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ERGODIC CONTROL OF STOCHASTIC NAVIER-STOKES EQUATION WITH LÉVY NOISE

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Abstract. In this work we consider the controlled stochastic Navier-Stokes equations perturbed by Lévy noise in a two dimensional bounded domain and a semimartingale formulation is used to characterize the probability law defining the space-time statistical solution. The existence and uniqueness of invariant measures or stationary measures is examined under suitable assumptions on the noise coefficient. Then we establish the existence of an ergodic control which is optimal in the class of all stationary measures for the system for a suitable class of cost functions. Thus for this system, it is possible to choose a stationary control that corresponds to a statistically stationary turbulent state with certain prescribed cost functional attaining a minimum.

1. Introduction

The optimal control theory of viscous time dependent deterministic and stochastic fluid dynamics has gained great attention in the past several decades (for example see [31, 15, 16], references therein). In the recent years, ergodicity results for the two and three dimensional stochastic Navier-Stokes equation (SNSE) on various domains has been established in the literature in several papers. The existence and uniqueness of invariant measures for the SNSE with degenerate and non-degenerate Gaussian noise is established in [14, 5, 23, 6, 34, 27, 17, 28, 29, 8, 22], for example, using various methods. Ergodicity results for the SNSE driven by Lévy noise with non-degenerate Gaussian part is established in [10].

The statistically stationary dynamics of fluid turbulence under control action is considered in this paper. By using a semimartingale formulation, we describe the solution to the martingale problem for the controlled SNSE perturbed by Lévy noise in two dimensional bounded domains. The existence and uniqueness of invariant measures for the solution of this system is discussed under suitable assumptions on the Gaussian and jump noise coefficients. We then establish that, for the controlled SNSE driven by Lévy noise, it is possible to choose a stationary control that corresponds to a statistically stationary turbulent state with certain prescribed cost functional attaining a minimum. This work appears to be the first work regarding the optimal ergodic control for the SNSE perturbed Lévy noise,

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besides the Guassian noise case outlined in [33]. The theory described in this paper is easily generalizable to ergodic measures found in three dimensions (see for instance [6, 7, 25]) and also for degenerate noise (see for example [27, 17, 29]).

Let $\mathcal{O}$ be a bounded open domain in $\mathbb{R}^2$ with a smooth boundary $\partial\mathcal{O}$. Let $\mathbf{u} = \mathbf{u}(x,t) = (u_1(x,t), u_2(x,t))$ denotes the velocity field, the scalar valued function $p = p(x,t)$ denotes the pressure field and $\mathbf{f} = \mathbf{f}(t,x,\mathbf{u})$ denotes the external random forcing. Let T be an arbitrary but fixed positive number. For $t \in [0,T]$, let us consider the controlled stochastic Navier-Stokes equation as follows:

$$\begin{aligned}&\frac{\partial \mathbf{u}(x,t)}{\partial t} + (\mathbf{u}(x,t)\cdot\nabla)\mathbf{u}(x,t) - \nu\Delta\mathbf{u}(x,t) + \nabla p(x,t)\\ &\quad = \mathbf{N}(\mathbf{v}(x,t)) + \mathbf{f}(t,x,\mathbf{u}(x,t)) \text{ for } (x,t) \in \mathcal{O}\times(0,T),\end{aligned} \tag{1.1}$$

with the incompressibility condition

$$\nabla\cdot\mathbf{u}(x,t) = 0 \text{ for } (x,t) \in \mathcal{O}\times(0,T), \tag{1.2}$$

the non-slip boundary condition

$$\mathbf{u}(x,t) = 0 \text{ for } (x,t) \in \partial\mathcal{O}\times(0,T), \tag{1.3}$$

and the initial condition

$$\mathbf{u}(x,0) = \mathbf{u}_0(x) \text{ for } x \in \mathcal{O}. \tag{1.4}$$

Here ν is the kinematic viscosity, $\mathbf{v}(\cdot,\cdot)$ is the control that takes values in some metrizable Lusin space $\mathbb{U}$ and $\mathbf{N}(\cdot)$ is a linear or nonlinear operator representing possible nonlinearities in the actuator term. For slightly conducting fluids, Lorentz force appears as a special case of such control forces.

Our aim in this paper is to find a control $\mathbf{v}$ which minimizes

$$\limsup_{t\to\infty}\frac{1}{t}\int_0^t \mathbb{E}[\mathcal{K}(\mathbf{u}(s),\mathbf{v}(s))]\mathrm{d}s,$$

among all stationary solutions, where the running cost $\mathcal{K} : \mathbb{V}\times\mathbb{U}\to[0,\infty]$ ($\mathbb{V}$ is defined in section 2) satisfies the following conditions:

(i) $\mathcal{K}(\cdot,\cdot) : \mathbb{V}\times\mathbb{U}\to[0,\infty]$ is measurable,

