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Complex Dynamics for Sierpinski Curve using Jungck Ishikawa Orbit

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Abstract : In this paper we are describing the structure and dynamical behaviour of the relational maps F(z) = $z^n - z^d + c$ and F($z = z^n - z^d + c/z^n$ where $n >= 2$, $d = 1$ and c € C on the Julia set of F(z). We have presented the generalisation of these relational maps for $n >= 2$ using Jungck Ishikawa iterative scheme. This paper also mathematically analysed the characteristics of the different images which are formed in the complex plane c by using Jungck Ishikawa Iteration. Escape criteria theorems are also proved for the given relational maps using above iteration.

Keywords: *Complex dynamics, Jungck –Ishikawa Iteration, Sierpinski curve.*

1. INTRODUCTION

We have seen a lot of research papers in previous years where the structure of various relational maps families such as $z \rightarrow z^n - z^d + c$ and z -> $z^n - z^d + c/z^n$ where $n> = 2$, $d > =1$ and c is having complex values. We have also seen that there are many iterative schemes used for approximating the fixed point. Here we are exploring Jungck-Ishikawa iterative scheme which is two step methods in which we break the function into two sub functions. Here our main motive is to a discuss Mandelbrot and Julia set in the parameter plane.

2. PRELIMINARIES

2.1. Ishikawa Iteration [1]

Let X be a subset of real or complex numbers and T : X $\rightarrow$ X for $x_0 \in$ X, we have the sequences $\{x_n\}$ and $\{y_n\}$ in X in the following manner:

$$\left.\begin{aligned} x_{n+1} &= \alpha_n T y_n + (1-\alpha_n)x_n \\ y_n &= \beta_n T x_n + (1-\beta_n)x_n \end{aligned}\right\}$$

where $0 \leq \beta_n \geq 1$ and $0 \leq \alpha_n \geq 1$ and α_n & β_n both convergent to non zero number.

2.2. Jungck Ishikawa Iteration [1]

Let (X, ||.||) be a Banach space and Y an arbitrary set. Let S, T: Y $\rightarrow$ X be two non self mappings such that T(Y) $\subseteq$ S(Y), S(Y) is a complete subspace of X and S is injective. Then for $x_o \in$ Y, define the sequence $\{Sx_n\}$ iteratively by

$$\left.\begin{aligned} Sx_{n+1} &= \alpha_n Ty_n + (1-\alpha_n)Sx_n \\ Sy_n &= \beta_n Tx_n + (1-\beta_n)Sx_n \end{aligned}\right\}$$

where $n = 0, 1 \ldots\ldots$ and $0 \leq \beta_n \geq 1$ and $0 \leq \alpha_n \geq 1$ and α_n & β_n both convergent to non zero number.

2.3. Mandelbrot Set [7]

The Mandelbrot set arises as the parameter plane for the simple quadratic family $z^2 + c$ and is the set of all complex parameters for which the orbit of the point 0 does not escape to ∞. In 1980 this set was given by Benoit Mandelbrot. Mandelbrot set is one of the most interesting objects of research in mathematics nowadays. It is the set of all values of c for which the Julia set is a connected set. Multibrot set is the set of values in the complex plane whose absolute value remains below some finite value throughout iterations by a member of the general monic univariate polynomial family $z^d + c$ where $n >= 2$.

2.4. Julia Sets [2]

For the family $z^2 + c$, the set of all points whose orbit tends to infinity is called the basin of infinity. The Julia set is the boundary of this basin. In other word Julia set is boundary of the set of points that escapes to infinity. The filled Julia set is the set of all point whose orbit does not tend to infinity.

3. EXPERIMENTAL ANALYSIS OF THE FUNCTION F(z) = $z^2 - z + c$

For the function $F(z) = z^2 - z + c$ it is seen that c plane lies in the Mandelbrot set if point 0 orbit does not tend to infinity. Figure 1, Figure 2, Figure 3 shows c plane fractal using Jungck Ishikawa iterative scheme where different values of α and β are chosen. In theorem 1, escape criteria for $F(z) = z^n - z^d + c$. is proved when $n = 2$ and $d = 1$.

