L}_\theta f = g$. In fact, by Fatou's lemma and Parseval's identity, we have that

$$\begin{aligned} & \sum_{m,n=0}^{\infty} (m^2 + n^2 + 2mn \cos 2\theta) |\langle f, J_{m,n} \rangle|^2 \\ & \leq \liminf_{k \rightarrow \infty} \sum_{m,n=0}^{\infty} (m^2 + n^2 + 2mn \cos 2\theta) |\langle f_k, J_{m,n} \rangle|^2 \\ & = \liminf_{k \rightarrow \infty} \|\mathcal{L}_\theta f_k\|^2 \\ & = \|g\|^2 < \infty. \end{aligned}$$

Thus $f \in \mathcal{D}(\mathcal{L}_\theta)$. For each $m, n \geq 0$, we have that

$$\begin{aligned} & \lim_{k \rightarrow \infty} \langle [(m+n) \cos \theta + i(m-n) \sin \theta] f_k + g, J_{m,n} \rangle \\ & = \langle [(m+n) \cos \theta + i(m-n) \sin \theta] f + g, J_{m,n} \rangle \end{aligned}$$

It follows from Parseval's identity and Fatou's lemma that

$$\begin{aligned} \|\mathcal{L}_\theta f - g\|^2 &= \sum_{m,n=0}^{\infty} |\langle [(m+n) \cos \theta + i(m-n) \sin \theta] f + g, J_{m,n} \rangle|^2 \\ &\leq \liminf_{k \rightarrow \infty} \sum_{m,n=0}^{\infty} |\langle [(m+n) \cos \theta + i(m-n) \sin \theta] f_k + g, J_{m,n} \rangle|^2 \\ &= \liminf_{k \rightarrow \infty} \|\mathcal{L}_\theta f_k - g\|^2 = 0. \end{aligned}$$

Thus $\mathcal{L}_\theta f = g$. □

Remark 4.6. Suppose that $H_{m,n}(x, y) = H_m(x)H_n(y)$ is the Hermite polynomial of two variables. Then it follows from Proposition 2.3 that

$$\sum_{m+n=l} |\langle f, J_{m,n} \rangle|^2 = \sum_{m+n=l} |\langle f, H_{m,n} \rangle|^2.$$

Together with

$$(m+n)^2 \geq m^2 + n^2 + 2mn \cos 2\theta = (m+n)^2 \cos^2 \theta + (m-n)^2 \sin^2 \theta \geq (m+n)^2 \cos^2 \theta,$$

we deduce that

$$\begin{aligned} \mathcal{D}(\mathcal{L}_\theta) &= \left\{ f \in L^2(\gamma), \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (m+n)^2 |\langle f, J_{m,n} \rangle|^2 < \infty \right\} \\ &= \left\{ f \in L^2(\gamma), \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (m+n)^2 |\langle f, H_{m,n} \rangle|^2 < \infty \right\}, \end{aligned} \tag{4.5}$$

that is to say, $\mathcal{D}(\mathcal{L}_\theta)$ is independent to θ . In fact, the right hand side of (4.5) is exact the Sobolev weighted space H_γ^2 , please refer to [12] for details.

Proposition 4.7. For any $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$, \mathcal{L}_θ is a normal operator on $\mathcal{D}(\mathcal{L}_\theta)$ such that $1 \in \mathcal{D}(\mathcal{L}_\theta)$ and $\mathcal{L}_\theta 1 = 0$.

Proof. Suppose that $f, g \in \mathcal{D}(\mathcal{L}_\theta)$. It follows from Parseval's identity that

$$\begin{aligned} \langle \mathcal{L}_\theta f, g \rangle &= - \sum_{m,n}^{\infty} [(m+n) \cos \theta + i(m-n) \sin \theta] \langle f, J_{m,n} \rangle \overline{\langle g, J_{m,n} \rangle} \\ &= - \sum_{m,n}^{\infty} \langle f, J_{m,n} \rangle \overline{[(m+n) \cos \theta - i(m-n) \sin \theta] \langle g, J_{m,n} \rangle} \\ &= \langle f, \mathcal{L}_{-\theta} g \rangle. \end{aligned}$$

Thus, the adjoint operator of \mathcal{L}_θ is $\mathcal{L}_\theta^* = \mathcal{L}_{-\theta}$. The equality (4.5) implies that

$$\mathcal{D}(\mathcal{L}_\theta) = \mathcal{D}(\mathcal{L}_{-\theta}) = \mathcal{D}(\mathcal{L}_\theta^*).$$

And for each $f \in \mathcal{D}(\mathcal{L}_\theta)$ such that $\mathcal{L}_\theta f \in \mathcal{D}(\mathcal{L}_\theta^*)$, we have that

$$\mathcal{L}_\theta^* \mathcal{L}_\theta f = \sum_{m,n} (m^2 + n^2 + 2mn \cos 2\theta) \langle f, J_{m,n} \rangle J_{m,n}(z) = \mathcal{L}_\theta \mathcal{L}_\theta^* f.$$

Therefore, \mathcal{L}_θ is a normal operator on $\mathcal{D}(\mathcal{L}_\theta)$. $1 \in \mathcal{D}(\mathcal{L}_\theta)$ and $\mathcal{L}_\theta 1 = 0$ is trivial. \square

Proposition 4.8. Denote by $\mathcal{D} = \text{span}\{J_{m,n}, m, n \geq 0\}$ the linear span (also called the linear hull) of complex Hermite polynomials. Then

$$\mathcal{D} \subset \left\{ \phi \in \mathcal{D}(\mathcal{L}_\theta) \cap L^4(\gamma) : \mathcal{L}_\theta \phi \in L^4(\gamma), |\phi|^2 \in \mathcal{D}(\mathcal{L}_\theta) \right\} \quad (4.6)$$

and $\text{graph}(\mathcal{L}_\theta|_{\mathcal{D}})$ is dense in $\text{graph}(\mathcal{L}_\theta)$.

