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BUCKLING THEORY OF ELASTIC CONTINUA WITH ISOTROPIC DAMAGE

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ABSTRACT

The way in which buckling occurs depends on how the structure is loaded and on its geometrical and material properties, in which many investigations have shown that the buckling load is mostly sensitive to the influence of the geometric imperfection. For the geometrical influence, Koiter (1945) has developed a post-buckling theory based on the energy approach. In this paper, by adopting Lemaitre's strain equivalence principle and Budiansky's virtual-work approach, we extend the Koiter's buckling theory to the case of isotropic damage body. Similar to Budiansky's buckling parameter $\rho = -\frac{\sigma_1 \tilde{\varepsilon}\,\hat{u}}{\sigma_c \varepsilon'' u_1^2}, \tilde{\varepsilon}' \equiv \varepsilon'[u]|_{u=0}$, for the first time, we have derived a damage-imperfection parameter $\rho = \frac{1-D}{\Gamma_c}\frac{\sigma_c \varepsilon_c^{*\prime} \hat{u} u_1}{\Xi}$, which shows that the damage parameter "1-D" has same status or sensitivity as geometric imperfection on buckling loads

***Keywords**:* buckling, elastic, isotropic, damage, imperfection, post-buckling

1. INTRODUCTION

As we known, for the thin wall structures the membrane stiffness is general several orders of magnitude greater than the bending stiffness. A thin wall structure can absorb a great deal of membrane strain energy without deforming too much. It must deform much more in order to absorb an equivalent amount of bending strain energy. If the structure is loaded in such a way that most of its strain energy is in the form of membrane compression, and if there is a way that this stored-up membrane energy can be converted into bending energy, the shell may fail rather dramatically in a process called "buckling", as it exchanges its membrane energy for bending energy. Very large deflections are generally required to convert a given amount of membrane energy into bending energy. [von Karman & Tsien (1939), Tsien (1942), Deuker (1943), Koiter (1945), Galletly *el al.* (1956,1959), Roorda (1965), Khot *el al.* (1969, 1970), van der Neut (1973), Bushnell (1985), Budiansky (1974), Hutchinson (1974), Arbocz (1974), Arbocz *el al.* (1987), Stumpf (1981), Sun (1989,1992) etc.]. The way in which buckling occurs depends on how the structure is loaded and on its geometrical and material properties. As was noted before, the initial geometric inperfections are often the main cause for the discrepancy between the test results and theoretical predictions for perfect shells. [Koiter (1945), Singer *el al.* (1971), Tvergaard (1973), Arbocz (1974), Danielson (1974), Seide (1974), Brush & Almroth (1975), Arbocz *el al.* (1987), etc]. The prebuckling process is often nonlinear if there is a reasonably large percentage of bending energy being stored in the structure throughout the loading history. According to the percentage of bending energy, the two basic way in which a conservative elastic system may loss its stability are: nonlinear collapse (snap-through, or over-the-lump) and bifurcation buckling. [Koiter (1945), Chien (1948), Budiansky & Hutchinson (1980), Arbocz *el al.* (1987) etc.]. Nonlinear collapse is predicted by means of a nonlinear analysis. The stiffness of the structure or the slope of the load-deflection curve, decrease with increasing load. At the collapse load the load-deflection curve has zero slope and, if the load is maintained as the structure deforms, failure of the

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structure is usually dramatic and almost instantaneous. This type of instability failure is often called "snap-through", a nomenclature derived from the many early tests and theoretical models of shallow arches, caps and cones. These very nonlinear systems initially deform slowly with increasing load. As the load approaches the maximum value, the rate of deformation increase until, reaching a status of neutral equilibrium in which the average curvature is almost zero, these shallow structures subsequently "snap-through" to a post-buckled state which resembles the original structure in an inverted form. The term "bifurcation buckling" refers to a different kind of failure, the onset of which is predicted by means of an eigenvalue analysis. At the buckling load, or bifurcation point on the load-deflection path, the deformations begin to grow in a new pattern which is quite different from the prebuckling pattern. Failure, or unbounded growth of this new deflection mode, occurs if the post-bifurcation load-deflection curve has a negative slope and the applied load is independent of the deformation amplitude. In general, the shallowest shell will snap-through, while the deeper shells will bifurcate. In this paper we shall restrict our discussion to those structures, which lose their stability by bifurcation.

The general theory buckling and post-buckling behavior of elastic structures enunciated by Koiter (1945) has spawned a considerable amount of research in this field, e.g., Sewell (1968), Thompson (1969), Budiansky & Hutchinson (1964), Budiansky (1965,1969), Fitch (1968), Cohen (1968), Masur (1973), Arbocz (1974, 1987), Budiansky (1974), Badur (1982), etc.]. Koiter (1945) shown that the buckling loads is very sensitive to the influence of geometric imperfections. Within the framework of Koiter, there is little work has been done on the general consideration of damaged body. One common understanding is that materials imperfection-damage has less influence than geometric imperfection. However, for isotropic damage body, this paper shows that materials damage has same position as geometric imperfection on the buckling loads.

To make the paper self-contained we find it important to repeat part of the results obtained previously. Thus in Sec.II we introduce the generalized principle and generalized stability criterion. In Sec.III, IV we perform buckling and post-buckling analysis of generalized system. In Sec. V generalized load-"shortening" relation has be obtained and modified. In Sec. VI some simplifications have adopted for various particular problems. Further, in Sec. VII we introduce initial imperfection and the sensitive relation between the buckling coefficients and the magnitude of the imperfection have been obtained and also modified.

2. GENERALIZED VARIATIONAL PRINCIPLE AND STABILITY CRITERION

Variational Principle

The all most of generalized variational principles have been studied in detail by Reissner (1953), Chien (1979), Hu (1954,1981), Washisu (1982). Assume $\varphi[u;\lambda]$ is the potential energy of the system $\Re$, u is the generalized displacement vector, λ is load parameter vector. In order to modified the solution the Lagrangian multiplies L can be introduced, then we have a functional, [Budiansky (1974)], $F = \phi[u;\lambda]$, then we have the variational principle of the system as follows $\delta F = F'[u;\lambda]\delta u = \phi'[u;\lambda]\delta u = 0$.

