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HEAT TRANSFER IN THE AXISYMMETRIC FLOW OF A NON-NEWTONIAN SECOND-ORDER FLUID BETWEEN TWO ENCLOSED DISCS OSCILLATING WITH DIFFERENT ROTATORY TORSIONAL OSCILLATIONS

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Abstract

The heat transfer problem in the axisymmetric flow of a non-Newtonian incompressible second-order fluid between two enclosed discs oscillating with different rotatory torsional oscillations has been studied. The flow functions H, G, L, m and the energy functions ϕ, ψ are expanded in the powers of amplitude of the oscillation ε (taken small). The steady and unsteady parts of the velocity and energy functions have been calculated successfully. The behaviour of the dimensionless temperature T^*, Nu_a (average Nusselt's number on the lower disc) and Nu_b (average Nusselt's number on the upper disc) increasing or decreasing trend of τ_1(elastico-viscous parameter), τ (phase difference parameter), and N (rotatory torsional oscillation velocity ratio) is represented through graphs. The behaviour of the graphs is discussed along with the conclusions obtained.

Keywords: *Heat transfer, axisymmetric flow, second-order fluid, torsionally oscillating discs.*

1. INTRODUCTION

The non-Newtonian fluids differ from Newtonian fluids in at least two ways:(1) they exibit normal stress effects, such as rod climbing and die-swell and (2) they exibit shear thining or shear thickening which is the decrease or increase in viscosity with increasing shear rate respectively [1]. Both these phenomena introduce non-linearities into the equations. Non-Newtonian fluids models have

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been used extensively in many areas of chemical industries, polymer processing coal slurries based fuels etc.

The study of torsional oscillation of bodiew in fluid is of special interest as it provides a method of measuring the material constants of the fluids. The torsional oscillations of Newtonian fluids have been discussed by Rosenblat [2]. Sharma and Gupta [3] have considered the flow of a non-newtonian second-order fluid between two infinite torsionally discs. Sharma and Singh [4] have solved the problem of Sharma and Gupta [3] when the discs are subjected to uniform suction and injection. The flows past a torsionally oscillating plane, induced by torsional oscillation of an elliptic cylinder and past an oscillating cylinder have been discussed by Chawla [5], Riley, N. etl. [6] and Bluckburm [7] respectively.

Smith [8] have observed the experimental cooling of radial flow turbines. Soo et.al. [9] have investigated the nature heat transfer from an enclosed rotating disc for viscous fluid. Singh, Agrawal and Singh [10] have solved the problem of heat transfer in the flow of a non-Newtonian second-order fluid between two enclosed counter torsionally oscillating discs. Thereafter Singh and Agarwal [11] have discussed the problem of heat transfer in the flow of a non-Newtonian second-order fluid between two enclosed torsionally oscillating discs with different permeability in the presence of magnetic field. In the present paper the behaviour of temperature and Nusselt numbers on the lower and upper discs has been investigated when both the discs are enclosed in the cylindrical casing and performing different rotatory torsional oscillations in non-Newtonian second-order fluid.

2. FORMULATION OF THE PROBLEM

The constitutive equation of an incompressible non-Newtonian second-order fluid as suggested by Coleman and Noll [12] can be written as:

$$\tau_{ij} = -p\delta_{ij} + 2\mu_1 d_{ij} + 2\mu_2 e_{ij} + 4\mu_3 c_{ij} \tag{1}$$

where, $d_{ij} = 1/2\left(u_{i,j} + u_{j,i}\right)$

$$e_{ij} = 1/2\left(a_{i,j} + a_{j,i}\right) + u^m_{,i}\, u_{m,j} \tag{2}$$

and $c_{ij} = d_{im}\, d^m_{,j}$

p is the hydrostatic pressure, τ_{ij} is the stress-tensor, u_i and a_i are the velocity and acceleration vectors.

The constitutive equation (1) together with the momentum equation for no extraneous force:

$$\rho\left(\partial u_i/\partial t + u^m\, u_{i,m}\right) = \tau^m{}_{i,m} \tag{3}$$

and the equation of continuity for incompressible fluid

$$u^i_{,i} = 0 \qquad \text{.........} (4)$$

where, ρ is the density of the fluid and (,) represents covariant differentiation, form the set of governing equations.

In three dimensional cyclindrical set of co-ordinates (r, θ, z), the system consists of a finite disc of radius r_s (coinciding with the plane $z = 0$) performing rotatory oscillations of the type $r\Omega\, cos\tau$ of small amplitude $\varepsilon(= \Omega/n)$, about the perpendicular axis $r = 0$ with angular velocity Ω in an incompressible second-order fluid forming the part of a cylindrical casing or housing. The top of the casing (coinciding with the plane $z = z_0 < r_s$) performing rotatory oscillations of the type $Nr\Omega\, cos\tau$ placed parallel to and at a distance equal to gap-length z_0 from the lower disc. The symmetrical radial steady outflow has a small mass rate m of radial outflow ('$-m$'for net radial inflow). The inlet condition is taken as a simple radial source flow along z-axis starting from radius r_0. The lower disc $z = 0$ is maintained at constant temperature T_a while the upper disc $z = z_0$ at constant temperature T_b.

Assuming (u, v, w) as the velocity components to the cylindrical system of co-ordinates (r, θ, z), the boundary conditions of the problem are:

$$z = 0 \quad u = 0 \quad v = Real\; r\Omega\, e^{i\tau} \quad w = 0\; T = T_a$$

$$z = z_0 \quad u = 0 \quad v = Real\; Nr\Omega\, e^{i\tau} \quad w = 0\; T = T_b \tag{5}$$

Where the gap length z_0 is assumed small in comparision with the disc radius r_s. The velocity components for the axisymmetric flow compatible with the continuity criterion can be taken as:

$$U = -\xi H'(\zeta, \tau) + (R_m/R_z)M'(\zeta, \tau)/\xi$$

$$V = \xi G(\zeta, \tau) + (R_L/R_z)L(\zeta, \tau)/\xi$$

$$W = 2H(\zeta, \tau) \tag{6}$$

and for the temperature we take

$$T = T_b + (\nu_1\Omega/C_v)\{\phi(\zeta, \tau) + \xi^2\psi(\zeta, \tau)\} \tag{7}$$

where, $U = u/\Omega z_{0,}$ $V = v/\Omega z_{0,}$ $W = w/\Omega z_{0,}$ are dimensionless velocity components along the co-ordinate axis (r, θ, z) and $H(\zeta, \tau)$, $G(\zeta, \tau)$, $L(\zeta, \tau)$ and $M'(\zeta, \tau)$, $\phi(\zeta, \tau)$, $\psi(\zeta, \tau)$are the dimensionless functions of the dimensionless variables $\zeta = z/z_0$, $\xi = r/z_0$and $\tau = nt$. $R_m\,(= m/2\ \pi\rho z_0\,\tau_1)$, $R_L\,(= L/2\ \pi\rho z_0\,\tau_1)$ are dimensionless numbers to be called the Reynolds number of net radial outflow and circulatory flow respectively. $R_z (= \Omega {z_0}^2/\upsilon_1)$ is the flow Reynolds number and $\upsilon_1 (= \mu_1/\rho)$ is the kinematics Newtonian viscosity.