Proof. The equality [8, Theorem 2.5]

$$J_{m,n}(z) = (m!n!2^{m+n})^{-\frac{1}{2}} \sum_{r=0}^{m \wedge n} (-1)^r 2^r \frac{m!n!}{(m-r)!(n-r)!r!} z^{m-r} \bar{z}^{n-r}, \quad \forall m, n \geq \mathbb{N}$$

and the equality [8, Corollary 2.8]

$$z^m \bar{z}^n = \sum_{k=0}^{m \wedge n} \frac{m!n!}{(m-k)!(n-k)!k!} \sqrt{(m-k)!(n-k)!2^{m+n}} J_{m-k, n-k}(z), \quad \forall m, n \geq \mathbb{N}$$

imply that $\mathcal{D} = \text{span}\{J_{m,n}, m, n \geq 0\} = \text{span}\{z^m \bar{z}^n, m, n \geq 0\}$. But $f(z) = z^m \bar{z}^n$ belonging to the right hand side of (4.6) is trivial. Thus (4.6) holds.

Since \mathcal{D} is a dense subset of $L^2(\gamma)$ (see Proposition 2.2), \mathcal{D} is dense in $\mathcal{D}(\mathcal{L}_\theta)$. Note that \mathcal{L}_θ is a closed operator, we get that $\text{graph}(\mathcal{L}_\theta|_{\mathcal{D}})$ is dense in $\text{graph}(\mathcal{L}_\theta)$. \square

Proof of Theorem 4.2. First, it follows from Proposition 2.2 that (4.3) holds when $\phi \in \mathcal{D} = \text{span}\{J_{m,n}, m, n \geq 0\}$.

Second, suppose that $\phi \in L^2(\gamma)$ satisfies that the sequence

$$a_{m,n} = \langle \phi, H_m(x)H_n(y) \rangle$$

is rapidly decreasing, then we will show that $\phi \in \mathcal{D}(\mathcal{L}_\theta)$ and (4.3) is satisfied. In fact, it follows from Proposition 5.8 that the Hermite expansion $\phi(z) = \sum_{m,n=0}^\infty a_{m,n} H_m(x) H_n(y)$ satisfies that

$$\left\| x^{k_1} y^{k_2} \frac{\partial^{p_1+p_2}}{\partial x^{p_1} \partial y^{p_2}} (\phi - \phi_l) \right\|_{L^2(\gamma)} \rightarrow 0 \text{ as } l \rightarrow \infty, \quad \forall k_1, k_2, p_1, p_2 \in \mathbb{N},$$

where $\phi_l = \sum_{m+n \leq l} a_{m,n} H_m(x) H_n(y)$. Thus,

$$\left\| z^{k_1} \bar{z}^{k_2} \frac{\partial^{p_1+p_2}}{\partial z^{p_1} \partial \bar{z}^{p_2}} (\phi - \phi_l) \right\|_{L^2(\gamma)} \rightarrow 0 \text{ as } l \rightarrow \infty, \quad \forall k_1, k_2, p_1, p_2 \in \mathbb{N}.$$

It follows from Proposition 2.3 that

$$\sum_{m+n \leq l} a_{m,n} H_m(x) H_n(y) = \sum_{m+n \leq l} b_{m,n} J_{m,n}(z).$$

Thus, as $l \rightarrow \infty$, we have that in $L^2(\gamma)$, $\phi_l \rightarrow \phi$ and

$$\begin{aligned} \mathcal{L}_\theta \phi_l &= - \sum_{m+n \leq l} [(m+n) \cos \theta + i(m-n) \sin \theta] b_{m,n} J_{m,n}(z) \\ &= [4 \cos \theta \frac{\partial^2}{\partial z \partial \bar{z}} - e^{i\theta} z \frac{\partial}{\partial z} - e^{-i\theta} \bar{z} \frac{\partial}{\partial \bar{z}}] \phi_l \\ &\rightarrow [4 \cos \theta \frac{\partial^2}{\partial z \partial \bar{z}} - e^{i\theta} z \frac{\partial}{\partial z} - e^{-i\theta} \bar{z} \frac{\partial}{\partial \bar{z}}] \phi. \end{aligned}$$

Since \mathcal{L}_θ is closed, we have that $\phi \in \mathcal{D}(\mathcal{L}_\theta)$ and (4.3) is satisfied.