Stability Criterion

This problem is very difficulty. In our discussion the following stability criterion will be adopted. For all admissible $\delta u \neq 0'$, and if $F'\ \delta u = 0$ for the system in equilibrium state

$$\begin{aligned}\Delta F = F[u+\delta u;\lambda] - F[u;\lambda] &= F'[u;\lambda]\delta u + 0.5F''[u;\lambda](\delta u)^2 + \ldots \\ &= 0.5F''[u;\lambda](\delta u)^2 + R = \tfrac{1}{2}\phi''[u;\lambda](\delta u)^2 + R\end{aligned} \quad , \tag{1}$$

where R is the remainder of Taylor expansion. Then we have general stability criterion: (a) the equilibrium state is stability , if $\Delta F > 0$; (b) the equilibrium state is instability, if $\Delta F < 0$. For the finite-dimensional system, [Thompson (1969)], (a) and (b) can be translated into (i) the equilibrium state is stability, if $F''(\delta u)^2 > 0$; (ii) the equilibrium state is instability, if $F''(\delta u)^2 < 0$. Unfortunately, a non-negative second variation of F is only a necessary condition of stability for a continuous system. But (a) is a sufficiency and necessary condition of buckling for conservative system including finite-dimensional and continuous system. [Koiter (1963), Budiansky (1974)].

Principle of Virtual-Work

In most case, $F[u; \lambda]$ may be written as [Budiansky (1974)] $F[u; \lambda] = \Im[\varepsilon] - \lambda\Delta[u]$, in terms of the generalized shortening $\Delta[u]$, of the strains energy $\Im$. The strains energy considered to be a funtional of the strains ε. The variational principle is now $\delta F = F'[u;\lambda]\delta u = \Im'[\varepsilon]\delta\varepsilon - \lambda\Delta'[u]\delta u = 0$, where $\delta\varepsilon = \varepsilon'[u]\delta u$. If we introduce effective stress $\frac{\sigma}{1-D}$ for isotropic damage by using Lemaitre strain equivalent principle (Lemaitre, 1971), where $0< D <1$, is called damage parameter (Lemaitre, 1992), then we have principle of virtual works for isotropic damage body

$$(1-D)\sigma\delta\varepsilon - \Gamma\Delta'[u]\delta u = 0 \text{ or, } \sigma\delta\varepsilon - \frac{\Gamma}{1-D}\Delta'[u]\delta u = 0 \tag{2}$$

$$\sigma\delta\varepsilon - \lambda\Delta'[u]\delta u = 0,\ \lambda = \frac{\Gamma}{1-D}$$

For antisotropic damage model, the damage parameter D should be considered as a tensor(Murakami, 1981).

Formulae (2) is the form of principle of virtual work for isotropic damage model, stating that, in a equipbrium state, the damage of the potential energy of the loads associated with “virtual” displacement δu must equal to the internal virtual work of stress $(1 - D)\sigma$ acting through strain $\delta\varepsilon$ that are compatible.

According to Budiansky (1974), the symbol σ have three meanings: it is the stress state σ; it is a linear operator on stains, making $\sigma\delta\varepsilon$ the total work of the stresses acting through $\delta\varepsilon$; and it is a function $\sigma[\varepsilon]$ of strain. This understanding of the symbol σ is very useful in the deriving of buckling and post-buckling equations, and the buckling load of the practical problems.

3. BIFURCATION ANALYSIS

In order to discover conditions for bifurcation buckling, [Koiter (1945), Budiansky (1974)], we assume first that there exist a fundamental solution $u_0(\lambda)$, $\varepsilon_0(\lambda)$, $\sigma(\lambda)$ that varies smoothly with λ as the load increases from zero. The above variational equation requires that

$$\sigma_0\varepsilon'[u_0(\lambda)]\delta u - \lambda\Delta'[u_0(\lambda)] = 0, \tag{3}$$

for all δu. Now suppose that, for some range of λ, there is another solution

$$u = u_0(\lambda) + v(\lambda) \quad \sigma = \sigma_0(\lambda) + s(\lambda) \quad \varepsilon = \varepsilon_0(\lambda) + \tau(\lambda) \tag{4}$$

that intersects the fundamental one at generalized buckling load λ_c, is the since that $\lim_{\lambda\to\lambda_c}[v(\lambda)\ \ s(\lambda)\ \ \tau(\lambda)]^T = 0$. It will be further assumed that $v_0(\lambda)$, $\varepsilon_0(\lambda)$, $\sigma_0(\lambda)$ exist for λ greater than λ_c, so that a true bifurcation, rather than a limited point, is implied by (4) and the limitation. The bifurcation buckling mode will be defined as $u_1 = \lim_{\lambda\to\lambda_c}(v/\|v\|)$, and also define the associated stress ans strain modes by $[\sigma_1, \varepsilon_1] = \lim_{\lambda\to\lambda_c}[s/\|v\|, \tau/\|v\|]^T$, where $\|\ \|$ represents a suitable norm; note $\|u_1\| = 1$. Since the u, ε, σ given by eqn(4) must satisfy equilibrium

$$\{\sigma_0(\lambda) + s(\lambda)\}\varepsilon'[u_0(\lambda) + v(\lambda)]\delta u - \lambda\Delta'[u_0(\lambda) + v(\lambda)] = 0, \tag{5}$$

and, under the assumption that ε, σ, Δ and G are analytica in the vicinity of $u_0(\lambda)$, a Taylor-series expansion gives

$$\sigma = \sigma_0 + \sigma_0' s + \sigma_0'' s^2/2 + \ldots \quad \varepsilon' = \varepsilon_0' + \varepsilon_0'' v + \varepsilon_0''' v^2/2 + \ldots \Delta' = \Delta_0' + \Delta_0'' v + \Delta_0''' v^2/2 + \ldots \tag{6}$$

for sufficiently small $|\lambda-\lambda_c|$. Substitute eqn(6) into eqn(5) and note eqn(3), then gives

$$s\varepsilon_0'\delta u + \left[(\sigma_0 + s)\varepsilon_0'' - \lambda\Delta_0''\right]v\delta u + 0.5\left[(\sigma_0 + s)\varepsilon''' - \lambda\Delta_0'''\right]v^2\delta u \ldots = 0, \tag{7}$$

Dividing eqn(7) by $|v|$, and letting $\lambda \rightarrow \lambda_c$, then gives

$$\sigma_1 \varepsilon'[u_0(\lambda_c)]\delta u + \sigma[\varepsilon_c]\varepsilon''[u_0(\lambda_c)]u_1\delta u - \lambda \Delta''[u_0(\lambda_c)]u_1\delta u = 0, \tag{8}$$

as the generalized variational equations governing the buckling mode u_1 and the critical load λ_c. The notations $u_0(\lambda_c) = u_c$, $\varepsilon[u_0(\lambda_c)] = \varepsilon_c$, $\varepsilon'[u_0(\lambda_c)] = \varepsilon'_c$ $\sigma[\varepsilon_c] = \sigma_c$, $\Delta''[u_0(\lambda_c)] = \Delta''_c$ etc. are convenient; Dividing eqn(6) by $|v|$, we have relations $\sigma_1 = \sigma'_c\varepsilon_1$, $\varepsilon_1 = \varepsilon'_c u_1$, then eqn(8) becomes

$$\sigma_1 \varepsilon'_c \delta u + \sigma_c \varepsilon''_c u_1 \delta u - \lambda_c \Delta''_c u_1 \delta u = 0, \tag{9}$$

Using eqn(2)', we have

$$\Gamma_c = (1 - D)\lambda_c, \tag{9'}$$

Eqn(9) indicate the critical load of damage body is lower than of perfect one. In other words, the critical load of damage body is critical load by damage parameter (1-D), this conclusion is also valid for antisotropic damage body.