The small mass rate $'m'$ of the radial outflow is represented by

$$m = 2\pi\rho \int_0^{z_0} ru\, dz \tag{8}$$

Using the expression (6) and (7), the boundary condition (5) transform for boundary conditions for the velocity and energy functions as follows:

$$\begin{array}{ll}
G(0,\tau) = \text{real}\left(e^{i\tau}\right), & G(1,\tau) = N\ \text{real}\left(e^{i\tau}\right) \\
L(0,\tau) = 0, & L(1,\tau) = 0 \\
H(0,\tau) = 0, & H(1,\tau) = 0 \\
H^{'}(0,\tau) = 0, & H'(1,\tau) = 0 \\
\phi(0,\tau) = 1/E = S & \phi(1,\tau) = 0 \\
\psi(0,\tau) = 0 & \psi(1,\tau) = 0
\end{array} \tag{9}$$

where, $E[= \Omega v_1/\{c_v(T_a - T_b)\}]$ is the Eckert number.The conditions on M on the boundaries are obtainable from the equation (8) for m as follows:

$$M(0,\tau) - M(1,\tau) = 1 \tag{10}$$

Which on choosing the discs as streamlines reduces to

$$M(1,\tau) = 1 M(0,\tau) = 0 \tag{11}$$

Using equation (1) and expression (6) in equation (3) and neglecting the squares and higher powers of R_m/R_z (assumed small), the following set of dimensionless equations is obtained as:

$$\begin{aligned}
-(1/\rho z_0)\,\partial p/\partial\xi = {} & -n\Omega\, z_0\{\xi\partial H^{'} - (R_m/R_z)(\partial M^{'}/\xi)\} + \Omega^2 z_0 \xi \left(H^{'2} - 2HH^{'} - G^2\right) + \Omega^2 z_0 (R_m/R_z)(2HM^{''}/\xi) - \\
& \Omega^2 z_0 (R_L/R_z)(2LG/\xi) + v_1\,\Omega/z_0 \left\{H^{''}\xi - (R_m/R_z)(M^{'''}/\xi)\right\} - \\
& (2\,v_2/z_0)\big[(n\Omega/2)\{(R_m/R_z)(\partial M'''/\zeta) - \xi\partial H'''\} + \Omega^2\xi\left(H''^2 - HH^{iv}\right) + (R_m/R_z)(\Omega^2/\xi)\left(H^{'''}M^{'} + H^{''}M^{''} + H'M''' + HM^{iv}\right) - (R_L/R_z)(2\Omega^2/\xi)\left(L^{'}G^{'} + LG''\right)\big] - \\
& (4\,v_3\Omega^2/z_0)\left\{(R_m/R_z)(1/2\xi)\left(H^{'''}M^{'} + H^{'}M^{'''} + H''M''\right) - (R_L/R_z)(1/2\xi)\left(2L^{'}G^{'} + LG''\right) + (\xi/4)\left(H''^2 - G'^2 - 2H'H'''\right)\right\}
\end{aligned} \tag{12}$$

$$0 = -n\Omega\, z_0\{\xi\partial G - (R_L/R_z)(\partial L/\xi)\} - 2\Omega^2 z_0\xi(HG' - H'G) - \Omega^2 z_0(R_{\boldsymbol{m}}/R_z)(2M'G/\xi) - \Omega^2 z_0(R_L/R_z)(2HL'/\xi) + v_1\,\Omega/z_0\,\{\xi G'' + (R_L/R_z)(L''/\xi)\} + (2\,v_2/z_0)\big[(n\Omega/2)\{\xi G'' + (R_L/R_z)(\partial L''/\xi)\} + (R_L/R_z)(\Omega^2/\xi)\big(H''L' + H'''L + HL''' + H'L''\big) + (\Omega^2\xi)\big(HG''' - H''G'\big) + (R_{\boldsymbol{m}}/R_z)(2\Omega^2/\xi)\big(M'G'' + M''G'\big)\big] + (2\,v_3\Omega^2/z_0)\{\xi\big(H'G'' - H''G'\big) + (R_L/R_z)(1/\xi)\big(H''L' + H'''L + H'L''\big) + (R_{\boldsymbol{m}}/R_z)(1/\xi)\big(2M''G' + M'G''\big)\} \tag{13}$$

$$-(1/\rho z_0)\,\partial p/\partial\xi = 2n\Omega\, z_0\partial H + 4\Omega^2 z_0 HH' - 2\,v_1\Omega\, H''/z_0 - (2\,v_2/z_0)\{n\Omega\partial H'' + 2\Omega^2\xi^2\big(H''H''' + G'G''\big) + \Omega^2\big(22H'H'' + 2HH'''\big) - 2(R_m/R_z)\Omega^2\big(H''M''' + H'''M''\big) + 2(R_L/R_z)\Omega^2\big(L'G'' + L''G'\big)\} - (2\,v_3\Omega^2/z_0)\{\xi^2\big(H''H''' + G'G''\big) + 14HH'' - (R_{\boldsymbol{m}}/R_z)\big(H''M''' + H'''M''\big) + (R_L/R_z)(L'G''' + L''G')\} \tag{14}$$