Finally, it follows from Proposition 5.5 that if $\phi \in C_c^\infty(\mathbb{R}^2)$ then there exists an approximation of identity $B_\epsilon \phi \in C_c^\infty(\mathbb{R}^2)$ such that for all $p_1 + p_2 \leq 2$ and $k_1, k_2 \geq 0$, $x^{k_1} y^{k_2} \frac{\partial^{p_1+p_2}}{\partial x^{p_1} \partial y^{p_2}} (B_\epsilon \phi) \rightarrow x^{k_1} y^{k_2} \frac{\partial^{p_1+p_2}}{\partial x^{p_1} \partial y^{p_2}} \phi$ in $L^2(\gamma)$ as $\epsilon \rightarrow 0$. In addition, it follows from Proposition 5.6 that the sequence $\langle B_\epsilon \phi, H_m(x) H_n(y) \rangle$ is rapidly decreasing. Thus, as $\epsilon \rightarrow 0$, we have that in $L^2(\gamma)$, $B_\epsilon \phi \rightarrow \phi$ and

$$\begin{aligned} \mathcal{L}_\theta (B_\epsilon \phi) &= [4 \cos \theta \frac{\partial^2}{\partial z \partial \bar{z}} - e^{i\theta} z \frac{\partial}{\partial z} - e^{-i\theta} \bar{z} \frac{\partial}{\partial \bar{z}}] B_\epsilon \phi \\ &\rightarrow [4 \cos \theta \frac{\partial^2}{\partial z \partial \bar{z}} - e^{i\theta} z \frac{\partial}{\partial z} - e^{-i\theta} \bar{z} \frac{\partial}{\partial \bar{z}}] \phi. \end{aligned}$$

Since \mathcal{L}_θ is closed, we have that $\phi \in \mathcal{D}(\mathcal{L}_\theta)$ and (4.3) is satisfied. □

Proof of Theorem 4.3. First, it follows from the density argument (see Proposition 5.5) and Lebesgue's dominated convergence theorem that the Mehler formula (3.5) is still valid for $\varphi \in C_\uparrow^0(\mathbb{R}^2)$, i.e.,

$$T_t^2 \varphi(x) = \int_{\mathbb{C}} \varphi(e^{-(\cos \theta + i \sin \theta)t} x + \sqrt{1 - e^{-2t \cos \theta}} y) d\gamma(y), \quad \forall \varphi \in C_\uparrow^0(\mathbb{R}^2).$$

Then $T_t^2 \varphi(x) = P_t \varphi(x) = E_x[\varphi(Z_t)]$ for each $\varphi \in C_\uparrow^0(\mathbb{R}^2)$.

Second, using (3.1), it follows from Ito’s lemma and Theorem 4.2 that for each $\varphi \in C^2_{\uparrow}(\mathbb{R}^2)$,

$$\begin{aligned} \varphi(Z_t) &= \varphi(x) + \int_0^t \frac{\partial}{\partial z} \varphi(Z_s) dZ_s + \int_0^t \frac{\partial}{\partial \bar{z}} \varphi(Z_s) d\bar{Z}_s + \int_0^t \frac{\partial^2}{\partial z \partial \bar{z}} \varphi(Z_s) d\langle Z, \bar{Z} \rangle_s \\ &= \varphi(x) + \int_0^t \mathcal{L}_\theta \varphi(Z_s) ds - \sqrt{2 \cos \theta} \left(\int_0^t \frac{\partial \varphi}{\partial z} (Z_s) d\zeta_s + \int_0^t \frac{\partial \varphi}{\partial \bar{z}} (Z_s) d\bar{\zeta}_s \right). \end{aligned}$$

Then

$$\begin{aligned} T_t^2 \varphi(x) &= E_x[\varphi(Z_t)] = \varphi(x) + E_x \left[\int_0^t \mathcal{L}_\theta \varphi(Z_s) ds \right] \\ &= \varphi(x) + \int_0^t E_x[\mathcal{L}_\theta \varphi(Z_s)] ds \quad (\text{by Fubini Theorem}) \\ &= \varphi(x) + \int_0^t T_s^2 \mathcal{L}_\theta \varphi(x) ds, \end{aligned}$$

and

$$\begin{aligned} \mathcal{A}_2 \varphi &= \lim_{t \downarrow 0} \frac{T_t^2 \varphi - \varphi}{t} = \lim_{t \downarrow 0} \frac{1}{t} \int_0^t T_s^2 \mathcal{L}_\theta \varphi(x) ds \\ &= \mathcal{L}_\theta \varphi \quad (\text{in } L^2(\gamma)), \end{aligned}$$

where to get the last equality we use the continuity of $t \rightarrow T_t^2 \varphi$ for any $\varphi \in L^2(\gamma)$ (see [15, Corollary 2.3] or part (a) of [15, Theorem 2.4]). Therefore, $\mathcal{A}_2 = \mathcal{L}_\theta$ on $C^2_{\uparrow}(\mathbb{R}^2)$,

Third, since $\text{graph}(\mathcal{L}_\theta|_{C^2_{\uparrow}(\mathbb{R}^2)})$ is dense in $\text{graph}(\mathcal{L}_\theta)$ (see Proposition 4.8) and \mathcal{A}_2 is closed, we have that $\mathcal{L}_\theta \subseteq \mathcal{A}_2$. It follows from Proposition 4.7 and Theorem 3.4 that both \mathcal{L}_θ and \mathcal{A}_2 are normal operators. Since \mathcal{L}_θ is maximally normal (see [19, Theorem 13.32]), we have that $\mathcal{A}_2 = \mathcal{L}_\theta$. \square