4. POST-BUCKLING ANALYSIS

In order to proceed with a determination of v in eqn(4) for $\lambda \neq \lambda_c$, introduce the scalar parameter ξ defined by [Koiter (1945), Budiansky (1974)] $\xi = \langle v, u_1 \rangle$, where the bracket symbol represents any bilinear inner product, the only restriction on this inner product is that $\langle u_1, u_1 \rangle \neq 0$. Indeed, it is particularly convenient to choose the norm $|v| = \langle v, v \rangle^{1/2}$, for then $\langle u_1, u_1 \rangle = 1$, and this choice will be assumed henceforth. Then it follows that $v = \xi u_1 + \tilde{v}$, so, we have $\langle \tilde{v}, u_1 \rangle = 0$, $\tilde{v}/\xi_c \rightarrow 0$, for $\lambda \rightarrow \lambda_c$. The parameter ξ is therefore a measure of the "amount" of buckling mode contained in the difference $u(\lambda)$-$u_0(\lambda)$ between the displacements on the bifurcated and the fundamental path, at a given value of λ. Similarly, s and τ can be written as $s = \xi\sigma_1 + \tilde{s}$, $\tau = \xi\varepsilon_1 + \tilde{\tau}$. Substitute these into eqn(4), gives

$$u = u_0(\lambda) + \xi u_1 + \tilde{v} \quad \sigma = \sigma_0(\lambda) + \xi\sigma_1 + \tilde{s} \quad \varepsilon = \varepsilon_0(\lambda) + \xi\varepsilon_1 + \tilde{\tau}, \tag{10}$$

The aim of the post-buckling analysis is now to discover how eqn(10) varies with λ along the bifurcated path, and, with $u_0(\lambda)$ considered known, this would appear to dictate a search for $\xi(\lambda)$, $\tilde{v}(\lambda)$, $\tilde{s}(\lambda)$, $\tilde{\tau}(\lambda)$. A small change in view-point is, however, very effective; consider ξ to be the independent variable, and look for ξ and $\tilde{v}(\lambda)$, $\tilde{s}(\lambda)$, $\tilde{\tau}(\lambda)$ as function of ξ. To this end, we will anticipate at least the asymptotic validity, for small ξ, of the perturbation expansions

$$\lambda = \lambda_c + \lambda_1\xi + \lambda_2\xi^2 + \ldots \quad \tilde{v} = \xi^2 u_2 + \xi^3 u_3 \ldots \quad \tilde{s} = \xi^2\sigma_2 + \xi^3\sigma_3 + \ldots \quad \tilde{\tau} = \xi^2\varepsilon_2 + \xi^3\varepsilon_3 + \ldots, \tag{11}$$

The full asymptotic expansion for the displacements is then

$$u = u_0(\lambda) + \xi u_1 + \xi^2 u_2 + \ldots \quad \sigma = \sigma_0(\lambda) + \xi\sigma_1 + \xi^2\sigma_2 + \ldots \quad \varepsilon = \varepsilon_0(\lambda) + \xi\varepsilon_1 + \xi^2\varepsilon_2 + \ldots, \tag{12}$$

Although we know that $\tilde{v}/\xi \rightarrow 0$ for $\xi \rightarrow 0$ or $\lambda \rightarrow \lambda_c$, there is of course no guarantee that these asymptotic expansions must involve only integral powers of ξ; but it will turn out that this prescription, will rare exceptions, is self-consistent, in the sense that a systematic procedure can be developed for the calculation of the terms in eqn (11). [Budiansky (1974)].

Each term in (7) of the variational equation of the problem will now, in turn, be expanded about λ_c

$$\sigma_0 = \sigma_c + (\lambda - \lambda_c)\dot{\sigma}_c + \frac{1}{2}(\lambda - \lambda_c)^2\ddot{\sigma}_c + \ldots \quad \varepsilon_0 = \varepsilon_c + (\lambda - \lambda_c)\dot{\varepsilon}_c + \frac{1}{2}(\lambda - \lambda_c)^2\ddot{\varepsilon}_c + \ldots$$
$$\Delta''_0 = \Delta''_c + (\lambda - \lambda_c)\dot{\Delta}''_c + \frac{1}{2}(\lambda - \lambda_c)^2\ddot{\Delta}''_c + \ldots. \tag{13}$$

in which the notations are

$$\sigma_0^{(n)} = \sigma^{(n)}[u_0(\lambda)] \quad \sigma_c^{(n)} = \sigma_0^{(n)}|_{\lambda=\lambda_c} \quad \dot{\sigma}_0^{(n)} = \frac{d}{d\lambda}\sigma^{(n)}[u_0(\lambda)] \quad \dot{\sigma}_c^{(n)} = \dot{\sigma}_c^{(n)}|_{\lambda=\lambda_c}, \tag{14}$$

etc. Then, we have

$$\begin{aligned} &\xi\sigma_1 + \xi^2\sigma_2 + \xi^3\sigma_3 + \ldots = \sigma_0' s + \frac{1}{2}\sigma_0'' s^2 + \ldots = \\ &= \left[\sigma_c' + (\lambda - \lambda_c)\dot{\sigma}_c' + \frac{1}{2}(\lambda - \lambda_c)^2\ddot{\sigma}_c' + \ldots\right](\xi\varepsilon_1 + \xi^2\varepsilon_2 + \ldots) \\ &+ \frac{1}{2}\left[\sigma_c'' + (\lambda - \lambda_c)\dot{\sigma}_c'' + \ldots\right](\xi\varepsilon_1 + \ldots)^2 + \ldots \end{aligned}, \tag{15}$$

From eqn(16), we get the following relations

$$\begin{aligned} &\sigma_2 = \sigma_c\varepsilon_2 + \lambda_1\dot{\sigma}_c'\varepsilon_1 + \frac{1}{2}\sigma_c''\varepsilon_1^2 \quad \sigma_3 = \sigma_c'\varepsilon_3 + \lambda_1\dot{\sigma}_c'\varepsilon_2 + \lambda_2\dot{\sigma}_c'\varepsilon_1 + \sigma_c''\varepsilon_1\varepsilon_2 + \frac{1}{2}\lambda_1\dot{\sigma}_c''\varepsilon_1^2 \ \ldots. \\ &\varepsilon_2 = \varepsilon_c' u_2 + \lambda_1\dot{\varepsilon}_c u_1 + \frac{1}{2}\varepsilon_c'' u_1^2 \quad \varepsilon_3 = \varepsilon_c' u_3 + \lambda_1\dot{\varepsilon}_c' u_2 + \lambda_2\dot{\varepsilon}_c' u_1 + \varepsilon_c'' u_1 u_2 + \frac{1}{2}\lambda_1\dot{\sigma}_c u_1^2 \ \ldots \end{aligned}, \tag{16}$$