The energy equation in terms of cylindrical set of co-ordinates (r, θ, z) for axisymmetric flow can be written as:

$$\rho C_v(\partial T/\partial t + u\,\partial T/\partial r + w\,\partial T/\partial z) = k\{\partial^2 T/\partial r^2 + (1/r\,)\,\partial T/\partial r + \partial^2 T/\partial z^2\} + \Phi \tag{15}$$

where,
$$\Phi = \tau_j^i\, d_i^j \tag{16}$$

C_v is the specific heat at constant volume, Φ be the viscous – dissipation function, τ_j^i is the mixed deviatoric stress tensor, k is the thermal conductivity, ρ is the density of the fluid. On differentiating equation (12) w.r.to ζ and (14) w. r. to ξ partially and then eliminating $\partial^2 p/\partial\zeta\partial\xi$ from the equation thus obtained, we get:

$$-n\Omega\, z_0\{\xi\partial H'' - (R_{\boldsymbol{m}}/R_z)(\partial M''/\xi)\} - 2\,\Omega^2 z_0\xi(HH'' + GG') + (R_{\boldsymbol{m}}/R_z)(2\,\Omega^2\, z_0/\xi)(H'M'' + HM''') - (\mathrm{R_L}/\mathrm{R_z})(2\,\Omega^2\, z_0/\xi)(LG' + L'G) - (v_1\,\Omega/z_0)\{(R_{\boldsymbol{m}}/R_z)\big(M^{iv}/\xi\big) - \xi H^{iv}\} - (2v_2/z_0)\big[(n\Omega/2)\{(R_{\boldsymbol{m}}/R_z)\big(\partial M^{iv}/\xi\big) - \xi H^{iv}\} - \Omega^2\xi\big(2H''H''' + H'H^{iv} + HH^{v} + 4G'G''\big) + (R_{\boldsymbol{m}}/R_z)(\Omega^2/\xi)\big(2H'''M'' + H^{iv}M' + 2H''M''' + 2H'M^{iv} + HM^{v}\big) - (R_L/R_z)(2\Omega^2/\xi)(2L'G'' + L''G' + \mathrm{L}G''')\big] - (2\,v_3\Omega^2/z_0)\{(R_{\boldsymbol{m}}/R_z)(1/\xi)\big(H^{iv}M' + 2H'''M'' + 2H''M''' + H'M^{iv}\big) - (R_L/R_z)(1/\xi)\big(3L'G'' + 2\mathrm{L}''\mathrm{G}' + \mathrm{L}\mathrm{G}'''\big) - \xi\big(H'H^{iv} + 3\mathrm{G}'\mathrm{G}'' + 2H''H'''\big)\} \tag{17}$$

On equating the coefficient of ξ and $1/\xi$ from the equation (13) and (17), the obtained equations are:

$$G'' = R\partial G + 2\varepsilon R(HG' - H'G) - \tau_1 \partial G'' - 2\,\varepsilon\tau_1\left(HG''' - H''G'\right) - 2\,\varepsilon\tau_2\left(H'G'' - H''G'\right) \tag{18}$$

$$L'' = R\partial L + 2\varepsilon R(M'G - HL') - \tau_1 \partial L'' - 2\,\varepsilon\tau_1\left(H''L' + H'''L + HL''' + H'L'' + 2M'G'' + 2M''G'\right) - 2\,\varepsilon\tau_2\left(H''L' + H'''L + H'L'' + 2M''G' + M'G''\right) \tag{19}$$

$$H^{iv} = R\partial H'' + 2\varepsilon R(HH''' + GG') - \tau_1 \partial H^{iv} - 2\,\varepsilon\tau_1\left(H'H^{iv} + HH^{v} + 2H''H''' + 4G'G''\right) - 2\,\varepsilon\tau_2\left(H'H^{iv} + 2H''H''' + 3G'G''\right) \tag{20}$$

$$M^{iv} = R\partial M'' + 2\varepsilon R(H'M'' + HM''' - LG' - L'G) - \tau_1 \partial M^{iv} - 2\,\varepsilon\tau_1\left(2H'''M'' + H^{iv}M' + 2H''M''' + 2H'M^{iv} + HM^{v} - 4L'G'' - 2L''G' - 2LG'''\right) - 2\,\varepsilon\tau_2\left(H^{iv}M' + 2H'''M'' + 2H''M''' + H'M^{iv} - 3L'G'' - 2L''G' - LG'''\right) \tag{21}$$

On substituting the expression (7) of temperature in equations (15), (16) and then equating the coefficient of ξ^2 and independent of ξ^2, the differential equations thus obtained are:

$$\psi'' = \varepsilon R P_r\left[\partial\psi/\varepsilon - 2H'\psi + 2H\psi' - H''^2 - G'^2 - \tau_1\left(H''\partial H'' + G'\partial G''\right) - 2\varepsilon\tau_1\left(H''H''^2 + H'G'^2 + HH''H''' + HG'G''\right) - 3\varepsilon\tau_2\left(H'H''^2 + H'G'^2\right)\right] \tag{22}$$

$$4\psi + \phi'' = \varepsilon R P_r[\partial\phi/\varepsilon + 2M'\psi + 2H\phi' - 12H'^2 + 2(R_m/R_Z)H''M'' - 2(R_L/R_Z)L'G' - \tau_1\{12H'\partial H' - (R_m/R_Z)\left(H''\partial M'' + M''\partial H_0''\right) + (R_L/R_Z)\left(G'\partial L'\right\}] \tag{23}$$

where, $P_r = \mu_1 C_v/k$ is the Prandtl number.

3. SOLUTION OF THE PROBLEM

To determine the value of the energy and velocity function Φ, Ψ and G, L, H and M we expand these functions in the ascending powers of the amplitude of the oscillation ε (assumed small) as:

$$G(\zeta,\tau) = \sum \varepsilon^{n} G_{n}(\zeta,\tau)$$

$$L(\zeta,\tau) = \sum \varepsilon^{n} L_{n}(\zeta,\tau)$$

$$H(\zeta,\tau) = \sum \varepsilon^{n} H_{n}(\zeta,\tau)$$

$$M(\zeta,\tau) = \sum \varepsilon^{n} M_{n}(\zeta,\tau)$$

$$\phi(\zeta,\tau) = \sum \varepsilon^{n} \phi_{n}(\zeta,\tau)$$

$$\psi(\zeta,\tau) = \sum \varepsilon^{\boldsymbol{n}} \psi_{\boldsymbol{n}}(\zeta,\tau) \tag{24}$$