Corollary 4.9. *For any $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$, $\mathcal{L}_\theta \subseteq \mathcal{A}_1$ (i.e., \mathcal{A}_1 is an extension of \mathcal{L}_θ to $L^1(\gamma)$) and*

$$\mathcal{D}(\mathcal{L}_\theta) = \{ \phi \in D(\mathcal{A}_1) \cap L^2(\gamma) : \mathcal{A}_1 \phi \in L^2(\gamma) \}. \tag{4.7}$$

Proof. The proof is similar to the real case [6, p. 19]. In detail,

$$\mathcal{D}(\mathcal{L}_\theta) = \mathcal{D}(\mathcal{A}_2) \subset \{ \phi \in D(\mathcal{A}_1) \cap L^2(\gamma) : \mathcal{A}_1 \phi \in L^2(\gamma) \}$$

is trivial. Now suppose that $\phi \in D(\mathcal{A}_1) \cap L^2(\gamma)$ and $\mathcal{A}_1 \phi \in L^2(\gamma)$, then as $t \rightarrow 0$,

$$\begin{aligned} \frac{T_t^2 \phi - \phi}{t} &= \frac{T_t^1 \phi - \phi}{t} \\ &= \frac{1}{t} \int_0^t T_s^1 \mathcal{A}_1 \phi ds \quad (\text{by the semigroup equation}) \\ &= \frac{1}{t} \int_0^t T_s^2 \mathcal{A}_1 \phi ds \rightarrow \mathcal{A}_1 \phi \quad (\text{in } L^2(\gamma)), \end{aligned}$$

where we use again the continuity of $t \rightarrow T_t^2 \varphi$ for the last equality. Thus $\phi \in \mathcal{D}(\mathcal{A}_2) = \mathcal{D}(\mathcal{L}_\theta)$ and (4.7) holds. \square

Proof of Theorem 4.4. By Proposition 4.5, \mathcal{L}_θ is closed. 1)-3) and 6) of Theorem 4.4 are shown by Proposition 4.7-4.8 and Corollary 4.9 respectively. Since $F \in C_\uparrow^2(\mathbb{C}^n)$ and $\vec{\phi} = (\phi_1, \dots, \phi_n) \in \mathcal{D}^n$, then $F \circ \vec{\phi} \in C_\uparrow^2(\mathbb{R}^2)$. By the complex version of the chain rule [21, p. 27], it follows from Theorem 4.2 that

$$\begin{aligned}
 \mathcal{L}_\theta(F \circ \vec{\phi}) &= [4 \cos \theta \frac{\partial^2}{\partial z \partial \bar{z}} - e^{i\theta} z \frac{\partial}{\partial z} - e^{-i\theta} \bar{z} \frac{\partial}{\partial \bar{z}}](F \circ \vec{\phi}) \\
 &= -e^{i\theta} z \sum_{i=1}^n \frac{\partial}{\partial z} \phi_i \frac{\partial F}{\partial z_i} \circ \vec{\phi} + \frac{\partial}{\partial z} \bar{\phi}_i \frac{\partial F}{\partial \bar{z}_i} \circ \vec{\phi} \\
 &\quad - e^{-i\theta} \bar{z} \sum_{i=1}^n \frac{\partial}{\partial \bar{z}} \phi_i \frac{\partial F}{\partial z_i} \circ \vec{\phi} + \frac{\partial}{\partial \bar{z}} \bar{\phi}_i \frac{\partial F}{\partial \bar{z}_i} \circ \vec{\phi} \\
 &\quad + 4 \cos \theta \sum_{i=1}^n \frac{\partial^2 \phi_i}{\partial z \partial \bar{z}} \frac{\partial F}{\partial z_i} \circ \vec{\phi} + \frac{\partial^2 \bar{\phi}_i}{\partial z \partial \bar{z}} \frac{\partial F}{\partial \bar{z}_i} \circ \vec{\phi} \\
 &\quad + 4 \cos \theta \sum_{i,j=1}^n \frac{\partial}{\partial z} \phi_i \left(\frac{\partial}{\partial \bar{z}} \phi_j \frac{\partial^2 F}{\partial z_i \partial z_j} \circ \vec{\phi} + \frac{\partial}{\partial \bar{z}} \bar{\phi}_j \frac{\partial^2 F}{\partial z_i \partial \bar{z}_j} \circ \vec{\phi} \right) \\
 &\quad + 4 \cos \theta \sum_{i,j=1}^n \frac{\partial}{\partial z} \bar{\phi}_i \left(\frac{\partial}{\partial \bar{z}} \phi_j \frac{\partial^2 F}{\partial \bar{z}_i \partial z_j} \circ \vec{\phi} + \frac{\partial}{\partial \bar{z}} \bar{\phi}_j \frac{\partial^2 F}{\partial \bar{z}_i \partial \bar{z}_j} \circ \vec{\phi} \right) \\
 &= 4 \cos \theta \sum_{i,j=1}^n \frac{\partial}{\partial z} \phi_i \frac{\partial}{\partial \bar{z}} \phi_j \frac{\partial^2 F}{\partial z_i \partial z_j} \circ \vec{\phi} + \frac{\partial}{\partial z} \bar{\phi}_i \frac{\partial}{\partial \bar{z}} \bar{\phi}_j \frac{\partial^2 F}{\partial \bar{z}_i \partial \bar{z}_j} \circ \vec{\phi} \\
 &\quad + 4 \cos \theta \sum_{i,j=1}^n \left(\frac{\partial}{\partial z} \phi_i \frac{\partial}{\partial \bar{z}} \bar{\phi}_j + \frac{\partial}{\partial \bar{z}} \phi_i \frac{\partial}{\partial z} \bar{\phi}_j \right) \frac{\partial^2 F}{\partial z_i \partial \bar{z}_j} \circ \vec{\phi} \\
 &\quad + \sum_{i=1}^n \mathcal{L}_\theta \phi_i \frac{\partial F}{\partial z_i} \circ \vec{\phi} + \mathcal{L}_\theta \bar{\phi}_i \frac{\partial F}{\partial \bar{z}_i} \circ \vec{\phi}. \tag{4.8}
 \end{aligned}$$