Substitute eqn(12) and eqn(14) into eqn(7). The term of order ξ in this expression simply reproduce the buckling equation eqn(9). The terms of order ξ^2 give the result

$$\begin{aligned} &\left(\sigma_2\varepsilon_c' + \sigma_c\varepsilon_c'' u_2 - \lambda_c\Delta_c'' u_2 + \sigma_1\varepsilon_c'' u_1 + \frac{1}{2}\sigma_c\varepsilon_c''' u_1^2 - \frac{1}{2}\lambda_c\Delta_c''' u_1^2\right)\delta u \\ &+\lambda_1\left(\dot{\sigma}_c\varepsilon_c'' u_1 + \sigma_c\dot{\varepsilon}_c'' u_1 + \sigma_1\dot{\varepsilon}_c' - \Delta_c'' u_1 - \lambda_c\dot{\Delta}_c'' u_1\right)\delta u = 0 \end{aligned}, \tag{17}$$

set $'u = u_1$, and $\sigma_2\varepsilon_c' u_1 + \sigma_c\varepsilon_c'' u_2 u_1 + G_c'' u_2 u_1 - \lambda_c\Delta_c'' u_2 u_1 = \lambda_1\left(\sigma_1\dot{\varepsilon}_c' u_1 + \dot{\sigma}_c'\varepsilon_1^2\right) + 0.5\sigma_1\varepsilon_c'' u_1^2 + 0.5\sigma_c''\varepsilon_1^2$

Consequently, the coefficient λ_1 can be found as

$$\lambda_1 = -\frac{[3\sigma_1\varepsilon_c' u_1^2 + \sigma_c\varepsilon_c''' u_1^3 + \sigma_c''\varepsilon_1^3 - \frac{\Gamma_c}{1-D}\Delta''' u_1^3]}{2\Xi}, \tag{18}$$

where $\Xi = \dot{\sigma}_c\varepsilon_c'' u_1^2 + \sigma_c\dot{\varepsilon}_c'' u_1^2 + 2\sigma_1\dot{\varepsilon}_c' u_1 + \dot{\sigma}_c'\varepsilon_1^2 - \Delta_c'' u_1^2 - \frac{\Gamma_c}{1-D}\dot{\Delta}_c'' u_1^2$ or rewritten as

$$\Xi = \underline{\dot{\sigma}_c\varepsilon_c'' u_1^2 + \sigma_c\dot{\varepsilon}_c'' u_1^2 + 2\sigma_1\dot{\varepsilon}_c' u_1 + \dot{\sigma}_c'\varepsilon_1^2 - \Delta_c'' u_1^2 - \Gamma_c\dot{\Delta}_c'' u_1^2} + \underline{\Gamma_c\dot{\Delta}_c'' u_1^2 - \frac{\Gamma_c}{1-D}\dot{\Delta}_c'' u_1^2} = \underline{\Xi} - D(1-D)^{-1}\Gamma_c\dot{\Delta}''_c u_1^2$$

where the materials perfect body $\underline{\Xi} = \underline{\dot{\sigma}_c\varepsilon_c'' u_1^2 + \sigma_c\dot{\varepsilon}_c'' u_1^2 + 2\sigma_1\dot{\varepsilon}_c' u_1 + \dot{\sigma}_c'\varepsilon_1^2 - \Delta_c'' u_1^2 - \Gamma_c\dot{\Delta}_c'' u_1^2}$.

$$\Gamma_1 = -\frac{(1-D)(3\sigma_1\varepsilon_c' u_1^2 + \sigma_c\varepsilon_c''' u_1^3 + \sigma_c''\varepsilon_1^3) - \Gamma_c\Delta''' u_1^3}{2(\underline{\Xi} - (1-D)^{-1}D\Gamma_c\dot{\Delta}''_c u_1^2)} \tag{18'}$$

Since none of the functional $\sigma[\varepsilon]$, $\varepsilon[u]$, and $\Delta[u]$ depend explicitly on λ, we can write $\dot{\sigma}_c' = \sigma_c''\dot{\varepsilon}_c \quad \dot{\varepsilon}_c' = \varepsilon_c''\dot{u}_c \quad \dot{\Delta}_c' = \Delta_c''\dot{u}_c$. Hence Ξ can be expressed in the alternative form $\Xi = \dot{\sigma}_c\varepsilon_c'' u_1^2 + \sigma_c\varepsilon_c''' u_1^2\dot{u}_c + 2\sigma_1\varepsilon_c'' u_1\dot{u}_c + \sigma_c''\varepsilon_1^2\dot{\varepsilon}_c - \Delta_c'' u_1^2 - \frac{\Gamma_c}{1-D}\Delta_c''' u_1^2\dot{u}_c$. If $\lambda_1 = 0$, we have

$$\lambda_2 = -\frac{[2\sigma_1\varepsilon_c''u_1u_2 + \sigma_2\varepsilon_c''u_1^2 + \frac{2}{3}\sigma_1\varepsilon_c'''u_1^3 + \sigma_c''\varepsilon_1^2\varepsilon_c'u_2 + \sigma_c\varepsilon_c'''u_1^2u_2 + \frac{1}{6}\sigma_c'''\varepsilon_1^4 + \frac{1}{6}\sigma_c\varepsilon^{IV}u_1^4 - \frac{\Gamma_c}{1-D}\Delta_c'''u_1^2u_2 - \frac{1}{6}\frac{\Gamma_c}{1-D}\Delta_c^{IV}u_1^4]}{\Xi} \quad (19)$$

$$\Gamma_2 = -\frac{(1-D)[2\sigma_1\varepsilon_c''u_1u_2 + \sigma_2\varepsilon_c''u_1^2 + \frac{2}{3}\sigma_1\varepsilon_c'''u_1^3 + \sigma_c''\varepsilon_1^2\varepsilon_c'u_2 + \sigma_c\varepsilon_c'''u_1^2u_2 + \frac{1}{6}\sigma_c'''\varepsilon_1^4 + \frac{1}{6}\sigma_c\varepsilon^{IV}u_1^4] - \Gamma_c\Delta_c'''u_1^2u_2 - \frac{1}{6}\Gamma_c\Delta_c^{IV}u_1^4]}{\underline{\Xi} - (1-D)^{-1}D\Gamma_c\dot{\Delta}''_c u_1^2} \quad (19)'$$