Substituting the expression (24) into equations (18) to (23) and neglecting the coefficient of ε^2 (assumed negligible small) and equating the terms independent of ε and coefficient of ε, the set of equations is obtained as follows:

$$G_0{}'' = R\partial G_0 - \tau_1 \partial G_0{}'' \tag{25}$$

$$G_1{}'' = R\partial G_1 - 2R(H_0{}'G_0{}' - H_0G_0{}') - \tau_1\partial G_1{}'' - 2\tau_1(H_0G_0{}''' - H_0{}''G_0{}') - 2\,\tau_2(H_0{}'G_0{}'' - H_0{}''G_0{}') \tag{26}$$

$$L_0{}'' = R\partial L_0 - \tau_1\partial L_0{}'' \tag{27}$$

$$L_1{}'' = R\partial L_1{}' - 2R\left(G_0M_0{}' + H_0L_0{}'\right) - \tau_1\partial L_1{}'' - 2\tau_1(L_0H_0{}''' + H_0{}''L_0{}' + H_0{}''L_0{}'' + H_0L_0{}''' + 2G_0{}'M_0{}'' + 2M_0{}'G_0{}'') - 2\,\tau_2\left(H_0{}'''L_0 + H_0{}''L_0' + H_0{}'L_0'' + 2G_0\partial M_0'' + 2M_0'G_0{}''\right) \tag{28}$$

$$H_0{}^{iv} = R\partial H_0{}'' - \tau_1\partial H_0{}^{iv} \tag{29}$$

$$H_1{}^{iv} = R\partial H_1'' + 2R(H_0H_0{}''' + G_0G_0{}') - \tau_1\partial H_1{}^{iv} - 2\tau_1(H_0{}'H_0{}^{iv} + H_0H_0{}^{v} + H_0H_0''' + 4G_0{}'G_0{}'' - 2\,\tau_2\left(3G_0{}'G_0{}'' + H_0{}'H_0{}^{iv} + 2H_0^{''H_0'''}\right) \tag{30}$$

$$M_0{}^{iv} = R\partial M_0'' - \tau_1\partial M_0{}^{iv} \tag{31}$$

$$M_1{}^{iv} = R\partial M_1'' + 2R(H_0{}'M_0'' + H_0M_0'' - L_0'G_0 - L_0G_0{}') - \tau_1\partial M_1{}^{iv} - 2\tau_1\left(2H_0{}'''M_0'' + +2H_0{}'M_0{}^{iv} + 2H_0{}''M_0{}''' + H_0{}^{iv}M_0' + H_0M_0{}^{v} - 4L_0{}'G_0'' - 2L_0{}''G_0' - 2L_0G_0'''\right) - 2\,\tau_2\left(2H_0{}'''M_0'' + +2H_0{}''M_0''' + H_0{}^{iv}M_0' + H_0'M_0{}^{iv} - 3L_0'G_0'' - 2L_0''G_0' - L_0G_0'''\right) \tag{32}$$

$$\psi_0'' = RP_r\partial\psi_0 \tag{33}$$

$$\psi_1'' = RP_r[\partial\psi_1 - 2H_0{}'\psi_0 + 2H_0\psi_0' - H_0''^2 - G_0'^2 - \tau_1(H_0{}''\partial H_0'' - G_0{}'\partial G_0{}')] \tag{34}$$

$$4\psi_0 + \phi_0'' = RP_r\partial\phi_0 \tag{35}$$

$$4\psi_1 + \phi_1'' = RP_r[\partial\phi_1 + 2(R_m/R_L)M_0'\psi_0 + 2H_0\phi_0' - 12H_0'^2 + 2(R_m/R_z)H_0{}''M_0'' - 2(R_L/R_z)L_0{}'G_0' - \tau_1\{12H_0{}'\partial H_0' - (R_m/R_z)(H_0{}''\partial M_0'' + M_0''\partial H_0{}'') + (R_L/R_z)(G_0{}'\partial L_0{}')\}] \tag{36}$$

Taking

$$G_{\boldsymbol{n}}(\zeta,\tau) = G_{\boldsymbol{ns}}(\zeta) + e^{i\tau}G_{\boldsymbol{nt}}(\zeta)L_{\boldsymbol{n}}(\zeta,\tau) = L_{\boldsymbol{ns}}(\zeta) + e^{i\tau}L_{\boldsymbol{nt}}(\zeta)$$

$$H_{\boldsymbol{n}}(\zeta,\tau) = H_{\boldsymbol{ns}}(\zeta) + e^{2i\tau}H_{\boldsymbol{nt}}(\zeta)M_{\boldsymbol{n}}(\zeta,\tau) = M_{\boldsymbol{ns}}(\zeta) + e^{i\tau}M_{\boldsymbol{nt}}(\zeta)$$

$$\phi_{\boldsymbol{n}}(\zeta,\tau) = \phi_{\boldsymbol{ns}}(\zeta) + e^{2i\tau}\phi_{\boldsymbol{nt}}(\zeta)\psi_{\boldsymbol{n}}(\zeta,\tau) = \psi_{\boldsymbol{ns}}(\zeta) + e^{2i\tau}\psi_{\boldsymbol{nt}}(\zeta) \quad (37)$$

And then using expression (24) and the expression (37), the boundary conditions (9) and (11) for $\boldsymbol{n} = 0,1$ are transformed as follows:

$$\begin{array}{llll}
G_{0s}(0) = 0 & G_{0t}(0) = 1 & G_{1s}(0) = 0 & G_{1t}(0) = 0 \\
G_{0s}(1) = 0 & G_{0t}(0) = N & G_{1s}(1) = 0 & G_{1t}(1) = 0 \\
H_{0s}(0) = 0 & H_{0t}(0) = 0 & H_{1s}(0) = 0 & H_{1t}(0) = 0 \\
H_{0s}(1) = 0 & H_{0t}(1) = 0 & H_{1s}(1) = 0 & H_{1t}(1) = 0 \\
H'_{0s}(0) = 0 & H'_{0t}(0) = 0 & H'_{1s}(0) = 0 & H'_{0t}(0) = 0 \\
H'_{0s}(0) = 0 & H'_{0t}(0) = 0 & H'_{1s}(0) = 0 & H'_{0t}(0) = 0 \\
L_{0s}(0) = 0 & L_{0t}(0) = 0 & L_{1s}(0) = 0 & L_{1t}(0) = 0 \\
L_{0s}(1) = 0 & L_{0t}(1) = 0 & L_{1s}(1) = 0 & L_{1t}(1) = 0 \\
M_{0s}(0) = 0 & M_{0t}(0) = 0 & M_{1s}(0) = 0 & M_{1t}(0) = 0 \\
M_{0s}(1) = 1 & M_{0t}(1) = 0 & M_{1s}(1) = 0 & M_{1t}(1) = 0 \\
M'_{0s}(0) = 0 & M'_{1t}(0) = 0 & M'_{1s}(0) = 0 & M'_{1t}(0) = 0 \\
M'_{0s}(1) = 0 & M'_{1t}(1) = 0 & M'_{1s}(1) = 0 & M'_{0t}(0) = 0 \\
\psi_{0s}(0) = 0 & \psi_{0t}(0) = 0 & \psi_{1s}(0) = 0 & \psi_{1t}(0) = 0 \\
\psi_{0s}(1) = 0 & \psi_{0t}(1) = 0 & \psi_{1s}(1) = 0 & \psi_{1t}(1) = 0 \\
\phi_{0s}(0) = S & \phi_{0t}(0) = 0 & \phi_{1s}(0) = 0 & \phi_{1t}(0) = 0 \\
\phi_{0s}(1) = 0 & \phi_{0t}(1) = 0 & \phi_{1s}(1) = 0 & \phi_{1t}(1) = 0
\end{array} \quad (38)$$