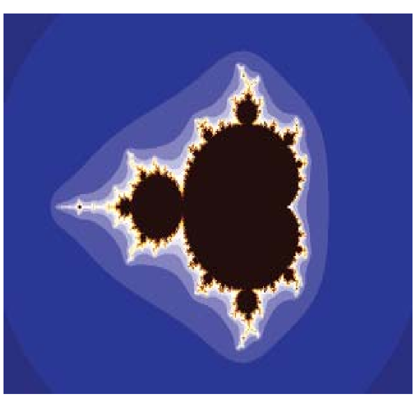

Figure 1: Mandelbrot set for $\alpha = 1$, $\beta = 1$ and $n = 2$

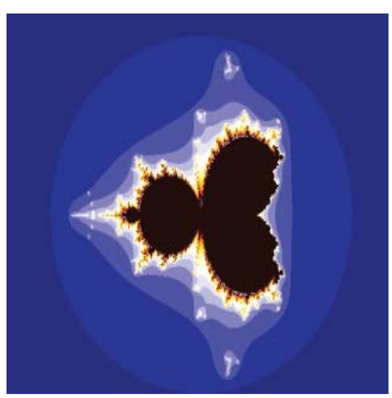

Figure 2: Mandelbrot set for $\alpha = .5$, $\beta = .5$ and $n = 2$

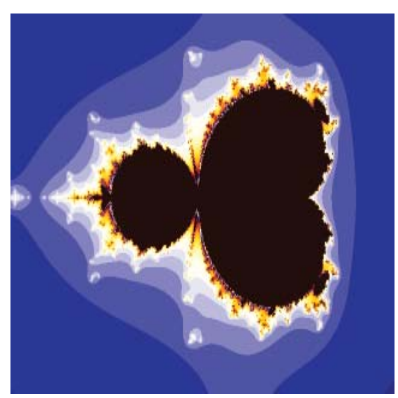

Figure 3: Mandelbrot set for $\alpha = .75$, $\beta = .5$ and $n = 2$

JULIA SETS :

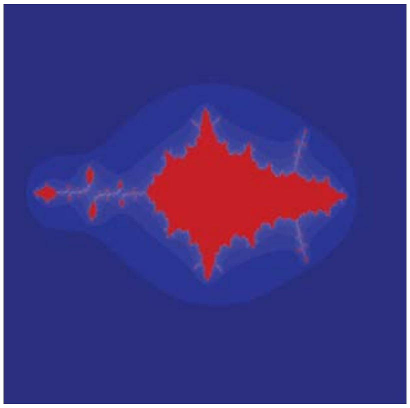

Figure 4: When α = .5, β = .5 and $c = -4.125 + 0.1i$

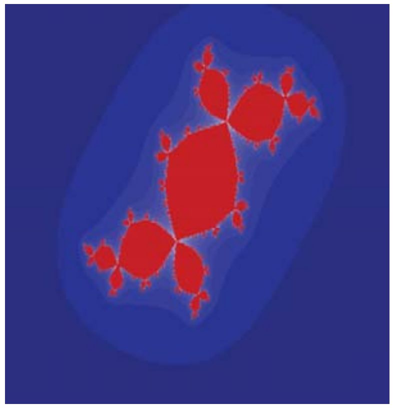

Figure 5: When α = 1, β = 1 and $c = -0.125 - 0.725i$

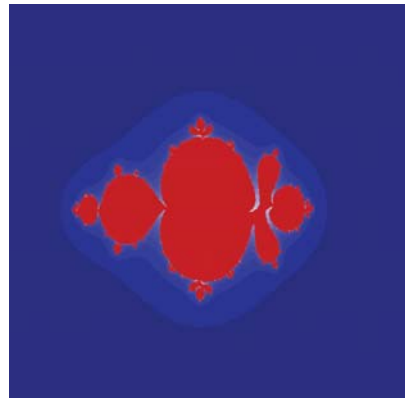

Figure 6: α = .75, β = .5 and $c = -1.9125 - 0.025i$

Theorem 1:

Assume that $|z| \geq |c| > \dfrac{2}{\alpha}$ and $|z| \geq |c| > \dfrac{2}{\beta}$ where $0 < \alpha < 1$ and $0 < \beta < 1$, c is a complex parameter. Define

$$\begin{aligned} S(z_1) &= (1-\alpha)S(z) + \alpha T(y), \\ &\vdots \\ S(z_n) &= (1-\alpha)S(z_{n-1}) + \alpha T(y_{n-1}) \end{aligned}$$

and

$$\begin{aligned} S(y) &= (1-\beta)S(z) + \beta T(z) \\ &\vdots \\ S(y_{n-1}) &= (1-\beta)S(z_{n-1}) + \beta T(z_{n-1}) \end{aligned}$$

Here Sz is injective, Tz is a quadratic polynomial and $n >= 2$ then $|z_n| \to \infty$ as $n \to \infty$

Proof : For quadratic complex polynomials $F(z) = z^2 - z + c$, we will choose $T(z) = z^2 + c$ and $S(z) = z$, c € C.