Taking $F(z_1, z_2) = z_1 \bar{z}_2$ in the above equation, we have that

$$\begin{aligned}
 (\phi, \psi)_\theta &= \frac{1}{2 \cos \theta} [\mathcal{L}_\theta(\phi \bar{\psi}) - \bar{\psi} \mathcal{L}_\theta(\phi) - \phi \mathcal{L}_\theta(\bar{\psi})] \\
 &= 2 \left[\frac{\partial \phi}{\partial z} \frac{\partial \bar{\psi}}{\partial \bar{z}} + \frac{\partial \phi}{\partial \bar{z}} \frac{\partial \bar{\psi}}{\partial z} \right] = 2 \left[\frac{\partial \phi}{\partial z} \frac{\partial \bar{\psi}}{\partial z} + \frac{\partial \phi}{\partial \bar{z}} \frac{\partial \bar{\psi}}{\partial \bar{z}} \right]. \tag{4.9}
 \end{aligned}$$

Hence, $(\phi, \psi)_\theta$ is a non-negative definite bilinear form on the field \mathbb{C} . Substituting (4.9) into (4.8), we show (4.4). Thus, 4)-5) of Theorem 4.4 are obtained. \square

5. Appendix

To be self-contained, we list the necessary results of functions slowly increasing at infinity. Some results which can not be found in textbooks will be shown shortly here. In this section all functions will be complex-valued and defined on \mathbb{R}^n .

Definition 5.1. Denote by $C_c^\infty(\mathbb{R}^n)$ the space of smooth and compactly supported functions on \mathbb{R}^n [5, p. 5]. Denote by $S(\mathbb{R}^n)$ the space of C^∞ functions rapidly decreasing at infinity [5, p. 105]. We say that a continuous function $f(x)$ is *slowly*

increasing at infinity if there exists an integer k such that $(1+r^2)^{-\frac{k}{2}}f(x)$ is bounded in \mathbb{R}^n with $r = |x|$ [5, p. 110]. Denote by $C_{\uparrow}^m(\mathbb{R}^n)$ the space of all functions having slowly increasing at infinity continuous partial derivatives of order $\leq m$.

Notation 1. Denote by γ the n -dimensional standard Gaussian measure:

$$d\gamma(x) = (2\pi)^{-\frac{n}{2}} \exp\left\{-\frac{|x|^2}{2}\right\} dx, \quad x \in \mathbb{R}^n.$$

Denote the density function by $\rho(x) = \frac{d\gamma(x)}{dx}$.

5.1. Approximation of identity of $C_{\uparrow}^m(\mathbb{R}^n)$ in $L^q(\gamma)$.

Notation 2. Set

$$\phi(x) = e^{-\frac{1}{1-|x|^2}} \mathbf{1}_{\{|x|<1\}}, \quad x \in \mathbb{R}^n, \quad (5.1)$$

where $|x| = \sqrt{x_1^2 + \cdots + x_n^2}$ and $\mathbf{1}_B$ the characteristic function of set B . Divide this function by its integral over the whole space to get a function $\alpha(x)$ of integral one which is called a mollifier. Next, for every $\epsilon > 0$, define [5, p. 5]

$$\alpha_{\epsilon}(x) = \frac{1}{\epsilon^n} \alpha\left(\frac{x}{\epsilon}\right). \quad (5.2)$$

Let $L_{loc}^1(\mathbb{R}^n)$ be the space of locally integrable function on \mathbb{R}^n . If $u \in L_{loc}^1(\mathbb{R}^n)$, the function

$$u_{\epsilon}(x) = \int_{\mathbb{R}^n} u(x-y)\alpha_{\epsilon}(y)dy = \int_{\mathbb{R}^n} \alpha_{\epsilon}(x-y)u(y)dy \quad (5.3)$$

is said to be the convolution of u and α_{ϵ} [5, Definition 1.4]. It is also denoted by the convolution operator $A_{\epsilon}u = (u * \alpha_{\epsilon})(x)$.