(ii) $\mathcal{K}(\cdot,\cdot) : \mathbb{V}_{\mathrm{w}}\times\mathbb{U}\to[0,\infty]$ is lower semicontinuous, where $\mathbb{V}_{\mathrm{w}}$ is the space $\mathbb{V}$ endowed with the weak topology,

(iii) $\mathcal{K}(\cdot,\cdot)$ is bounded below by

$$\mathcal{K}(\mathbf{u},\mathbf{v}) \geq C\big(\|\mathbf{u}\|_{\mathbb{V}}^2 + \ell^4(\mathbf{v})\big), \text{ for all } (\mathbf{u},\mathbf{v}) \in \mathbb{V}\times\mathbb{U}, \tag{1.5}$$

where $\ell : \mathbb{U}\to\mathbb{R}^+$ is an inf-compact function such that

$$\|\mathbf{N}(\mathbf{v})\|_{\mathbb{H}} \leq \ell(\mathbf{v}) + C, \text{ for all } \mathbf{v} \in \mathbb{U}. \tag{1.6}$$

An example of such a running cost $\mathcal{K}(\cdot,\cdot)$ is

$$\mathcal{K}(\mathbf{u},\mathbf{v}) = \|\mathrm{curl}\ \mathbf{u}\|_{\mathbb{H}}^2 + \|\mathbf{v}\|_{\mathbb{H}}^4,$$

which is the *enstrophy minimization problem*, with $\mathbb{U} = \mathbb{H}$ and $\mathbf{N}(\mathbf{v}) = \mathbf{L}_{\mathbf{N}}\mathbf{v}$, where $\mathbf{L}_{\mathbf{N}} \in \mathrm{L}(\mathbb{H})$ and $\mathrm{L}(\mathbb{H}) := \mathrm{L}(\mathbb{H},\mathbb{H})$ denotes the space of all bounded linear operators on $\mathbb{H}$.

The organization of the paper is as follows. In section 2, we describe the semi-martingale framework of the stochastic Navier-Stokes system (1.1)-(1.4) and the

martingale problem is formulated. The existence and uniqueness of invariant measures for the above system is examined in section 3. The existence of an ergodic control which is optimal in the class of all stationary measures for the system (1.1)-(1.4) is proved in section 4.

2. Semimartingale Formulation and Martingale Problem

Let us begin with the abstract formulation of the stochastic Navier-Stokes system (1.1)–(1.4) by making use of the following conventional notations. Let us denote

$$\mathcal{V} = \{\mathbf{u} \in \mathrm{C}_0^\infty(\mathcal{O}) : \nabla \cdot \mathbf{u} = 0\}.$$

Let $\mathbb{H}$ and $\mathbb{V}$ be the completion of $\mathcal{V}$ in $\mathbb{L}^2(\mathcal{O})$ and $\mathbb{H}^1(\mathcal{O})$ norms respectively. Then, we have

$$\mathbb{H} := \big\{\mathbf{u} \in \mathbb{L}^2(\mathcal{O}) : \nabla \cdot \mathbf{u} = 0, \mathbf{u} \cdot \mathbf{n}\big|_{\partial\mathcal{O}} = 0\big\}, \tag{2.1}$$

with the norm $\|\mathbf{u}\|_{\mathbb{H}} := \big(\int_{\mathcal{O}} |\mathbf{u}(x)|^2 \mathrm{d}x\big)^{1/2}$, where $\mathbf{n}$ is the outward drawn normal to $\partial\mathcal{O}$ and

$$\mathbb{V} := \big\{\mathbf{u} \in \mathbb{H}_0^1(\mathcal{O}) : \nabla \cdot \mathbf{u} = 0\big\}, \tag{2.2}$$

with the norm $\|\mathbf{u}\|_{\mathbb{V}} := \big(\int_{\mathcal{O}} |\nabla\mathbf{u}(x)|^2 \mathrm{d}x\big)^{1/2}$. Let $\mathbb{V}'$ be the dual of $\mathbb{V}$. Then, we have the dense, continuous and compact embedding

$$\mathbb{V} \hookrightarrow \mathbb{H} \equiv \mathbb{H}' \hookrightarrow \mathbb{V}'. \tag{2.3}$$

The inner product in the Hilbert space $\mathbb{H}$ is denoted by $(\cdot,\cdot)$ and the induced duality pairing, for instance between the spaces $\mathbb{V}$ and $\mathbb{V}'$, by $\langle\cdot,\cdot\rangle$. Let $\mathbf{P}_{\mathbb{H}} : \mathbb{L}^2(\mathcal{O}) \to \mathbb{H}$ be the Helmholtz-Hodge orthogonal projection operator. We define the Stokes operator

$$\mathbf{A} : \mathrm{D}(\mathbf{A}) \to \mathbb{H} \text{ with } \mathbf{A}\mathbf{u} = -\mathbf{P}_{\mathbb{H}}\Delta\mathbf{u}, \tag{2.4}$$

where $\mathrm{D}(\mathbf{A}) = \mathbb{V}\cap\mathbb{H}^2(\mathcal{O}) = \big\{\mathbf{u} \in \mathbb{H}_0^1(\mathcal{O}) \cap \mathbb{H}^2(\mathcal{O}) : \nabla \cdot \mathbf{u} = 0\big\}$ is the domain of the operator $\mathbf{A}$. The Stokes operator is a positive selfadjoint operator with a compact resolvent and if $0 < \lambda_1 \le \lambda_2 \le \cdots$ are the eigenvalues of $\mathbf{A}$, then we have

$$\|\mathbf{u}\|_{\mathbb{V}}^2 \ge \lambda_1 \|\mathbf{u}\|_{\mathbb{H}}^2, \text{ for all } \mathbf{u} \in \mathbb{V}.$$