For the polynomial $T(z) = z^2 + c$ by triangular inequality, we have

$$|T(z)| \geq |z^2| + |c|$$

$$\geq |z^2| + |z|$$

since $$|z| \geq |c|$$

$$= |z|\,(|z| - 1) \tag{1}$$

$$|S(y)| = |(1-\beta)S(z) + \beta T(z)|$$

$$= |(1-\beta)|z| + \beta T(|z|\,(|z| - 1)) \quad \text{(usng (1))}$$

$$|S(y)| \geq |z|\,(1 - \beta + \beta|z| + \beta)$$

i.e., $$|S(y)| \geq |z|\,(\beta\,|z| - 1)$$

i.e., $$y \geq |z|\,(\beta\,|z| - 1)$$

Since $|z| > \frac{2}{\beta}$ implies

$$
\begin{aligned}
|z|^2(\beta|z|-1)^2 &> |z|^2 > \beta|z|^2 \\
|y^2| &> |z|^2 (\beta|z|-1)^2 \\
|y^2| &> |z|^2 > \beta|z|^2 \qquad (2)
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
T(y) &= y^2 + c \\
|T(y) &= |y^2| + c \\
|T(y) &> \beta|z^2| + |z|
\end{aligned}
$$

Since

$$
\begin{aligned}
|z| &\geq |c| \\
|T(y)| &= |z|(\beta|z| + 1) \qquad (3) \\
S(z_n) &= (1-\alpha)\, S(z_{n-1}) + \alpha T(\gamma_{n-1})
\end{aligned}
$$

So

$$
\begin{aligned}
|S(z_1)| &= |(1-\alpha)\, S(z) + \alpha T(y)| \\
|S(z_1)| &\geq |(1-\alpha)|z| + \alpha(|z|)(\beta|z| + 1)) \\
|S(z_1)| &\geq |z|\,(1-\alpha) + \alpha(\beta|z| + 1)) \\
|S(z_1)| &\geq |z|\,(1-\alpha) + \alpha\beta|z| + \alpha)) \\
|S(z_1)| &\geq |z|\,(\alpha\beta|z| - 1) \\
|z_1| &\geq |z|\,(\alpha\beta|z| - 1)
\end{aligned}
$$

Since $|z| \geq |c| > \frac{2}{\alpha}$ and $|z| \geq |c| > \frac{2}{\beta}$ so that $|z| > \frac{2}{\alpha\beta}$. Therefore there exists $\lambda > 0$ such that $(\alpha\beta\,|z| - 1) > (1 + \lambda)$. Consequently $|z_1 > (1 + \lambda)|z|$. So we apply the same argument repeatedly to find $|z_n| > (1 + \lambda)^n|z|$.

Thus this proves that orbit of Jungck Ishikawa Iteration under quadratic polynomial tends to ∞.

4. ANALYSISING SINGULARLY PERTURBED RALATIONAL MAP ($z \to z^n - z^d + c / z^n$)

Consider another relational map $F(z) = z^n - z^d + c/z^n$ where $n = 2$ and $d = 1$. Here it is seen that if α and β value is 0 the Julia set converges to a unit disk. But if we increase the value of α and β then the Julia sets formed are quite different. See Figure 9, Figure 10, Figure 11.

In Figure 9 we can see that external region contains parameter for which the Julia set is similar to Cantor set, and the holes represents Sierpinski holes. It is also seen that if we take both α and β values 1 then the Julia set is symmetrical about both x and y axis which is not in any other cases(Figure 9). If we take α = .5 and β = .5 then the necklaces are symmetrical about only x axis. Most interesting fractal is Figure 11 when α =.7 and β = .5 as it contains four parts and boundary of each itself represents Cantor set.