Applying (37) and (38) in the equations (25) to (36), the values of the dimensionless velocity functions $G_{0s}(\zeta)$, $G_{0t}(\zeta)$, $L_{0s}(\zeta)$, $L_{0t}(\zeta)$, $H_{0s}(\zeta)$, $H_{0t}(\zeta)$, $M_{0s}(\zeta)$ and $M_{0t}(\zeta)$ are obtained as follows:

$$G_{0s}(\zeta) = L_{0s}(\zeta) = L_{0t}(\zeta) = H_{0s}(\zeta) = H_{0t}(\zeta) = M_{0t}(\zeta) = 0$$

$$G_{0t}(\zeta) = (\sinh a(1-\zeta) + \sinh a\zeta)/\sinh a$$

$$M_{0s}(\zeta) = 3\zeta^2 - 2\zeta^3 \quad (39)$$

where, $a = A + iB$

$$A = \left[R\{\tau_1 + (1+\tau_1^2)^{1/2}\}/2(1+\tau_1^2)\right]^{1/2}$$

$$B = \left[R\{(1+\tau_1^2)^{1/2} - \tau_1\}/2(1+\tau_1^2)\right]^{1/2}$$

$$G_0(\zeta,\tau) = \text{real part of } \{e^{i\tau}G_{0t}(\zeta)\} \tag{40}$$

To find the expression of the dimensionless temperature $T^* = \frac{T-T_b}{T_a-T_b}$, the dimensionless energy functions $\phi_0(\zeta,\tau)$, $\phi_1(\zeta,\tau)$, $\psi_0(\zeta,\tau)$ and $\psi_1(\zeta,\tau)$ are to be calculated from the differential equations (33) and (36) which do not contain $G_{1s}(\zeta)$, $G_{1t}(\zeta)$, $L_{1s}(\zeta)$, $L_{1t}(\zeta)$, $H_{1s}(\zeta)$, $H_{1t}(\zeta)$, $M_{1s}(\zeta)$ and $M_{1t}(\zeta)$. Hence there is no need to calculate velocity functions.

Using the values of velocity functions (39) and expression (37) the differential equations (33) to (36) are converted into the differential equations in terms of steady and unsteady part as follows:

$$\psi_{0s}(\zeta) = 0; \qquad \psi_{0t}'' - \left(\sqrt{2iRP_r}\right)^2 \psi_{0t} = 0 \tag{41}$$

$$\psi_{1s}(\zeta) = 0; \qquad \psi_{1t}'' - \left(\sqrt{2iRP_r}\right)^2 \psi_{1t} = -RP_r(1+i\tau_1){G_{0t}'}^2 \tag{42}$$

$$\phi_{0s}'' = 0; \qquad \phi_{0t}'' - \left(\sqrt{2iRP_r}\right)^2 \phi_{0t} = 0 \tag{43}$$

$$\phi_{1s}'' = 0; \qquad \phi_{1t}'' - \left(\sqrt{2iRP_r}\right)^2 \phi_{1t} = -4\psi_{1t} \tag{44}$$

The solutions of the above set of differential equations (41) to (44) subject to the boundary conditions (38) are given as follows:

$$\psi_{0s}(\zeta) = \psi_{0t}(\zeta) = \psi_{1s}(\zeta) = 0$$

$$\psi_{1t}(\zeta) = c_1\cosh a_1\zeta + c_2\sinh a_1\zeta - \frac{a^2RP_r(1+i\tau_1)}{2sinh^2a}\frac{1}{(4a^2-a_1^2)}[\cosh 2a(1-\zeta) + N^2\cosh 2a\zeta - 2N\cosh a(1-2\zeta)\,] + \frac{a^2RP_r(1+i\tau_1)}{2a_1^2sinh^2a}(1+N^2-2N\cosh a) \tag{45}$$

$$\phi_{0s}(\zeta) = S(1-\zeta)\,, \qquad \phi_{0t}(\zeta) = 0, \qquad \phi_{1s}(\zeta) = 0$$

$$\phi_{1t}(\zeta) = c_1\cosh a_1\zeta + c_2\sinh a_1\zeta - 4\left[\frac{c_1\zeta}{2a_1}\sinh a_1\zeta + \frac{c_2\zeta}{2a_1}\cosh a_1\zeta - \frac{a^2RP_r(1+i\tau_1)}{2sinh^2a}\frac{1}{(4a^2-a_1^2)^2}\{\cosh 2a(1-\zeta) + N^2\cosh 2a\zeta - 2N\cosh a(1-2\zeta)\}\right] + \frac{a^2RP_r(1+i\tau_1)}{2a_1^4sinh^2a}(1+N^2-2N\cosh a) \tag{46}$$

where, $a_1 = A_1(1+i) = \sqrt{2RP_r}(1+i)$

The constants c_1, c_2, c_3 and c_4 used in the set of differential equations (45) to (46) are determined with the help of boundary conditions (38) in terms of energy functions.