Lemma 5.2. *Suppose that $f(x) \in C_{\uparrow}^0(\mathbb{R}^n)$. Then $A_{\epsilon}f \in C_{\uparrow}^{\infty}(\mathbb{R}^n)$ (the space of C^{∞} functions slowly increasing at infinity) and for any $q \geq 1$ and any $k \in \mathbb{N}^n$, $\lim_{\epsilon \rightarrow 0} x^k A_{\epsilon}f = x^k f$ in $L^q(\gamma)$.*

Proof. First, for any $\epsilon > 0$, since $\alpha_{\epsilon} \in C_c^{\infty}(\mathbb{R}^n) \subset S(\mathbb{R}^n)$ and $f(x) \in C_{\uparrow}^0(\mathbb{R}^n) \subset S'(\mathbb{R}^n)$ (tempered distributions, see Example 4 in [5, 110]), it follows from Theorem 4.9 of [5, p. 133] that $A_{\epsilon}f = f * \alpha_{\epsilon} \in C_{\uparrow}^{\infty}(\mathbb{R}^n)$.

Second, since any polynomial $P(x)$, $x \in \mathbb{R}^n$, is in $L^q(\gamma)$, we have $f, A_{\epsilon}f \in L^q(\gamma)$. Thus,

$$\|A_{\epsilon}f - f\|_q^q = \lim_{n \rightarrow \infty} \int_{B_a} |A_{\epsilon}f - f|^q d\gamma(x),$$

where $B_a = \{x \in \mathbb{R}^n, |x| \leq a\}$.

Finally, given $\sigma > 0$, there exists B_a such that

$$\|A_{\epsilon}f - f\|_q^q \leq \int_{B_a} |A_{\epsilon}f - f|^q d\gamma(x) + \frac{\sigma}{2}. \quad (5.4)$$

Note that

$$\int_{B_a} |A_{\epsilon}f - f|^q d\gamma(x) \leq \sup_{x \in B_a} |A_{\epsilon}f - f|^q \gamma(K_n) \leq \sup_{x \in B_a} |A_{\epsilon}f - f|^q. \quad (5.5)$$

It follows from [5, Theorem 1.1] that $A_\epsilon f \rightarrow f$ uniformly on B_a as $\epsilon \rightarrow 0$. Thus there exists $\epsilon_0 > 0$ such that $\sup_{x \in B_a} |A_\epsilon f - f| \leq (\frac{\sigma}{2})^{\frac{1}{q}}$ for any $0 < \epsilon < \epsilon_0$. Together with (5.4) and (5.5), we have that $\|A_\epsilon f - f\|_q^q \leq \sigma$, which proves that $A_\epsilon f \rightarrow f$ in $L^q(\gamma)$, as $\epsilon \rightarrow 0$.

Similar to the above proof, it follows that for any $k \in \mathbb{N}^n$, $\lim_{\epsilon \rightarrow 0} x^k A_\epsilon f = x^k f$ in $L^q(\gamma)$. □

Corollary 5.3. *Suppose that $f(x) \in C_\uparrow^m(\mathbb{R}^n)$. Then for any $p, k \in \mathbb{N}^n$ such that $|p| \leq m$, $x^k \partial^p(A_\epsilon f) \rightarrow x^k \partial^p f$ in $L^q(\gamma)$, as $\epsilon \rightarrow 0$.*

Proof. First, if $f(x) \in C_\uparrow^m(\mathbb{R}^n)$ then $\partial^p f \in C_\uparrow^0(\mathbb{R}^n)$ for any $p \in \mathbb{N}^n$ such that $|p| \leq m$. It follows from Lemma 5.2 that $x^k A_\epsilon(\partial^p f) \rightarrow x^k \partial^p f$ in $L^q(\gamma)$ for any $k \in \mathbb{N}^n$. Second, for any $p \in \mathbb{N}^n$, if $u, \partial^p u \in L_{loc}^1(\mathbb{R}^n)$ then $\partial^p(A_\epsilon u) = A_\epsilon(\partial^p u)$. Finally, since $C_\uparrow^0(\mathbb{R}^n) \subset L_{loc}^1(\mathbb{R}^n)$, we have that $x^k \partial^p(A_\epsilon f) \rightarrow x^k \partial^p f$ in $L^q(\gamma)$. □

Notation 3. Let $a \in \mathbb{R}^+$ and denote by B_{a+1} and B_a concentric balls of radius $a + 1$ and a , respectively. It follows from Corollary 3 of [5, p. 9] that there exists a so-called (smooth) cutoff function $\beta_a(x) \in C_c^\infty(\mathbb{R}^n)$ such that: (i) $0 \leq \beta_a \leq 1$ and $\text{supp} \beta_a \subset B_{a+1}$, (ii) $\beta_a(x) = 1$ on B_a , (iii) for all $p \in \mathbb{N}^n$, $\sup_{x \in \mathbb{R}^n} |\partial^p \beta_a| \leq c(n, p)$.