In the case of $\lambda_1 = 0$, we have the governing equation of u_2

$$\sigma_2\varepsilon_c'\delta u + \sigma_c\varepsilon_c''u_2\delta u - (1-D)^{-1}\Gamma_c\Delta_c''u_2\delta u + \sigma_1\varepsilon_c''u_1\delta u + \frac{1}{2}\sigma_c\varepsilon_c'''u_1^2\delta u - \frac{1}{2}(1-D)^{-1}\Gamma_c\Delta_c'''u_1^2\delta u = 0, \quad (20)$$

This is the governing equation of post-buckling state, and it is a nonhomegeneous equation. There is an apparatus lack of uniqueness in the solution. Once u_2 is found, λ_2 as given by eqn(20) can be computed. In practice calculation, the u_i (i = 1, 2,...) can be found by perturbation procedure. It is seen that the post-buckling coefficients depend only on the buckling mode u_1, but the calculation of λ_2 would generally require the determination of u_2 as well, and so on. And an symmetric bifurcation corresponds to $\lambda_1 \neq 0$, and in symmetric bifurcation the load rises or falls during buckling according to the sign of λ_2. The results for λ_i are, of cause, not valid if $\Xi = \varphi_c''u_1^2 = 0$. The boundary influence coefficients will be zero if the boundary were satisfied.

5. GENERALIZED LOAD- "SHORTENING" RELATION

For some problems, the load- "shortening" relation may be interested. The load- "shortening" relation for perfect structure has been obtained by using the normal energy approaches [Budiansky (1974)]. Since $F[u; \lambda]$ is $F[u; \lambda] = \Im[u] - \lambda\Delta[u]$, where $\Delta[0] = 0$, and by analogy with simple model $\Delta[u]$ can be regarded as a generalized shortening. Assume further that _ does not depend explicitly on λ, i.e. $\Im = \Im[u]$, and it follows that $F[u; \lambda] = \Im[u] - \lambda\Delta[u]$, so we have generalized load- "shortening" $\Delta[u] = \partial_\lambda F[u; \lambda]$, but since $dF[u; \lambda]/d\lambda = F'[u; \lambda]du/d\lambda + \partial_\lambda F[u; \lambda]$, and $F'[u, \lambda]$ $(du/d\lambda)$ must vanish, then $\Delta[u] = -dF[u;\lambda]/d\lambda = -\dot{F}[u;\lambda]$. Notice along any equilibrium path $0 = dF'[u;\lambda]/d\lambda\delta u = \Delta'\delta u - F''u,_\lambda\delta u$. Now set $u = u_c$, and let $\delta u = u_1$, then we have $\Delta_c'u_1 = 0$, which says that at the critical load the generalized shortening is stationary with respect to the buckling displacement. Then in buckling state gives $F_c'u_1 = \sigma_c\varepsilon_c'u_1 - \lambda_c\Delta_c'u_1 = [\sigma_c\varepsilon_c']u_1 = 0$, which means that the stresses in inside and boundary of structures at bifurcation do no work through the initial strains of the buckling mode. With the prebuckling generalized shortening defined by $\Delta_0 = -dF[u_0(\lambda);\lambda]/d\lambda = -\dot{F}_0$, we have

$$\Delta - \Delta_0 = -d(F[u;\lambda] - F_0)d\lambda = [0.5\dot{F}_0''v^2 + (1/6)\dot{F}_0'''v^3 + ...], \quad (21)$$

and so, for small ξ, then $v \approx \xi u_1$ and we have

$$\Delta - \Delta_0 \approx -0.5\xi^2\dot{F}_c''u_1^2 = -\frac{1}{2}\xi^2\Xi = -\frac{1}{2}\xi^2\left\{\underline{\Xi} - D(1-D)^{-1}\Gamma_c\dot{\Delta}''_c u_1^2\right\}, \quad (22)$$

where $\Xi = \dot{\sigma}_c\varepsilon_c''u_1^2 + \sigma_c\varepsilon_c'''u_1^2\dot{u}_c + 2\sigma_1\varepsilon_c''u_1\dot{u}_c + \sigma_c''\varepsilon_1^2\dot{\varepsilon}_c - \Delta_c''u_1^2 - \frac{\Gamma_c}{1-D}\Delta_c'''u_1^2\dot{u}_c$ and the materials perfect body $\underline{\Xi} = \underline{\dot{\sigma}_c\varepsilon_c''u_1^2 + \sigma_c\dot{\varepsilon}_c''u_1^2 + 2\sigma_1\dot{\varepsilon}_c'u_1 + \dot{\sigma}_c'\varepsilon_1^2 - \Delta_c''u_1^2 - \Gamma_c\dot{\Delta}_c''u_1^2}$.

This relation is quite general including cases, in which the prebuckling variation of shortening with λ is not linear. [Budiansky (1974)]. Since damage parameter D must be always much smaller than 1, so "1-D" is less always less 1 and also positive; because Γ_c and $\dot{\Delta}''_c u_1^2$ are always positive, so we have

$$\underline{\Xi} - D(1-D)^{-1}\Gamma_c\dot{\Delta}''_c u_1^2 < \underline{\Xi}, \quad (23)$$

This relation indicates the general “shortening” of damage body is bigger than the perfect body, in other words, the damage body is little bit softer or weaker than the perfect one, will has the worst loading capability.

In the case of symmetric bifurcation, it is useful to define the initial postbuckling stiffness of the perfect structure as $K = \frac{d\lambda}{d\Delta}$ at $\lambda = \lambda_c$, and compare it to the corresponding prebuckling stiffness K_0 at $\lambda = \lambda_c$ on the fundamental path. we find $\frac{1}{K} = \frac{1}{K_0} - \frac{\Xi}{2\lambda_2}$, or $\frac{1}{K} = \frac{1}{K_0} - \frac{(1-D)}{2\Gamma_c}\Xi$, then we have the pre-buckling and post-buckling stiffness rate $\frac{K_0}{K} = 1 + \frac{(1-D)}{2\Gamma_2}\frac{\Xi}{\Delta_c' \dot{u}}$.