We define the nonlinear operator

$$\mathbf{B} : \mathrm{D}(\mathbf{B}) \subset \mathbb{H} \times \mathbb{V} \to \mathbb{H} \text{ with } \mathbf{B}(\mathbf{u}) = \mathbf{P}_{\mathbb{H}}(\mathbf{u} \cdot \nabla)\mathbf{u}. \tag{2.5}$$

For more details and properties of the Stokes operator $\mathbf{A}$ and the nonlinear operator $\mathbf{B}$, we refer readers to [21, 35, 36]. Let us now apply $\mathbf{P}_{\mathbb{H}}$ to the system (1.1) and take the external forcing to be Lévy noise to obtain

$$\left.\begin{aligned} \mathrm{d}\mathbf{u}(t) + [\nu\mathbf{A}\mathbf{u}(t) + \mathbf{B}(\mathbf{u}(t))]\mathrm{d}t &= \mathbf{N}\pi_t \mathrm{d}t + \sigma(t,\mathbf{u}(t))\mathrm{dW}(t) \\ &\quad + \int_{\mathrm{Z}} \gamma(t-,\mathbf{u}(t-),z)\widetilde{\mathcal{N}}(\mathrm{d}t,\mathrm{d}z), \\ \mathbf{u}(0) &= \mathbf{u}_0 \text{ in } \mathbb{H}, \end{aligned}\right\} \tag{2.6}$$

where $\mathbf{N}\pi_t = \int_{\mathbb{U}} \mathbf{N}(\mathbf{v})\pi_t(\mathrm{d}\mathbf{v})$. In (2.6), Z is a measurable subspace of some Hilbert space (for example measurable subspaces of $\mathbb{R}^2$, $\mathbb{L}^2(\mathcal{O})$ etc). Also $\mathrm{W}(t) =$

$\mathrm{W}(x,t)$ is an $\mathbb{L}^2-$valued Wiener process with a nuclear covariance operator Q and $\widetilde{\mathcal{N}}(\mathrm{d}t,\mathrm{d}z) := \mathcal{N}(\mathrm{d}t,\mathrm{d}z) - \lambda(\mathrm{d}z)\mathrm{d}t$ is a compensated Poisson random measure, where $\lambda(\mathrm{d}z)$ is a $\sigma-$finite Lévy measure on some Hilbert space (for instance $\mathbb{R}^2$, $\mathbb{L}^2(\mathcal{O})$ etc.) with an associated Poisson random measure $\mathcal{N}(\mathrm{d}t,\mathrm{d}z)$ such that $\mathbb{E}(\mathcal{N}(\mathrm{d}t,\mathrm{d}z)) = \lambda(\mathrm{d}z)\mathrm{d}t$. The processes $\mathrm{W}(\cdot)$ and $\widetilde{\mathcal{N}}(\mathrm{d}t,\mathrm{d}z)$ are mutually independent.

Let us assume the noise coefficients $\sigma(\cdot,\cdot)$ and $\gamma(\cdot,\cdot,\cdot)$, and the linear or nonlinear operator $\mathbf{N}(\cdot)$ satisfy the following hypothesis.

Hypothesis 2.1. *The noise coefficients $\sigma(\cdot,\cdot)$ and $\gamma(\cdot,\cdot,\cdot)$, and the operator $\mathbf{N}(\cdot)$ satisfy*

(H.1) *(Growth Condition) For all $\mathbf{u}\in\mathbb{H}$ and for all $t\in[0,T]$, there exists a positive constant K such that*

$$\|\sigma(t,\mathbf{u})\|^2_{\mathrm{L}_Q(\mathbb{H})} + \int_{\mathrm{Z}} \|\gamma(t,\mathbf{u},z)\|^2_{\mathbb{H}}\lambda(\mathrm{d}z) \leq K\big(1+\|\mathbf{u}\|^2_{\mathbb{H}}\big),$$

where $\mathrm{L}_Q(\mathbb{H})$ denotes the space of all Hilbert-Schmidt operators from $Q^{\frac{1}{2}}\mathbb{H}$ to $\mathbb{H}$.