JULIA SETS:

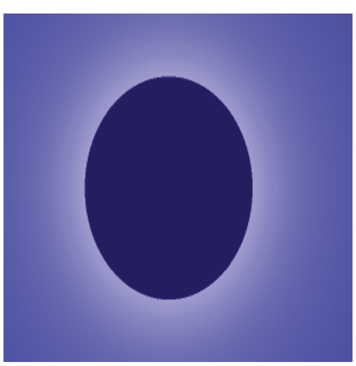

Figure 7: When α = 0, β = 0 , n = 2 and c = –0.001

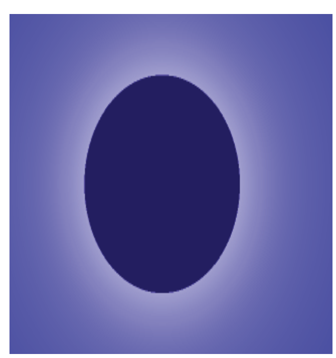

Figure 8: When α = 0, β = 0, n = 2 and c = 0.068750 – .0375i

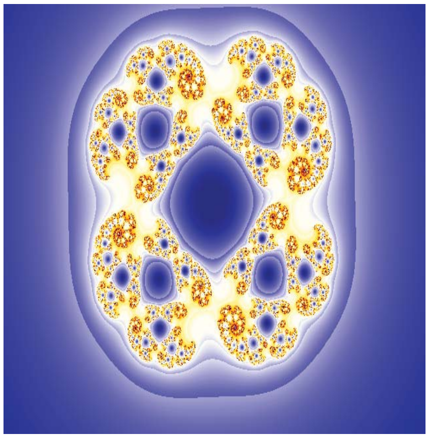

Figure 9: When $\alpha = 1$, $\beta = 1$, $n = 2$ $c = 0.1375 - 0.025\,i$

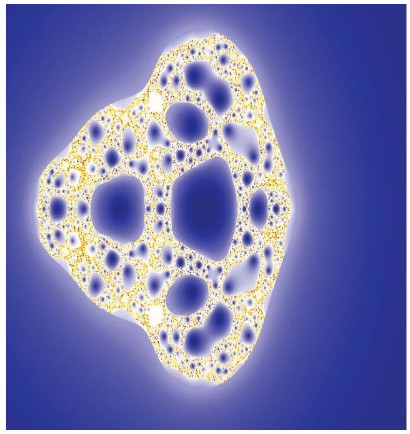

Figure 10: When $\alpha = .5$, $\beta = .5$, $n = 2$ and $c = -1.25$

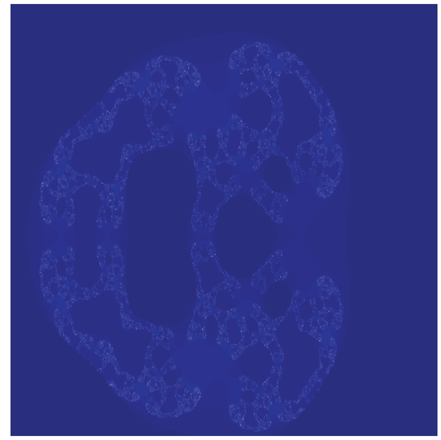

Figure 11: when $\alpha = .7$, $\beta = .5$ and $c = 0.40625 - 0.29375i$

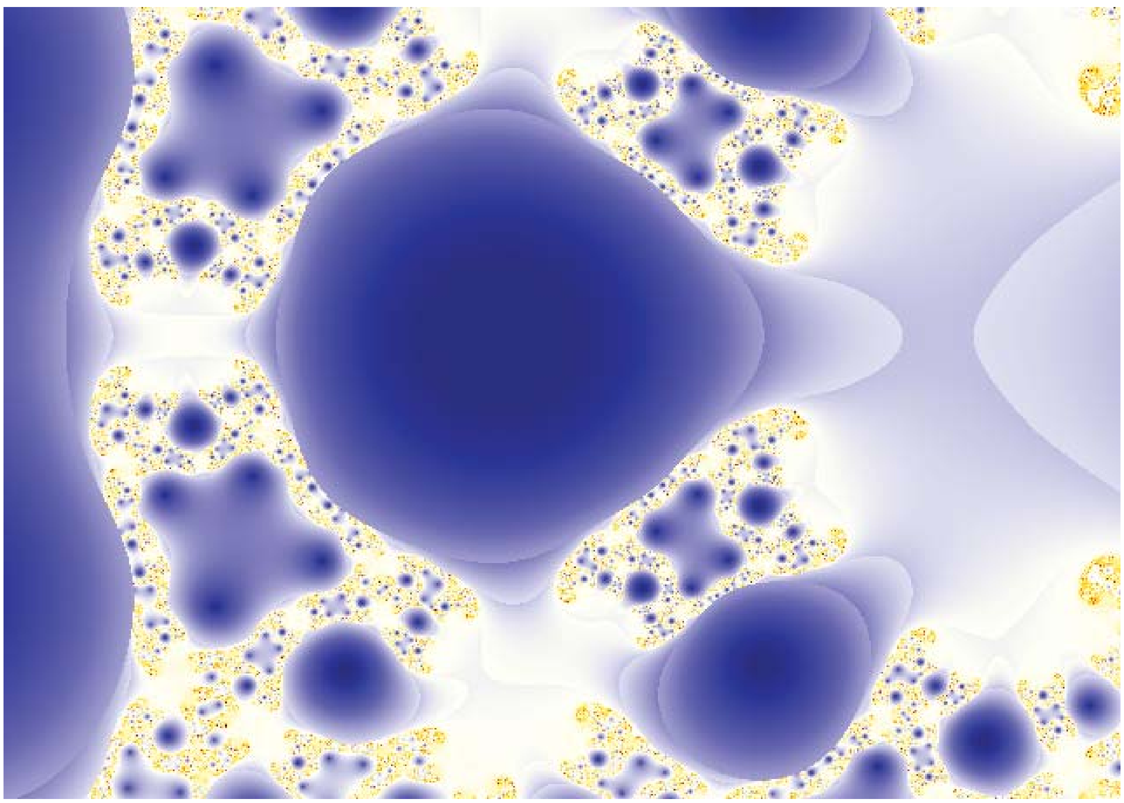

Figure 12: Magnification of Figure 11

We have seen that for many parameteric values, the Julia sets for the above map are Sierpinski curves. As we know that one of the important characteristics of Sierpinski curve Julia sets is the Fatou set which consists of infinitely many open disks, and each disk is bounded by a simple closed curve, and every pair of curves is disjoint.