6. SPECIAL CASE

The above results found are very general, for more so than would usually require, and in various particular problems one or more of the following simplifications can be invoked: (i) linear stress-strain relation; (ii) quadratic strain-displacement relation; (iii) linear shortening-displacement relation. The above simplification means $\sigma^{(n)} = 0$ for $n> = 2$, $\varepsilon^{(n)} = 0$ for $n> = 3$, $\Delta^{(n)} = 0$, for $n> = 2$; respectively. According to Budiansky (1974), we known, in very many problems all three of these conditions are met. Limitation to elastic strains makes (i) valid in all but rubber like or hyperelastic materials. A quadratic strain-displacement (ii) relation is, of course, obtained when the Lagrangian strain tensor, or a simplied variant thereof, is employed. The linearity of $\Delta[u]$ occurs in dead-loading situations(though not in hydrostatic loading) or for conservative system. In any event, with (i)-(iii) valid we get

$$\lambda_1 = -\frac{3\sigma_1 \varepsilon''_c u_2^{\ 2}}{\Xi}, \ \lambda_2 = -\frac{2\sigma_1 \varepsilon''_c u_1 u_2 + \sigma_2 \varepsilon''_c u_1^2}{\Xi}, \tag{24}$$

or

$$\Gamma_1 = -(1-D)\frac{3\sigma_1 \varepsilon''_c u_2^{\ 2}}{\Xi}, \ \Gamma_2 = -(1-D)\frac{2\sigma_1 \varepsilon''_c u_1 u_2 + \sigma_2 \varepsilon''_c u_1^2}{\Xi} \tag{24'}$$

The buckling equation eqn(9) also implies to $\sigma_1 \varepsilon'_c \delta u + \sigma_c \varepsilon''_c u_1 \delta u = 0$, and post-buckling equation eqn(20) simples to $\sigma_2 \varepsilon'_c \delta u + \sigma_c \varepsilon''_c u_2 \delta u + \sigma_1 \varepsilon''_c u_1 \delta u = 0$. Another simplifying assumption, independent of (i)-(iii) above, is that of a (iv) linear fundamental state.

This assumption provides the common situation associated with buckling problems in which the prebuckling displacement, stresses, and strains vary linearly with λ, and the load λ appears linearly in the bifurcation eigenvalue problem. We will define assumption (iv) to mean a special kind of linearity associated with the function $\sigma[\varepsilon]$, $\varepsilon[u]$ and $\Delta[u]$, and the fundamental solutions u_0, σ_0, and ε_0, in which not only do the identities $\sigma[k\varepsilon_0] = k\sigma[\varepsilon_0], \varepsilon[ku_0] = k\varepsilon[u_0], \Delta[ku_0] = k\Delta[u_0]$ hold, for any constant k, but, in addition, the operators $\sigma_0^{(n)}$, $\varepsilon_0^{(n)}$, $\Delta_0^{(n)}$ are all independent of λ for $n > -1$. This guarantees that the fundamental solution of the field equations will be linear in λ, and it also makes the buckling equation a linear eigenvalue problem. Furthermore, we have relation $\sigma_c = \lambda_c \dot{\sigma}_c, \dot{\varepsilon}_0^{(n)} = \dot{\sigma}_0^{(n)} = \dot{\Delta}_0^{(n)} = 0$, for $n \geq 1$. If all the assumptions (i)-(iv) can be invoked, and set $\delta u = u_1$ in the buckling equation, and note the relation $\sigma_c \varepsilon''_c u_1^2 = \lambda_c \dot{\sigma}_c \varepsilon''_c u_1^2 = -\sigma_1 \varepsilon_1, \lambda_c \frac{\dot{\sigma}_c \varepsilon''_c u_1^2}{\sigma_1 \varepsilon_1} = -1$, then the buckling loads can be obtained in explicitly

$$\Gamma_c = (1-D)\frac{\sigma_1 \varepsilon_c u_1}{\dot{\sigma}_c \varepsilon''_c u_1^2} = -(1-D)\frac{\sigma_1 \varepsilon_1}{\dot{\sigma}_c \varepsilon''_c u_1^2}, \tag{25}$$

Γ_c is called damage buckling loads. The results for Γ_i (i = 1, 2,..) become

$$\frac{\Gamma_1}{\Gamma_c} = -(1-D)\frac{3\sigma_1\ddot{\varepsilon}_c u_1^2}{2\sigma_c\ddot{\varepsilon}_c u_1^2}, \tag{26}$$

for $\lambda_1 = 0$, we have

$$\frac{\Gamma_2}{\Gamma_c} = -(1-D)\frac{2\sigma_1\ddot{\varepsilon}_c u_1 u_2 + \sigma_2\ddot{\varepsilon}_c u_1^2}{\sigma_c\ddot{\varepsilon}_c u_1^2}, \tag{27}$$

and the generalized load- “shortening” is $\Delta - \Delta_0 \approx -0.5\xi^2\dot{\sigma}_c\ddot{\varepsilon}_c u_1^2$, and stiffness rate is $\frac{K_0}{K} = 1 - \frac{\Gamma_c}{\Gamma_2}\frac{\dot{\sigma}_c\ddot{\varepsilon}_c u_1^2 + 2\sigma_1\ddot{\varepsilon}_c u_1\dot{u}_c}{\sigma_c\dot{\varepsilon}_c}$.

If all the assumption (i)-(iv) hold, $\frac{K_0}{K} = 1 - \frac{\Gamma_c}{2\Gamma_2}\frac{\sigma_c\ddot{\varepsilon}_c u_1^2}{\sigma_c\varepsilon_c}$.

7. INITIAL IMPERFECTIONS

The real structures always have some imperfections. Koiter(1945) has show that small geometrically imperfections in some structures can be responsible for large reduction in their static buckling loads. As is well know, a this shell is often very imperfection-sensitive in this sense, with a perfect specimen some times having a “classical” buckling load several times higher than that of an imperfect one. Many analytical studies have sought to correlate reductions in buckling loads with assumed initial imperfections of various sizes and shapes. [Arbocz *el al.* (1987), Bushnell (1985), etc.]. If the structure under analysis is not quite perfect, in that it contains a displacement u before the application of load. The strain-displacement relations are [Budiansky (1974)] $\bar{\varepsilon} = \varepsilon[u + \bar{u}] - \varepsilon[\bar{u}]$, giving the strain ε of the imperfect structure in terms of the additional displacement u and the old function ε[u] of the perfect structure. Similarly, the generalized shortening is $\bar{\Delta} = \Delta[u + \bar{u}] - \Delta[\bar{u}]$. The stress-strain relation is $\sigma = \sigma[\mu]$, and so the variational equation for imperfect system is now

$$\sigma[\bar{\varepsilon}]\varepsilon'[u + \bar{u}]\delta u - \lambda\Delta'[u + \bar{u}]\delta u = 0, \tag{28}$$