(H.2) *(Lipschitz Condition) For all $\mathbf{u}_1,\mathbf{u}_2\in\mathbb{H}$ and for all $t\in[0,T]$, there exists a positive constant L such that*

$$\|\sigma(t,\mathbf{u}_1)-\sigma(t,\mathbf{u}_2)\|^2_{\mathrm{L}_Q(\mathbb{H})} + \int_{\mathrm{Z}} \|\gamma(t,\mathbf{u}_1,z)-\gamma(t,\mathbf{u}_2,z)\|^2_{\mathbb{H}}\lambda(\mathrm{d}z) \leq L\|\mathbf{u}_1-\mathbf{u}_2\|^2_{\mathbb{H}}.$$

(H.3) *We fix the measurable subset Z_m of Z with $\mathrm{Z}_m\uparrow\mathrm{Z}$ and $\lambda(\mathrm{Z}_m)<\infty$ such that*

$$\sup_{\|\mathbf{u}\|_{\mathbb{H}}\leq M}\int_{\mathrm{Z}_m^c} \|\gamma(t,\mathbf{u},z)\|^2_{\mathbb{H}}\lambda(\mathrm{d}z)\to 0, \text{ as } m\to\infty, \text{ for any fixed } M>0.$$

(H.4) *$\mathbf{N}:\mathbb{U}\to\mathbb{H}$ is continuous and satisfies (1.6).*

See section 3.1, [32] for more details of the operator $\mathbf{N}$. The controls we consider are in the class $\mathcal{A}$, where $\mathcal{A}$ is the collection of all $\pi\in\mathcal{P}$ such that

$$\|\pi\|^4_{\ell} := \int_0^T\int_{\mathbb{U}} \ell^4(\mathbf{v})\pi(\mathrm{d}\mathbf{v},\mathrm{d}t) < \infty, \tag{2.7}$$

and $\mathcal{P}$ is the space of probability measures on $[0,T]\times\mathbb{U}$. This class is referred to as *admissible relaxed controls*.

In the following definition, we give the formal generator for some class of test functions.

Definition 2.2. (Formal Generator) For $f\in\mathrm{D}(\mathscr{L})$, the formal generator $\mathscr{L}f$ is given by

$$\begin{aligned}\mathscr{L}f = &-\left\langle \nu\mathbf{A}\mathbf{u}+\mathbf{B}(\mathbf{u})-\mathbf{N}\pi, \frac{\partial f}{\partial\mathbf{u}}\right\rangle + \frac{1}{2}\,\mathrm{Tr}\left(\sigma(t,\mathbf{u})Q\sigma^*(t,\mathbf{u})\frac{\partial^2 f}{\partial\mathbf{u}^2}\right)\\ &+\int_{\mathrm{Z}}\left\{f(\mathbf{u}+\gamma(t,\mathbf{u},z))-f(\mathbf{u})-\left\langle\gamma(t,\mathbf{u},z),\frac{\partial f}{\partial\mathbf{u}}\right\rangle\right\}\lambda(\mathrm{d}z),\end{aligned} \tag{2.8}$$

for all $\mathbf{u}\in\mathrm{D}(\mathbf{A})$.

As an example for the functions $f \in \mathrm{D}(\mathscr{L})$, one may consider the following:

Example 2.3. *The test functions* $f(\cdot)$ (cylindrical (tame) function) *of the form*

$$f(\mathbf{u}) := \phi(\langle e_1, \mathbf{u}\rangle, \cdots, \langle e_m, \mathbf{u}\rangle),\ \mathbf{u} \in \mathbb{H}, \tag{2.9}$$

where $\phi(\cdot) : \mathbb{R}^m \to \mathbb{R}$ *is a smooth function with compact support in* $\mathbb{R}^m$, $e_k \in \mathrm{D}(\mathbf{A}), k = 1, \cdots, m$.

2.1. Martingale problem. Let us solve the system (2.6) by the method of martingale problems and so we give the following definition. Let $\mathcal{P}(\mathbb{U})$ denotes the space of probability measures on $\mathbb{U}$.

Definition 2.4. A $\mathbb{V}\times\mathcal{P}(\mathbb{U})-$valued process $(\mathbf{u}, \pi)$ defined on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ with a probability measure $\mathbb{P} : \mathscr{B}(\Omega) \to [0, 1]$ such that $\mathbb{P}\{\omega \in \Omega : \mathbf{u}(0) = \mathbf{u}_0\} = 1$, is said to be a *solution to the relaxed controlled martingale problem for* $(\mathscr{L}, \mathscr{P})$ with respect to a filtration $\{\mathscr{F}_t\}$ if

(i) $(\mathbf{u}, \pi)$ is $\mathscr{F}_t-$progressive,
(ii) $\mathcal{L}(\mathbf{u}(0)) = \mathscr{P}$, where $\mathcal{L}(\mathbf{u}(0))$ denotes the law of $\mathbf{u}(0)$,
(iii) for $f \in \mathrm{D}(\mathscr{L})$,

$$\mathbf{M}_t^f := f(\mathbf{u}(t)) - f(\mathbf{u}(0)) - \int_0^t \int_{\mathbb{U}} \mathscr{L} f(\mathbf{u}(s), \mathbf{v})\pi_s(\mathrm{d}\mathbf{v})\mathrm{d}s$$

is a real valued locally square integrable $(\Omega, \mathscr{F}, \mathscr{F}_t, \mathbb{P})-$local càdlàg martingale.