Also each pair of Sierpinski curves are homeomorphic [9] in nature.

SIERPINSKI CURVE JULIA SETS:

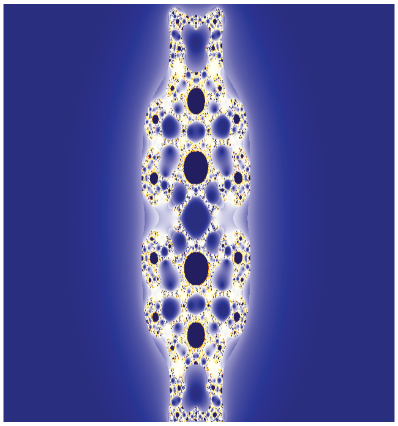

Figure 13: When α = .5, β = .1, n = 3 and c = 0.63125 + 0.01875i

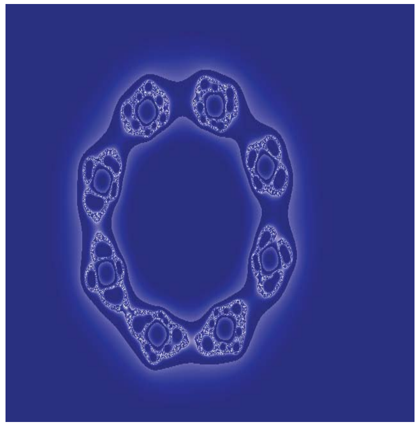

Figure 14: When α = .5, β = 1, n = 4 and c = 0.375 + 0.20625i

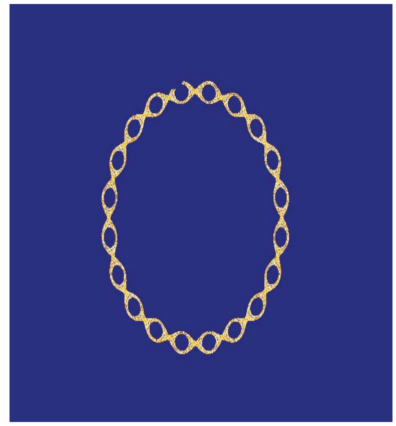

Figure 14: When α = 1, β = 1, n = 10 and c = 0.3 + 0.005i

In above Figure 13, Figure 14 and Figure 15 we have generated c – plane for $n = 3$, $n = 4$, $n = 10$ by taking different values of α and β. Here the external region is called as Cantor set locus where the Julia sets are Cantor sets and the central region is called the McMullen domain [8]. In theorem 2 we have proved the escape criteria for

$$F(z) = z^n - z^d + c/z^n.$$

when $n = 2$

and $d = 1.$

Theorem 2: Assume that $|z| \geq |c| > \frac{2}{\alpha}$ and $|z| \geq |c| > \frac{2}{\beta}$ where $0 < \alpha < 1$ and $0 < \beta < 1$, c is a complex parameter.