Lemma 5.4. *Let the cutoff function β_a prevail. Suppose that $g \in C_\uparrow^\infty(\mathbb{R}^n)$ and set $g_a = g\beta_a$. Then $g_a \in C_c^\infty(\mathbb{R}^n)$, and for any $k, p \in \mathbb{N}^n$, $\lim_{a \rightarrow \infty} x^k \partial^p g_a = x^k \partial^p g$ in $L^q(\gamma)$ for any $q \geq 1$.*

Proof. The Leibniz’s rule implies that

$$\partial^p g_a = \sum_{l \leq p} \frac{p!}{l!(p-l)!} \partial^l \beta_a \partial^{p-l} g.$$

Denote $G_a = \mathbb{R}^n - B_a$, it follows from (i)-(iii) of Notation 3 that

$$\begin{aligned} |\partial^p g_a - \partial^p g| &= \mathbf{1}_{G_a} \left| (\beta_a - 1) \partial^p g + \sum_{0 < l \leq p} \frac{p!}{l!(p-l)!} \partial^l \beta_a \partial^{p-l} g \right| \\ &\leq c \times \mathbf{1}_{G_a} \sum_{l \leq p} |\partial^{p-l} g|, \end{aligned} \tag{5.6}$$

where $c = p! \max_{0 < l \leq p} c(n, p-l)/l!(p-l)!$.

Since $g \in C_\uparrow^\infty(\mathbb{R}^n)$, we have $h(x) := x^k \sum_{l \leq p} |\partial^{p-l} g| \in C_\uparrow^\infty(\mathbb{R}^n) \subset L^q(\gamma)$. Therefore, $h\mathbf{1}_{G_a} \rightarrow 0$ in $L^q(\gamma)$ as $a \rightarrow \infty$. Together with (5.6), we have that $\lim_{a \rightarrow \infty} x^k \partial^p g_a = x^k \partial^p g$ in $L^q(\gamma)$. □

Proposition 5.5. (Approximation of identity of $C_\uparrow^m(\mathbb{R}^n)$ in $L^q(\gamma)$)

Suppose that $f(x) \in C_\uparrow^m(\mathbb{R}^n)$. Denote

$$B_\epsilon f = \beta_{\frac{1}{\epsilon}} \times A_\epsilon f.$$

Then $B_\epsilon f \in C_c^\infty(\mathbb{R}^n)$, and for $q \geq 1$ and $k, p \in \mathbb{N}^n$ such that $|p| \leq m$, $x^k \partial^p(B_\epsilon f) \rightarrow x^k \partial^p f$ in $L^q(\gamma)$, as $\epsilon \rightarrow 0$.

Proof. Lemma 5.2 implies that $A_\epsilon f \in C_\uparrow^\infty(\mathbb{R}^n)$. Then it follows from Lemma 5.4 that $B_\epsilon f \in C_c^\infty(\mathbb{R}^n)$ and $\|x^k \partial^p(B_\epsilon f) - x^k \partial^p(A_\epsilon f)\|_q \rightarrow 0$. Corollary 5.3 implies that

$$\|x^k \partial^p(A_\epsilon f) - x^k \partial^p f\|_q \rightarrow 0.$$

By the triangle inequality, we have that as $\epsilon \rightarrow 0$,

$$\begin{aligned} \|x^k \partial^p(B_\epsilon f) - x^k \partial^p f\|_q &\leq \|x^k \partial^p(B_\epsilon f) - x^k \partial^p(A_\epsilon f)\|_q + \|x^k \partial^p(A_\epsilon f) - x^k \partial^p f\|_q \\ &\rightarrow 0. \end{aligned}$$

□

5.2. The N-representation theorem for $S(\mathbb{R}^n)$ in $L^2(\gamma)$. Suppose

$$H_l(x) = \frac{(-1)^l}{\sqrt{l!}} e^{x^2/2} \frac{d^l}{dx^l} e^{-x^2/2}$$

is the l -th Hermite polynomial of one variable. It is well known that the set of Hermite polynomials of several variables

$$\left\{ \mathbf{H}_m := \prod_{k=1}^n H_{m_k}(x_k), \quad m = (m_1, \dots, m_n) \in \mathbb{N}^n \right\} \quad (5.7)$$

is an orthonormal basis of $L^2(\gamma)$. Thus, every function $u \in L^2(\gamma)$ has a unique series expression

$$u = \sum_{m \in \mathbb{N}^n} a_m \mathbf{H}_m, \quad (5.8)$$

where the coefficients a_m are given by

$$a_m = \int_{\mathbb{R}^n} u(x) \mathbf{H}_m(x) d\gamma(x).$$

Proposition 5.6. $u \in L^2(\gamma)$ satisfies that $a_m = \int_{\mathbb{R}^n} u(x) \mathbf{H}_m(x) d\gamma(x)$ is rapidly decreasing (i.e., for $r \in \mathbb{N}^n \geq 0$, $a_m = O(m^{-r})$ as $|m| \rightarrow \infty$) if and only if $u = f\rho^{-\frac{1}{2}}$ with $f \in S(\mathbb{R}^n)$.

Proof. Denote the Hermite functions $\mathcal{H}_m(x) = \mathbf{H}_m(x)\rho^{\frac{1}{2}}$, then

$$\int_{\mathbb{R}^n} u(x) \mathbf{H}_m(x) d\gamma(x) = \int_{\mathbb{R}^n} f(x) \mathcal{H}_m(x) dx.$$

The desired conclusion is followed from Theorem 3.5 and Exercise 3 of [10, p. 135].