For the relaxed controlled problem, the cost functional becomes:

$$\limsup_{t\to\infty} \frac{1}{t} \int_0^t \mathbb{E}\bigg(\int_{\mathbb{U}} \mathcal{K}(\mathbf{u}(s), \mathbf{v})\pi_s(\mathrm{d}\mathbf{v})\bigg)\mathrm{d}s.$$

Furthermore, if $(\mathbf{u}, \pi)$ is a stationary process, then the cost functional reduces to

$$\langle \mathscr{K}, \mu\rangle = \int \mathscr{K}\pi \mathrm{d}\mu,$$

where $\mathscr{K}\pi := \int_{\mathbb{U}} \mathcal{K}(\mathbf{u}, \mathbf{v})\pi(\mathrm{d}\mathbf{v})$ and $\mu = \mathcal{L}(\mathbf{u}(t), \pi(t)) \in \mathcal{P}(\mathbb{V}) \times \mathcal{P}(\mathcal{P}(\mathbb{U}))$ for all $t \geq 0$. Here $\mathcal{L}(\mathbf{u}(t), \pi(t))$ denotes the law of $(\mathbf{u}(t), \pi(t))$.

Let us define

$$\widetilde{\Omega} = \mathbb{D}(0, T; \mathbb{V}')_{\mathrm{J}} \cap \mathbb{L}^\infty(0, T; \mathbb{H})_{w^*} \cap \mathbb{L}^2(0, T; \mathbb{V})_w \cap \mathbb{L}^2(0, T; \mathbb{H}),$$

where $\mathbb{D}(\cdot;\cdot)$ is the class of càdlàg functions from $[0, T]$ into $\mathbb{V}'$. Let us endow $\widetilde{\Omega}$ with the supremum topology $\mathcal{T} := \mathcal{T}_1 \vee \mathcal{T}_2 \vee \mathcal{T}_3 \vee \mathcal{T}_4$, where

$\mathcal{T}_1 := \mathbb{D}(0, T; \mathbb{V}')_{\mathrm{J}}$, where J denotes the extended Skorohod topology (see [24]),
$\mathcal{T}_2 := \mathbb{L}^\infty(0, T; \mathbb{H})_{w^*}$, where w^* denotes the weak-star topology,
$\mathcal{T}_3 := \mathbb{L}^2(0, T; \mathbb{V})_w$, where w denotes the weak topology, and
$\mathcal{T}_4 := \mathbb{L}^2(0, T; \mathbb{H})$, equipped with the strong topology.

Then, from Proposition 1, page 63 of [24], it is clear that the intersection of these spaces $\widetilde{\Omega}$, endowed with the supremum topology $\mathcal{T}$ is a Lusin space. Let $\Omega = \widetilde{\Omega} \times \mathcal{P}$, where $\mathcal{P}$ is the space of probability measures on $[0, T] \times \mathbb{U}$. For any $\omega \in \Omega$, let $\omega = (\omega_1, \omega_2)$, where $\omega_1 \in \widetilde{\Omega}$ and $\omega_2 \in \mathcal{P}$. Let ξ be the mapping from $[0, T] \times \Omega \to \mathbb{V}' \times \mathcal{P}$ defined by $\xi(t, \omega) := \omega(t)$ and let $\mathscr{F}_t$ be the canonical filtration

generated by the functions $\xi(t,\omega)$ on Ω, that is $\mathscr{F}_t = \sigma\{\xi(s,\omega) : 0 \le s \le t\}$ for all $t \in [0,T]$. Note that the solution of the relaxed controlled martingale problem for the system (2.6) is equivalently a probability measure $\mathbb{P}$ on $\Omega = \widetilde{\Omega} \times \mathcal{P}$ such that for all $f \in \mathrm{D}(\mathscr{L})$,

$$f(\omega_1(t)) - f(\omega_1(0)) - \int_0^t \int_{\mathbb{U}} \mathscr{L} f(\omega_1(s), \mathbf{v}) \omega_2(s)(\mathrm{d}\mathbf{v})\mathrm{d}s,$$

is a $\mathbb{P}-$martingale with respect to $\mathscr{F}_t$, where $\mathscr{L}f$ is defined in (2.8).