Define

$$P(z_1) = (1-\alpha)P(z) + \alpha Q(y),$$
$$\vdots$$
$$P(z_n) = (1-\alpha)P(z_{n-1}) + \alpha Q(y_{n-1})$$

and

$$Py_1 = (1-\beta)Pz + \beta Qz,$$
$$\vdots$$
$$Py_{n-1} = (1-\beta)Pz_{n-1} + \beta Qz_{n-1}$$

Here Pz is injective, Qz is a quadratic polynomial and $n >= 2$ then $|z_n| \to \infty$ as $n \to \infty$

Proof : For quadratic complex polynomials

$$F(x) = z^2 - z + c/z^2,$$

we will choose $Q(z) = z^2 + c/z^2$

and $P(z) = z, c € C.$

If $|z| \geq 2$, then we have

$$|Q(z)| \geq |z^2| - \frac{|c|}{|z^2|}$$

$$\geq |z||z| - \frac{1}{|z||z|}$$

$$\geq 2|z| - \frac{1}{4}$$

$$\geq \frac{3}{2}|z| \tag{1}$$

Now $|P(y)| = |(1-\beta)P(z) + \beta Q(z)|$

$$= (1-\beta)|z| + \frac{3}{2}\beta\,|z| \text{ using (1)}$$

$$= |z| - \beta|z| + \frac{3}{2}\beta\,|z|$$

$$= |z| + \frac{1}{2}\beta\,|z|$$

$$\geq \frac{3}{2}\beta|z|$$

i.e. $|P(y)| \geq \frac{3}{2}\beta\,|z|$

i.e. $|y| \geq \frac{3}{2}\beta\,|z|$

$$|y^2| \geq \frac{3}{2}\beta|z^2|$$

Now $|Q(y)| = |y^2| + |c|$

$\because$ $|z| \geq |c|$

$$|Q(y)| \geq \frac{3}{2}\beta\,|z^2| + |z|$$

$$|Q(y)| \geq |z|\left(\frac{3}{2}\beta|z| + 1\right)$$

Since $P(z_n) = (1-\alpha)\,P(z_{n-1}) + \alpha Q(y_{n-1})$

We have $|P(z_1)| = |(1-\alpha)\,P(z) + \alpha Q(y)$

$$\geq (1-\alpha)|z| + \alpha|z|\left(\frac{3}{2}\beta|z| + 1\right)$$

$$\geq |z|\left(1 - \alpha + \alpha\frac{3}{2}\beta|z| + \alpha\right)$$

$$|z_1| \geq |z|\left(1 + \alpha\frac{3}{2}\beta|z|\right)$$

Since $|z| \geq 2$ and $0 < \beta < 1$ and $0 < \beta < 1$

Therefore, $$|z_1| \geq \alpha\beta\frac{3}{2}|z|$$

or $$|z_1| \geq \frac{3}{2}|z|$$

By Induction, we have $$|z_1| \geq \left(\frac{3}{2}\right)^n |z|$$

Hence we have proved that orbit of Jungck Ishikawa iteration escapes to infinity.

5. CONCLUSION

Here Devaney's work has been extended and a series of Julia sets for special case when n>=2 and d=1 for the given relational maps is constructed. We have also mathematically analysed the symmetry of Mandelbrot and Julia sets which comes out to be beautiful images and all follows some symmetry. Jungck Ishikawa iterative scheme is also used to prove the escape criteria for given functions.

6. FUTURE SCOPE

In this paper we have shown generalisation of the function $F(z) = z^n - z^d + c$ and $F(z) = z^n - z^d + c / z^n$ for $n >= 2$, $d = 1$. In our next papers we will be exploring Jungck Ishikawa Iterative schemes for trigonometric and logarithmic relational maps as they always gives different and interesting Mandelbrot and Julia sets and fixed points as comparison to others.

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