where the Frechet derivative is with respect to u. It is appropriate, in many problem, to assert that in the pressure of an initial imperfection $\bar{u} = \xi\hat{u}$, where $\|\hat{u}\| = <\hat{u}, \hat{u}>^{1/2} = 1$, and then ξ is a good measure of the magnitude of the imperfection. A search will now be instituted for relations among the function u and the scalars λ and ξ that, in the vicinity of $\lambda = \lambda_c$, are dictated by the equilibrium condition eqn(28). Accordingly we write, in terms of the fundamental solution $u_0(\lambda)$ of the perfect structure $u = u_0(\lambda) + \bar{v}$, where now $\bar{v}$ can be made as small as desired by making both ξ and $|\lambda - \lambda_c|$ sufficiently small. No prejudgment is made concerning the uniqueness of $\bar{v}$ for given values of ξ and λ, but the double limit $\lim_{\lambda\to\lambda_C}\lim_{\xi\to 0}\frac{\bar{v}}{<\bar{v},\bar{v}>^{1/2}} \equiv v_1$ must be the same for each single-valued branch of v under consideration; it will be verified shortly that $v_1 = u_1$. For a sufficiently small v. Similarly, the associated stress and strain, and their buckling mode are

$$\sigma = \sigma_0(\lambda) + \bar{s} \quad \varepsilon = \varepsilon_0(\lambda) + \bar{\tau} \quad \lim_{\lambda\to\lambda_c}\lim_{\lambda\to\lambda_c}\left[\bar{s}/\|\bar{v}\|\ \bar{\tau}/\|\bar{v}\|\right] = [s_1\ \tau_1], \tag{29}$$

It will also be verified shortly that $s_1 = \sigma_1$, $\tau_1 = \varepsilon_1$. we now expand eqn (28) into

$$\begin{aligned}&[\sigma_0 + \bar{s}]\ \{\varepsilon'[u_0 + \bar{u}] + \varepsilon''[u_0 + \bar{u}]\bar{v} + 0.5\varepsilon'''[u_0 + \bar{u}]\bar{v}^2 + \ldots\}\delta u\\ &-\lambda\{\Delta'[u_0 + \bar{u}] + \Delta''[u_0 + \bar{u}]\bar{v} + 0.5\Delta'''[u_0 + \bar{u}]\bar{v}^2 + \ldots\}\delta u = 0\text{'}\end{aligned} \tag{30}$$

Now introduce the notation $\varepsilon^*[u_0, u]$ to denote the Frechet derivative of μ with respect to u, then

$$\varepsilon'[u_0+\bar{u}] = \varepsilon'[u_0] + \bar{\xi}\varepsilon'^*[u_0]\hat{u} + \frac{1}{2}\bar{\xi}^2\varepsilon'^{**}[u_0]\hat{u}^2 + \ldots = \varepsilon'_0 + \bar{\xi}\varepsilon'^*_0\hat{u} + \frac{1}{2}\bar{\xi}^2\varepsilon'^{**}_0\hat{u}^2 + \ldots, \tag{31}$$

and similarly introduce the notation g'^{*} and $\Delta'^{*}$to denote the Frechet derivative of g' and Δ' with respect to u, etc. Then eqn (30) becomes

$$\begin{aligned}&[\sigma_0+\bar{s}]\{\varepsilon'_0+\bar{\xi}\,\varepsilon'^*_0\hat{u}+0.5\bar{\xi}^2\varepsilon'^{**}\hat{u}^2+\ldots+[\varepsilon''_0+\bar{\xi}\varepsilon''^*_0\hat{u}+\ldots]\bar{v}+0.5[\varepsilon'''_0 v+\bar{\xi}\varepsilon'''^*_0\hat{u}+\ldots]\bar{v}^2+\ldots\}\delta u\\&-\lambda\{\Delta'_0+\bar{\xi}\Delta'^*_0\hat{u}+0.5\bar{\xi}^2\Delta'^{**}_0\hat{u}^2+..+[\Delta''_0+\bar{\xi}\Delta''^*_0\hat{u}+\ldots]\bar{v}+0.5[\Delta'''_0+\bar{\xi}\Delta'''^*_0\hat{u}+\ldots]\bar{v}^2+\ldots\}\delta u=0\,,\end{aligned} \tag{32}$$

Dividing by $<\bar{v},\bar{v}>^{1/2}$, letting $\bar{\xi}\to 0$ and then letting $\lambda \to \lambda_c$, and note eqn(3) gives

$$s_1\varepsilon'_c\delta u + \sigma_c\varepsilon''_c v_1\delta u - \lambda_c\Delta''_c v_1\delta u = 0, \tag{33}$$

and this verifies that v_1, s_1 are indeed the same as u_1, σ_1. We now introduce $\xi =< \bar{v}, v_1 >$ as in the perfect case, and try to find a relationship among λ, ξ and $\bar{\xi}$ that is uniformly valid for suitably restricted small values of ξ, $\bar{\xi}$ and $|\lambda-\lambda_c|$. More precisely, we will look for a dependence of λ on ξ and $\bar{\xi}$ along curves $\bar{\xi} = \alpha_0\xi^\gamma$, where α_0 is a scalar parameter, and the exponent γ will be chosen to suit our convenience. Expanding the operators in (33) about $\lambda = \lambda_c$ just like (14), gives

$$\begin{aligned}&[\sigma_c+(\lambda-\lambda_c)\dot{\sigma}_c+0.5(\lambda-\lambda_c)^2\ddot{\sigma}_c+\ldots]\{\alpha_0\xi^\gamma[\varepsilon'^*_c+(\lambda-\lambda_c)\dot{\varepsilon}'^*_c+\ldots]\hat{u}+0.5\bar{\xi}^2\varepsilon'^{**}_c\hat{u}^2+\ldots\\&+[\varepsilon''_c+(\lambda-\lambda_c)\dot{\varepsilon}''_c+\ldots]\bar{v}+\alpha_0\xi^\gamma[\varepsilon''^*_c\hat{u}+\ldots]\bar{v}+0.5[\varepsilon'''_c+(\lambda-\lambda_c)\dot{\varepsilon}'''_c+\ldots]\bar{v}^2+\ldots\}\delta u\\&+\bar{s}\ \{\varepsilon'_c+(\lambda-\lambda_c)\dot{\varepsilon}'_c+0.5(\lambda-\lambda_c)^2\ddot{\varepsilon}'_c+\ldots+\alpha_0\xi^\gamma[\varepsilon'^*_c+(\lambda-\lambda_c)\dot{\varepsilon}'^*_c]\hat{u}+[\varepsilon''_c+(\lambda-\lambda_c)\dot{\varepsilon}''_c+\ldots]\bar{v}\\&+0.5[\varepsilon'''_c+(\lambda-\lambda_c)\dot{\varepsilon}'''_c+\ldots]\bar{v}^2\ldots\}\delta u-\lambda\{\ [\Delta''_c+(\lambda-\lambda_c)\dot{\Delta}''_c+0.5(\lambda-\lambda_c)^2\ddot{\Delta}''_c+\ldots]\bar{v}\\&+0.5[\Delta'''+(\lambda-\lambda_c)\dot{\Delta}'''_c+\ldots]\bar{v}^2+\alpha_0\xi^\gamma[\Delta'^*_c+\ldots]\hat{u}+\ldots\}\delta u=0\end{aligned}\,, \tag{34}$$