By the disintegration (see Corollary 3.1.2, [3]) of the measure $\mathbb{P}$ for the projection $\Omega \to \widetilde{\Omega}$,

$$\mathbb{P}(\mathrm{d}\mathbf{u}, \mathrm{d}\pi) = \mathbb{P}(\mathrm{d}\mathbf{u}, \pi)\Pi(\mathrm{d}\pi),$$

where $\mathbb{P}(\mathrm{d}\mathbf{u}, \pi)$ is the regular conditional distribution of $\mathbf{u}$ given π and Π is the law of π. Thus the martingale problem formulated above reduces to that of finding the martingale solution $\mathbb{P}(\mathrm{d}\mathbf{u}, \pi)$ for each fixed π. Hence, we need to find a Radon measure $\mathbb{P}(\mathrm{d}\mathbf{u}, \pi)$ (which is again denoted by $\mathbb{P}$) on $\mathscr{B}(\widetilde{\Omega})$ depending measurably on π such that

$$\mathbf{M}_t(\mathbf{u}) := \mathbf{u}(t) - \mathbf{u}_0 + \int_0^t \left\{ \nu \mathbf{A}\mathbf{u}(s) + \mathbf{B}(\mathbf{u}(s)) - \int_{\mathbb{U}} \mathbf{N}(\mathbf{v}(s))\pi_s(\mathrm{d}\mathbf{v}) \right\} \mathrm{d}s, \quad (2.10)$$

is a real valued locally integrable $(\widetilde{\Omega}, \mathscr{B}(\widetilde{\Omega}), \widetilde{\mathscr{F}_t}, \mathbb{P})-$local càdlàg martingale (where $\widetilde{\mathscr{F}_t} = \sigma\{\mathbf{u}(s), s \le t\}$) with the quadratic variation process

$$[[\mathbf{M}]]_t = \int_0^t \sigma(s, \mathbf{u}(s)) Q \sigma^*(s, \mathbf{u}(s)) \mathrm{d}s + \int_0^t \int_{\mathrm{Z}} (\gamma \otimes \gamma)(s-, \mathbf{u}(s-), z) \mathcal{N}(\mathrm{d}z, \mathrm{d}s).$$

Note that the Meyer process of $\mathbf{M}$ is given by

$$\ll \mathbf{M} \gg_t = \int_0^t \sigma(s, \mathbf{u}(s)) Q \sigma^*(s, \mathbf{u}(s)) \mathrm{d}s + \int_0^t \int_{\mathrm{Z}} (\gamma \otimes \gamma)(s, \mathbf{u}(s), z) \lambda(\mathrm{d}z) \mathrm{d}s,$$

so that $[[\mathbf{M}]]_t - \ll \mathbf{M} \gg_t$ is a local martingale. We use the following energy equality to get an a-priori estimate on the solutions.

Theorem 2.5. ([24]) *Let us consider a probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\ge 0}, \mathbb{P})$ and an $\mathscr{F}_t-$adapted processes $\widetilde{\mathbf{x}}$, $\widetilde{\mathbf{y}}$ and $\widetilde{\mathbf{M}}_t$ such that $\widetilde{\mathbf{M}}_t$ is an $\mathbb{H}-$valued square integrable, right-continuous martingale with $\widetilde{\mathbf{M}}_0 = 0$, $\widetilde{\mathbf{x}} \in \mathbb{L}^2(0,T;\mathbb{V})$, a.s., $\widetilde{\mathbf{y}} \in \mathbb{L}^2(0,T;\mathbb{V}')$, a.s. Let $\widetilde{\mathbf{x}}(0) = \widetilde{\mathbf{x}}_0 \in \mathbb{H}$ and*

$$\widetilde{\mathbf{x}}(t) = \widetilde{\mathbf{x}}(0) + \int_0^t \widetilde{\mathbf{y}}(s)\mathrm{d}s + \widetilde{\mathbf{M}}_t, \ t \in [0,T], \mathbb{P}-a.s.$$

Then the paths of $\widetilde{\mathbf{x}}$ are a.s. in $\mathbb{D}([0,T];\mathbb{H})$ ($\mathbb{H}-$valued Skorohod space) and the Itô formula applies for $\|\widetilde{\mathbf{x}}\|^2_{\mathbb{H}}$: for $\mathbb{P}-$a.s.,

$$\|\widetilde{\mathbf{x}}(t)\|^2_{\mathbb{H}} = \|\widetilde{\mathbf{x}}_0\|^2_{\mathbb{H}} + 2\int_0^t \langle \widetilde{\mathbf{y}}(s), \widetilde{\mathbf{x}}(s) \rangle \mathrm{d}s + 2\int_0^t \left\langle \widetilde{\mathbf{x}}(s-), \mathrm{d}\widetilde{\mathbf{M}}_s \right\rangle + \int_0^t \mathrm{Tr}\Big([[\widetilde{\mathbf{M}}]]_s \Big) \mathrm{d}s. \quad (2.11)$$

Proof. See Theorem 6.1, [24]. □

Let $\mathcal{M}$ be the collection of all probability measures on Ω which are solutions to the martingale problem, and whose support is in $\mathbb{L}^2(0,T;\mathbb{V})\cap\mathbb{L}^\infty(0,T;\mathbb{H})\times\mathcal{A}$. Then, we have the following a-priori energy estimates (see [32, 33, 30]):