The energy functions

$$\psi(\zeta,\tau) = \varepsilon \text{ real part of } \left\{e^{2i\tau}\psi_{\mathbf{1}t}(\zeta)\right\} \tag{47}$$

$$\phi(\zeta,\tau) = \phi_{0s}(\zeta) + \varepsilon \text{ real part of } \left\{e^{2i\tau}\phi_{\mathbf{1}t}(\zeta)\right\} \tag{48}$$

provide us the dimensionless temperature as:

$$T^* = \frac{T-T_b}{T_a-T_b} = \frac{\nu_1\Omega}{C_\nu(T_a-T_b)}\{\phi(\zeta,\tau)+\xi^2\psi(\zeta,\tau)\} = E\{\phi(\zeta,\tau)+\xi^2\psi(\zeta,\tau)\}$$

The average Nusselt's number Nu_a on the lower disc and Nu_b (Nusselt's number on the upper disc) are obtained as:

$$Nu_a = -E\left[\{\phi'(\zeta,\tau)\}_{\zeta=0} + \left\{\frac{(\xi^2+\xi_0{}^2)}{2}\{\psi'(\zeta,\tau)\}_{\xi=0}\right\}\right]$$

$$Nu_b = -E\left[\{\phi'(\zeta,\tau)\}_{\zeta=1} + \left\{\frac{(\xi^2+\xi_0{}^2)}{2}\{\psi'(\zeta,\tau)\}_{\xi=1}\right\}\right]$$

4. RESULTS AND DISCUSSION

The variation of the dimensionless temperature T^* with ζ for different values of elastico-viscous parameter $\tau_1 = 1,3,5$ at $\tau = \pi/3$, $P_r = 6$, $N = 2, E = 5$, $\xi = 2$, $\varepsilon = 0.01$, $R = 5$ is represented through Fig. 1. It is evident from this figure that the temperature decreases with an increase in τ_1 throughout the gap-length and attains maximum value 1.0 at the lower disc while minimum in the interval $0.5 \le \zeta \le 0.6$. The behaviour of temperature for different values of torsional oscillation parameter $N = 2,3,4$ at the rest identical parameter and numbers shown in Fig. 3 is similar to it's behaviour with τ_1 represented in Fig. 1.

Fig. 2 depicts the behaviour of T^* with ζ for different values of phase difference parameter $\tau = \pi/12\,,\pi/6\,,\pi/3$ at $\tau_1 = 3$, $P_r = 6, N = 2$, $E = 5$, $\varepsilon = 0.01$, $\xi = 2, R = 5$. It is observed from this figure that the temperature is maximum at the lower disc, decreases with an increase in ζ upto the approximate value of $\zeta = 0.6$ and start increasing thereafter upto the upper disc. It is also seen that T^* decreases with an increase in $\tau \in [\pi/12\,,\pi/6]$ wherever at $\tau = \pi/3$ the value of T^* lie between it's value at $\tau = \pi/12$ and $\tau = \pi/6$.

The behaviour of Nusselt's number Nu_a with ξ at the lower disc for different values of elastico-viscous parameter $\tau_1 = 1,3,5$ at $\tau = \pi/3$, $P_r = 6$, $N = 2$, $R = 5$, $E = 5, \xi_0 = 5$, $\varepsilon = 0.01$ is represented through Fig. 4 and for different values of $N = 2,3,4$ at $\tau_1 = 3$ with rest similar parameter values through Fig. 6. The average Nusselt's number Nu_a is increasing with an increase in τ_1 and N in Fig. 4 and Fig. 6 respectively. It is also evident from these figures that the heat is flowing from the lower disc to fluid and then fluid to the upper disc of the system. The behaviour of Nu_a with τ is shown in Fig. 5. It is clear from this figure that Nu_a is also increasing with increase in $\tau \in [\pi/12, \pi/3]$. Thus the heat flux is flowing from the lower disc toward the upper disc in this case also.

The variation of average Nusselt's number Nu_b with ξ for different values of elastico-viscous parameter $\tau_1 = 1,3,5$ at $\tau = \pi/3$, $P_r = 6$, $N = 2$, $R = 5$, $E = 5$, $\xi_0 = 5$, $\varepsilon = 0.01$ is represented through Fig. 7. It is evident from this figure that at $\tau_1 = 3$, Nu_b is positive throughout the gap-length which shows that at $\tau_1 = 3$ the heat is flowing from the lower disc to upper disc via fluid wherever the heat flux is flowing from upper disc towards the lower disc at $\tau_1 = 1, 5$. Fig.8 shows that heat is flowing from lower disc towards the upper disc at $\tau = 0, \pi/6, \pi/3$ and the value of Nu_b increases with an increase in ξ.

The variation of average Nusselt's number Nu_b with ξ for different values of elastico-viscous parameter $N = 2, 3, 4$ at $\tau = \pi/3$, $P_r = 6$, $N = 2$, $R = 5$, $E = 5$, $\xi_0 = 5$, $\varepsilon = 0.01$ is represented through Fig. 9. It is seen through this figure that Nu_b is positive at $N = 2$ wherever Nu_b is negative at $N = 3, 4$. It is also observed from this figure that Nu_b is negative for $N \geq 3$ and Nu_b is positive for $N = 2$. It is also exhibited from this figure that for $N \geq 3$, Nu_b decreases with an increase in N and heat is being flown from the upper disc towards the lower disc in the whole of the radial region.

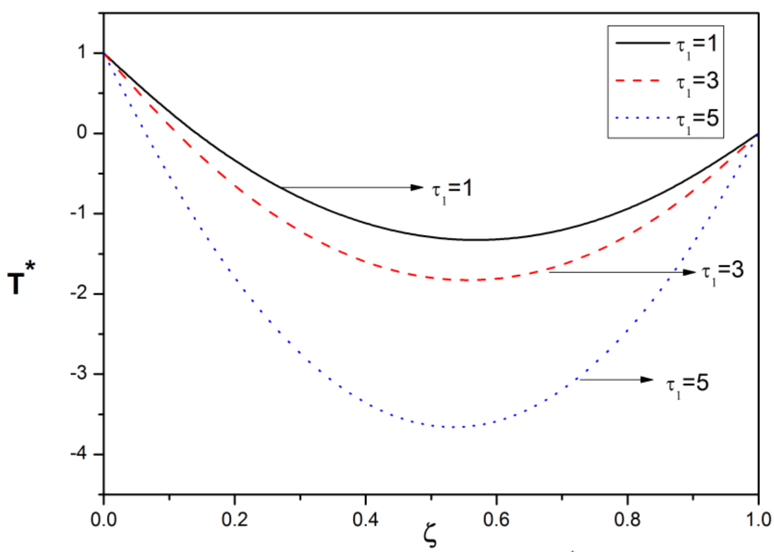

Fig.1 Variation of dimensionless temperature $\mathbf{T}^*$ with ζ for different values of elastico-viscous parameter τ_1