If the choice of an integer value for $\gamma > 2$ is anticipated, it is appropriate to assume expansions in the form

$$\lambda = \lambda_c + \bar{\lambda}_1\xi + \bar{\lambda}_2\xi^2 + \ldots \quad \bar{v} = \xi u_1 + \xi^2 v_2 + \xi^3 v_3 + \ldots \quad \bar{s} = \xi\sigma_1 + \xi^2 s_2 + \ldots, \tag{35}$$

where $<v_n, u_1>=0$, and λ_i (i = 1, 2, 3...), as well as the v_n, will depend on α_0. The detail expansion of (34) after substitute (35) into (33) can be found in Sun(1992). If we pick $\gamma = 2$, and set $\delta u = u_1$, the terms of order ξ^2 gives

$$\bar{\lambda}_1 - \hat{\lambda}_1 + \alpha_0\rho\lambda_c + \alpha\bar{\lambda}_1 + \beta_1 = 0, \tag{36}$$

where

$$\rho = \frac{\sigma_c\varepsilon'^*_c\hat{u}u_1}{\lambda_c\Xi}, \text{ or } \rho = (1-D)\frac{\sigma_c\varepsilon'^*_c\hat{u}u_1}{\Gamma_c\Xi} \tag{37}$$

which is called imperfection-damage parameter, can be viewed as the natural extension of Budiansky's buckling parameter ρ.

So, we have the relation $\lambda \approx \lambda_c + \lambda_1\xi - \bar{\xi}\rho\lambda_c\xi^{-1}$ for small enough value of ξ and $\bar{\xi}$. The approximation can be improved systematically, appears adequate for asymptotic estimates of snapping loads λ_s.

$$\frac{\Gamma_s}{\Gamma_c} \approx 1 - 2\left(-(1-D)\xi\rho_b\frac{\Gamma_1}{\Gamma_c}\right)^{1/2} \quad \text{i.e.} \quad \left(\frac{\Gamma_s}{\Gamma_c} - 1\right)^2 = \sqrt{2}\left(-(1-D)\bar{\xi}\rho_b\frac{\Gamma_1}{\Gamma_c}\right), \tag{38}$$

where $\rho = (1-D)\rho_b$, $\rho_b = \dfrac{\sigma_c \overset{*}{\varepsilon_c} \hat{u} u_1}{\Gamma_c \Xi}$ can be called as Budiansky buckling parameter for damage body.

If $\lambda_1 = 0$, we can pick $\gamma = 3$, and v_2 is identified as u_2, and with $\delta u = u_1$, the terms of order ξ^3 gives $\overline{\lambda}_2 = \lambda_2 - \alpha_0 \rho \lambda_c$, with α_0 now replaced by $\bar{\xi}/\xi^3$, gives $\lambda \approx \lambda_c + \lambda_2 \xi^2 - \bar{\xi}\rho\lambda_c \xi^{-1}$ and, for $\lambda_2 < 0$, the snapping load becomes

$$\left(1 - \frac{\Gamma_s}{\Gamma_c}\right)^{3/2} \approx \frac{3}{2}\sqrt{3}\left(-\frac{\Gamma_2}{\Gamma_c}\right)^{1/2} (1-D)\rho_b \left|\bar{\xi}\right|, \tag{39}$$

In the (38) and (39), we can find the sensitivity (imperfection and boundary) discussed is clearly not only dependent (via ρ) on the choice of u for the shape of the imperfection, but also the damage parameter D or (1-D), and their interaction ρ.

The above results are remarkable, because it shows that damage has same influence as geometrical imperfections on the buckling loads, which is contrary to the statement in the literature which believe that the geometric imperfection is most sensitive effect on buckling. Eqn(40) let the perfect parameter (1-D) receive same status as the geometric imperfection. This can be explained physically as follows: if the perfect body has a isotropic damage D, it looks a little bit softer, it can be considered as body with a geometric imperfections (1-D), so that the damage does reduce the buckling as any general geometric imperfection. It is perfect agree with our own experience, the softer body will has lower buckling loads.

If the assumptions (i)-(iv) hold, and take $\hat{u} = u_1$ - that is, we consider the influence of an imperfection in the shape of the buckling mode. Then we have $\Xi = \dot{\sigma}_c \varepsilon_c'' u_1^2$, $\sigma_c \overset{*}{\varepsilon_c} \hat{u} u_1|_{\hat{u}=u_1} = \sigma_c \varepsilon_c'' u_1^2 = \lambda_c \dot{\sigma}_c \varepsilon_c'' u_1^2 = \lambda_c \Xi$ and the generalized parameter ρ becomes $\rho = 1$. This remarkable results is quite general for structures. [Budiansky (1974)]. It should be point out the generalized imperfection parameter that we obtained is different from Budiansky's one (1974). Although Budiansky get this result for normal case, and the final conclusion is correct, but his formula $\left[\rho = -\dfrac{\sigma_1 \tilde{\varepsilon}\, \hat{u}}{\sigma_c \varepsilon'' u_1^2}, \tilde{\varepsilon}' \equiv \varepsilon'[u]\|_{u=0}\right]$ is much different from our and not easy to be used to practical problem. Here we derived the buckling parameter in a very nature way, without assumption.

$$\left(\frac{\Gamma_s}{\Gamma_c} - 1\right)^2 = \sqrt{2}\left(-\bar{\xi}\frac{\Gamma_1}{\Gamma_c}\right) \tag{40}$$

$$\left(1 - \frac{\Gamma_s}{\Gamma_c}\right)^{3/2} \approx \frac{3}{2}\sqrt{3}\left(-\frac{\Gamma_2}{\Gamma_c}\right)^{1/2} \left|\bar{\xi}\right|, \tag{41}$$

8. CONCLUSIONS

Based on the systematic discussion, we have following conclusions:

1. The critical load of isotropic damage body is $\Gamma_c = (1-D)\lambda_c$, where λ_c is the critical load of corresponding materials perfect body, and D is damage materials parameter.

2. $\Delta - \Delta_0 \approx -\dfrac{1}{2}\xi^2 \left\{\Xi - D(1-D)^{-1}\Gamma_c \dot{\Delta}''_c u_1^2\right\}$, the generalized load-"shortening" relation for damage body is bigger than one of the materials pefect body. The physical interpretation is that damage body is softer.

3. $\rho = \dfrac{1-D}{\Gamma_c}\dfrac{\sigma_c \overset{*}{\varepsilon_c} \hat{u} u_1}{\Xi}$, the imperfection-damage parameter has been introduced, which implies that interaction

between geometric imperfection and damage, which can be considered as natural extension of Budiansky's buckling parameter for the case of damage body.

4. The most remarkable result is that damage parameter receives same status as geometric imperfection on the effect of buckling loads.

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