Theorem 2.6. *Let the initial data satisfy $\mathbb{E}\|\mathbf{u}_0\|^2_{\mathbb{H}}<\infty$. Then, the class of probability measures $\mathbb{P}\in\mathcal{M}$ satisfy the a-priori energy estimate:*

$$\mathbb{E}^{\mathbb{P}}\left[\sup_{0\leq t\leq T}\|\mathbf{u}(t)\|^2_{\mathbb{H}}+\nu\int_0^T\|\mathbf{A}^{1/2}\mathbf{u}(s)\|^2_{\mathbb{H}}\mathrm{d}s\right]\leq C\big(\mathbb{E}\big[\|\mathbf{u}_0\|^2_{\mathbb{H}}\big]+\mathbb{E}^{\Pi}\|\pi\|^2_{\ell}+1\big). \tag{2.12}$$

Moreover, for $1<k<\infty$, if the initial data satisfy $\mathbb{E}\|\mathbf{u}_0\|^{2k}_{\mathbb{H}}<\infty$, there exists a constant K_1 such that

$$\|\sigma(t,\mathbf{u})\|^{2k}_{\mathrm{L}_Q(\mathbb{H})}+\int_{\mathrm{Z}}\|\gamma(t,\mathbf{u},z)\|^{2k}_{\mathbb{H}}\lambda(\mathrm{d}z)\leq K_1\big(1+\|\mathbf{u}\|^{2k}_{\mathbb{H}}\big), \tag{2.13}$$

and if the $2k-$order moments are finite for the chattering controls, i.e.,

$$\mathbb{E}^{\Pi}\left(\int_0^T\int_{\mathbb{U}}\ell(\mathbf{v})^{2k}\pi_t(\mathrm{d}\mathbf{v})\mathrm{d}t\right)<\infty.$$

Then, we have

$$\begin{aligned}&\mathbb{E}^{\mathbb{P}}\left[\sup_{0\leq t\leq T}\|\mathbf{u}(t)\|^{2k}_{\mathbb{H}}+\nu\int_0^T\|\mathbf{u}(s)\|^{2k-2}_{\mathbb{H}}\|\mathbf{A}^{1/2}\mathbf{u}(s)\|^2_{\mathbb{H}}\mathrm{d}s\right]\\&\leq C\left\{\mathbb{E}\big[\|\mathbf{u}_0\|^{2k}_{\mathbb{H}}\big]+\mathbb{E}^{\Pi}\left(\int_0^T\int_{\mathbb{U}}\ell(\mathbf{v})^{2k}\pi_t(\mathrm{d}\mathbf{v})\mathrm{d}t\right)+1\right\}.\end{aligned} \tag{2.14}$$

For the proof of Theorem 2.6, see Theorem 4.6, [32] and Theorem 3.1, [30]. Using the above estimates, the class of measures $\mathbb{P}$ in the above theorem which satisfy $\mathbb{E}^{\Pi}\|\pi\|^2_{\ell}\leq C$ for a fixed constant C can be shown to be tight (see Theorem 4.8, [32], Proposition 3.1,[30]). By the Galerkin approximation procedure, we get a sequence of measures in the tight class above (see section 4, [32] and section 3, [30]). Thus, we can extract a subsequence that weakly converges to a limit. We identify the limit as a solution of the martingale problem, posed by the stochastic Navier-Stokes equation perturbed by Lévy noise, using the continuity (Minty stochastic lemma, see Lemma 4.1, [30]) and uniform integrability (see Lemma 5.15, [32]) of the martingales involved (for more details see [31, 32, 30]). Also the solution to the martingale problem is unique (see Theorem 2.2, [30]).

3. Existence and Uniqueness of Invariant Measures

In this section, we discuss the existence and uniqueness of invariant measures for the controlled stochastic Navier-Stokes equation perturbed by Lévy noise following the ergodicity results obtained in [14, 10].

Definition 3.1. A probability measure μ on $(\mathrm{X},\mathscr{B}(\mathrm{X}))$ is called *an invariant measure or a stationary measure* for a given transition probability function $\mathrm{P}(t,\mathbf{x},\mathrm{d}\mathbf{y})$ if it satisfies

$$\mu(A)=\int_{\mathrm{X}}\mathrm{P}(t,\mathbf{x},A)\mathrm{d}\mu(\mathbf{x}),$$

for all $A \in \mathscr{B}(\mathrm{X})$ and $t > 0$. Equivalently, if for all $\phi \in \mathrm{C}_b(\mathrm{X})$ (the space of bounded continuous functions on X), and all $t \geq 0$,

$$\int_{\mathrm{X}} \phi(\mathbf{x}) \mathrm{d}\mu(\mathbf{x}) = \int_{\mathrm{X}} (\mathrm{P}_t\phi)(\mathbf{x}) \mathrm{d}\mu(\mathbf{x}),$$

where $(\mathrm{P}_t)_{t\geq 0}$ is defined by

$$\mathrm{P}_t\phi(\mathbf{x}) = \int_{\mathrm{X}} \phi(\mathbf{y}) \mathrm{P}(t, \mathbf{x}, \mathrm{d}\mathbf{y}).$$