□

Remark 5.7. Clearly, the smooth and compactly supported function satisfies the above condition. In fact,

$$C_c^\infty(\mathbb{R}^n) = \left\{ u = f\rho^{-\frac{1}{2}} : f \in C_c^\infty(\mathbb{R}^n) \right\} \subset \left\{ u = f\rho^{-\frac{1}{2}} : f \in S(\mathbb{R}^n) \right\}.$$

Proposition 5.8. If $u \in L^2(\gamma)$ satisfies that $a_m = \int_{\mathbb{R}^n} u(x) \mathbf{H}_m(x) d\gamma(x)$ is rapidly decreasing, then the Hermite expansion $u(x) = \sum_{m \in \mathbb{N}^n} a_m \mathbf{H}_m(x)$ satisfies that

$$\|x^k \partial^p(u - u_l)\|_{L^2(\gamma)} \rightarrow 0 \text{ as } l \rightarrow \infty, \quad \forall k, p \in \mathbb{N}^n,$$

where $u_l = \sum_{|m| \leq l} a_m \mathbf{H}_m(x)$.

Proof. Proposition 5.6 implies that $S(\mathbb{R}^n) \ni f = \sum_m a_m \mathcal{H}_m(x)$. Denote $f_l = u_l \rho^{\frac{1}{2}}$, then it follows from the N-representation theorem for $S(\mathbb{R}^n)$ (see Theorem V.13 of [17, p. 143]) that $f_l \rightarrow f$ in $S(\mathbb{R}^n)$ which means that: as $l \rightarrow \infty$,

$$\|x^m \partial^i (f - f_l)\|_{L^2(dx)} \rightarrow 0 \quad \forall m, i \in \mathbb{N}^n.$$

The Leibniz's rule implies that there exists a constant $c > 0$ such that

$$\|x^k \partial^p (u - u_l)\|_{L^2(\gamma)} \leq c \sum_{m \leq k+p, i \leq p} \|x^m \partial^i (f - f_l)\|_{L^2(dx)}.$$

Thus the desired conclusion follows. \square

References

1. Arató, M.: *Linear Stochastic Systems with Constant Coefficients. A Statistical Approach.* Lecture Notes in Control and Information Sciences, 45. Springer-Verlag, Berlin, 1982.
2. Arató, M.; Baran, S. and Ispány M.: Functionals of complex Ornstein-Uhlenbeck processes. *Comput. Math. Appl.* **37**(1999), no. 1, 1–13.
3. Arató, M.; Kolmogorov, A.N. and Sinay Yu.G.: Estimation of the parameters of a complex stationary Gaussian Markov process, *Dokl. Akad. Nauk. SSSR* **146**(1962), 747–750.
4. Balescu, R.: *Statistical Dynamics—Matter out of Equilibrium*, Imperial College Press, 1997.
5. Barros-Neto, J.: *An Introduction to the Theory of Distributions*, Marcel Dekker, Inc., New York. 1973.
6. Bell, D. R.: *The Malliavin Calculus*, Dover Publications, Inc., Mineola, New York. 2006.
7. Chen, Y.: Comment on “On the IR-ratio rule of power spectra”[*J. Math. Phys.* 50, 063301 (2009)], *J. Math. Phys.* **51**(2010), no.4, 044101.
8. Chen, Y. and Liu, Y.: On the eigenfunctions of the complex Ornstein-Uhlenbeck operators, *Kyoto J. Math.*, **54**(2014), no.3, 577–596.
9. Itô, K.: Complex multiple Wiener integral, *Japan J. Math.* **22**(1953), 63–86. Reprinted in: *Kiyosi Itô selected papers*, Edited by Daniel W. Stroock, S.R.S. Varadhan, Springer-Verlag, 1987.
10. Jones, D. S.: *The Theory of Generalised Functions*, 2nd ed, Cambridge University Press. 1982.
11. Kuo, H. H.: *Introduction to Stochastic Integration*, Springer. 2006.
12. Lunardi, A.: On the Ornstein-Uhlenbeck operator in L^2 spaces with respect to invariant measures, *Trans. Amer. Math. Soc.* **349**(1997), no.1, 155–169.
13. Malliavin, P.: *Stochastic Analysis*, Springer.1997.
14. Nualart, D.: *The Malliavin Calculus and Related Topics*, Springer-Verlag. 2006.
15. Pazy, A.: *Semigroups of Linear Operators and Applications to Partial Differential Equations.* AMS **44**. Springer. 1983.
16. Qian, M.-P. and Xie, J.-S.: On the IR-ratio rule of power spectra, *J. Math. Phys.* **50**(2009), no.6, 063301.
17. Reed, M. and Simon, B.: *Methods of Modern Mathematical Physics*, New York: Academic Press, Vol. 1. 1975
18. Riesz, F. and Nagy, B. S.: *Functional Analysis*, Dover Publications, Inc., New York. 1955.
19. Rudin, W.: *Functional Analysis*, 2nd Ed., McGraw-Hill Companies, Inc. 1991.
20. Shigekawa, I.: *Stochastic Analysis*, Translations of Mathematical Monographs, Vol.224. 1998.
21. Stein, E. M. and Shakarchi, R.: *Complex Analysis*, Princeton University Press, Princeton and Oxford.2003.
22. Stroock, D. W.: The Malliavin calculus, a functional analytic approach, *J. Funct. Anal.*, **44**(1981), no.2, 212–257.
23. Stroock, D. W.: Some applications of stochastic calculus to partial differential equations. In: *Ecole d'Eté de Probabilités de Saint-Flour XI-1981. Lecture Notes in Math.*, 976, 268–380, Springer-Verlag. 1983.

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