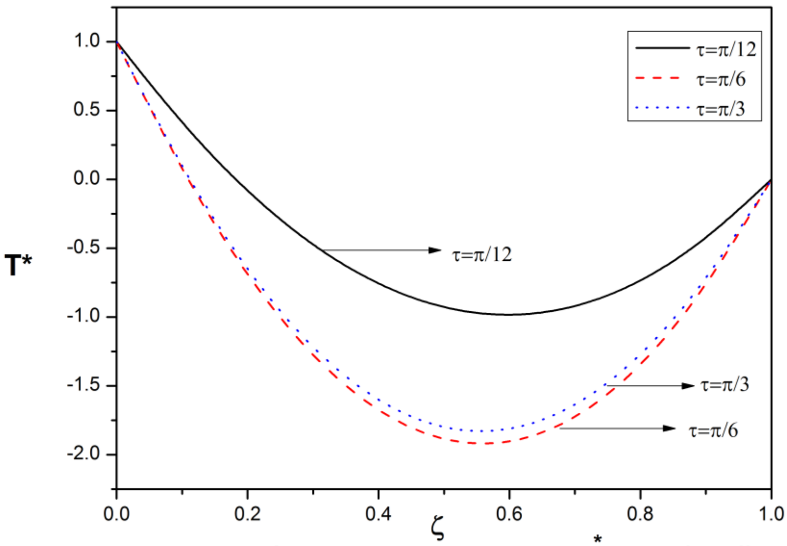

Fig. 2 Variation of dimensionless temperature $\mathbf{T}^*$ with ζ for different values of phase difference parameter τ

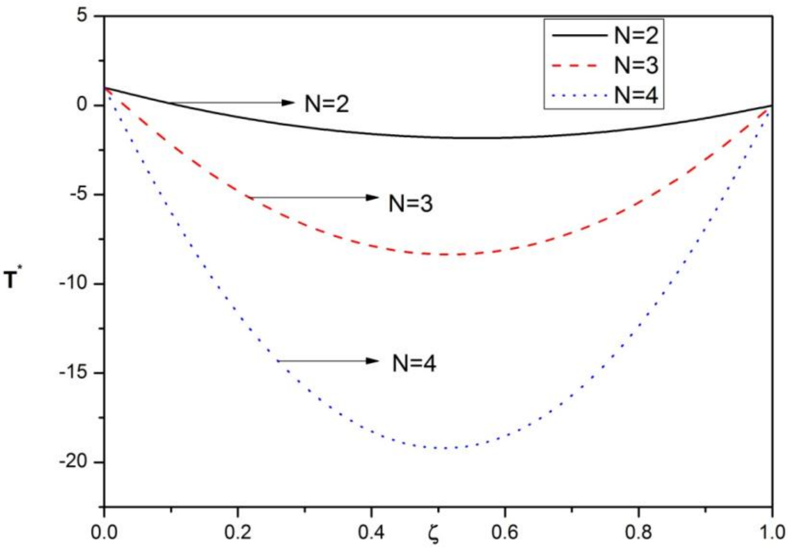

Fig. 3 Variation of dimensionless temperature $\mathbf{T}^*$ with ζ for different values of torsional oscillation parameter N

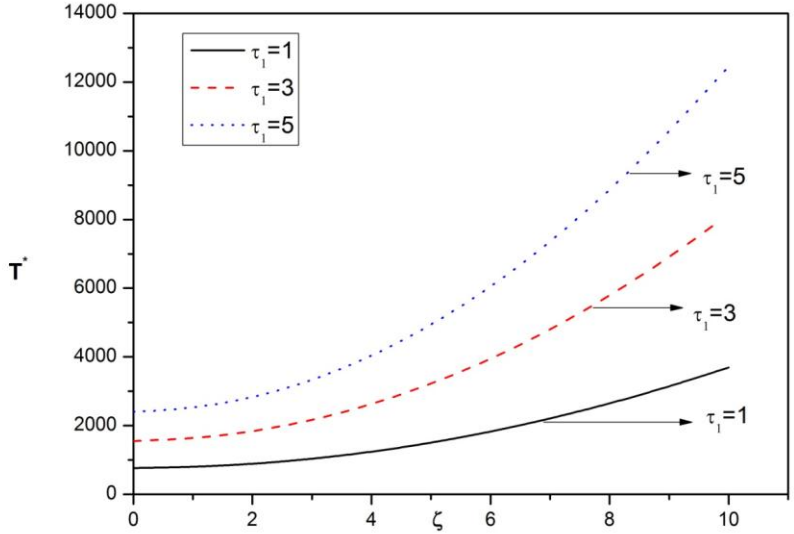

Fig.4 Variation of Nusselt's number $\mathbf{Nu_a}$ with ξ for different values of elastico-viscous parameter τ_1

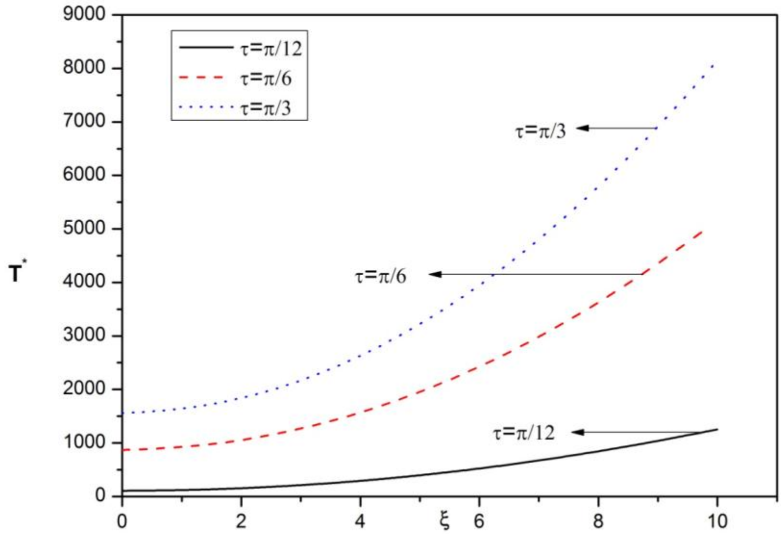

Fig.5 Variation of Nusselt's number $\mathbf{Nu_a}$ with ξ for different values of phase difference parameter τ

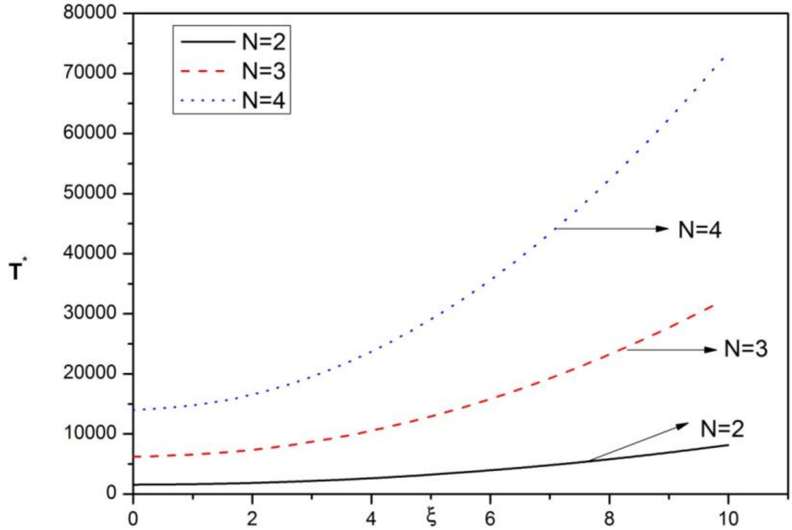

Fig.6 Variation of Nusselt's number $\mathbf{Nu_a}$ with ξ for different values of torsional oscillation parameter N.

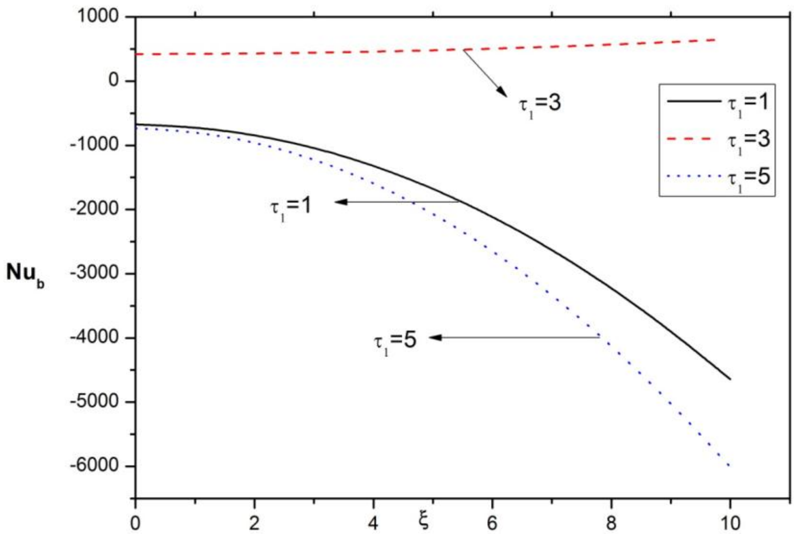

Fig.7 Variation of Nusselt's number $\mathbf{Nu_b}$ with ξ for different values of elastico-viscous parameter τ_1

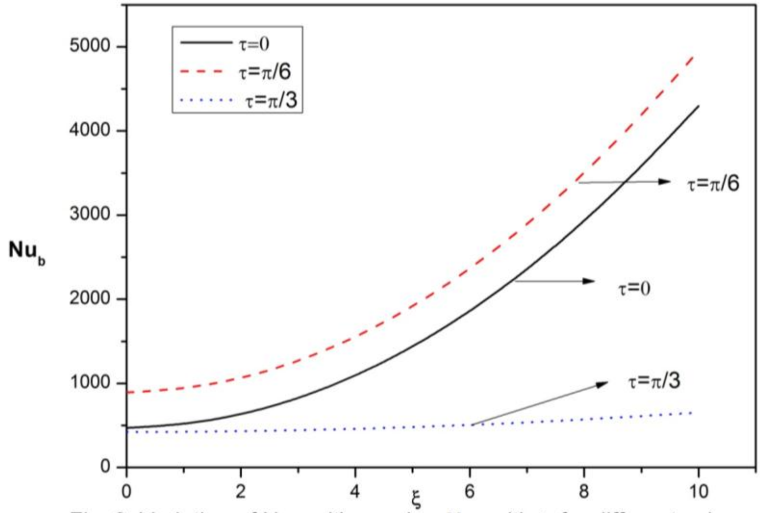

Fig. 8 Variation of Nusselt's number $\mathbf{Nu_b}$ with ξ for different values of phase difference parameter τ

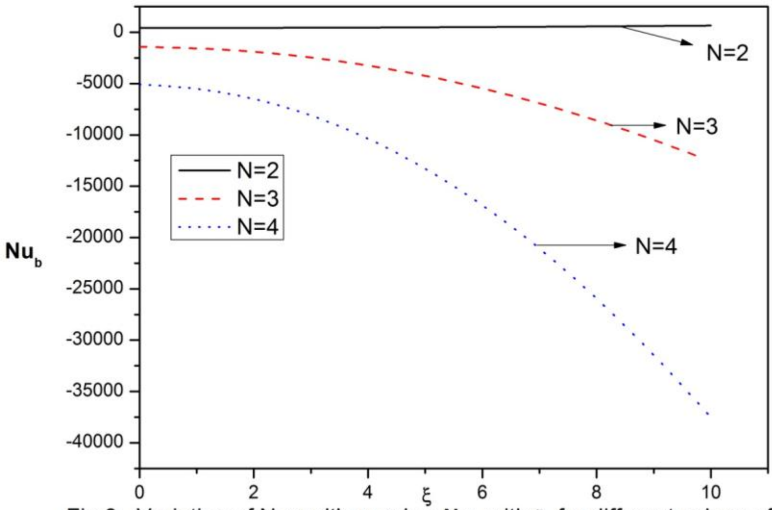

Fig.9 Variation of Nusselt's number $\mathbf{Nu_b}$ with ξ for different values of torsional oscillation parameter N

5. CONCLUSION

The results of the present paper at $N = -1$ are in good agreement with those of Singh and Vikas [14] for m (constant magnetic field) = 0.

From Fig. 1, 2, 3 and 4 it is concluded that temperature is minimum in the middle of the gap-length and decreases with an increase in τ_1 and N. Hence the fluid becomes more and more cooler with an increase in elastic-viscous parameter τ_1 and rotator oscillation ratio N. It is also evident from $Nu_a > 0$, that the heat flux is flowing from the lower disc in the fluid and fluid to the upper disc wherever Fig. 7 and Fig. 9 show that the heat is being transferred from upper disc to fluid and fluid to the lower disc at $\tau_1 = 1, 5$ and $N = 3, 4$ respectively. The behaviour of heat flux in Fig. 7, 8, 9 is reversed at $N = 2$, $\tau_1 = 3$ and $\tau = \pi/3$.

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