Let $\mathbf{u}(t;\mathbf{x})$ denotes the solution of the controlled stochastic Navier-Stokes equation (2.6) with the initial condition $\mathbf{x} \in \mathbb{H}$. Let $(\mathrm{P}_t)_{t\geq 0}$ be the Markov semigroup in the space $\mathrm{C}_b(\mathbb{H})$ associated to the system (2.6) defined by

$$\mathrm{P}_t\phi(\mathbf{x}) = \mathbb{E}^{\mathbb{P}}[\phi(\mathbf{u}(t;\mathbf{x}))] = \int_{\mathbb{H}} \phi(\mathbf{y}) \mathrm{P}(t, \mathbf{x}, \mathrm{d}\mathbf{y}) = \int_{\mathbb{H}} \phi(\mathbf{y}) \mu_{t,\mathbf{x}}(\mathrm{d}\mathbf{y}),\ \phi \in \mathrm{C}_b(\mathbb{H}), \tag{3.1}$$

where $\mathrm{P}(t, \mathbf{x}, \mathrm{d}\mathbf{y})$ is the transition probability of $\mathbf{u}(t;\mathbf{x})$ and $\mu_{t,\mathbf{x}}$ is the law of $\mathbf{u}(t;\mathbf{x})$. From (3.1), we have

$$\mathrm{P}_t\phi(\mathbf{x}) = \langle \phi, \mu_{t,\mathbf{x}} \rangle = \langle \mathrm{P}_t\phi, \mu \rangle, \tag{3.2}$$

where μ is the law of the initial data $\mathbf{x} \in \mathbb{H}$. Thus from (3.2), we have $\mu_{t,\mathbf{x}} = \mathrm{P}_t^*\mu$. We say that a probability measure μ on $\mathbb{H}$ is an invariant measure if

$$\mathrm{P}_t^*\mu = \mu, \text{ for all } t \geq 0. \tag{3.3}$$

Let $\{\pi_t\}$ be a given measure-valued process with law Π.

Theorem 3.2. *Let $\mathbb{E}\big[\|\mathbf{u}_0\|_{\mathbb{H}}^2\big] < \infty$, where $\mathbf{u}(0) = \mathbf{u}_0$. Suppose there exist constants $\widetilde{C} > 0$ and $T_0 > 0$ such that*

$$\sup_{t>T_0} \frac{1}{t} \mathbb{E}^{\Pi}\bigg(\int_0^t \int_{\mathbb{U}} \ell^2(\mathbf{v}) \pi_s(\mathrm{d}\mathbf{v}) \mathrm{d}s \bigg) \leq \widetilde{C}. \tag{3.4}$$

If $K < 2\nu\lambda_1$, where λ_1 is the first eigenvalue of $\mathbf{A}$ and K is the constant appearing in the linear growth hypothesis (H.1). Then, there exists an invariant measure for the stochastic Navier-Stokes equation (2.6) with support in $\mathbb{V}$.

Proof. Let us define the sequence of stopping times by

$$\tau_N := \inf\bigg\{ t > 0 : \|\mathbf{u}(t)\|_{\mathbb{H}}^2 + \int_0^t \|\mathbf{u}(s)\|_{\mathbb{V}}^2 \mathrm{d}s \geq N \bigg\}, \tag{3.5}$$

for $N \in \mathbb{N}$. By using the Itô's formula given in Theorem 2.5 (see 2.11), Cauchy-Schwarz inequality, Young's inequality and Hypothesis 2.1, we have

$$\begin{aligned}
&\|\mathbf{u}(t\wedge\tau_N)\|_{\mathbb{H}}^2 + 2\nu \int_0^{t\wedge\tau_N} \|\mathbf{A}^{1/2}\mathbf{u}(s)\|_{\mathbb{H}}^2 \mathrm{d}s \\
&= \|\mathbf{u}(0)\|_{\mathbb{H}}^2 + 2\int_0^{t\wedge\tau_N} (\mathbf{N}\pi_s, \mathbf{u}(s))_{\mathbb{H}} \mathrm{d}s + 2\int_0^{t\wedge\tau_N} (\sigma(s,\mathbf{u}(s))\mathrm{d}\mathrm{W}(s), \mathbf{u}(s))_{\mathbb{H}} \\
&\quad + 2\int_0^{t\wedge\tau_N} \int_{\mathrm{Z}} (\gamma(s-,\mathbf{u}(s-),z), \mathbf{u}(s-))_{\mathbb{H}} \widetilde{\mathcal{N}}(\mathrm{d}s, \mathrm{d}z)
\end{aligned}$$

$$
\begin{aligned}
&+\int_0^{t\wedge\tau_N}\mathrm{Tr}(\sigma(s,\mathbf{u}(s))Q\sigma^*(s,\mathbf{u}(s)))\mathrm{d}s+\int_0^{t\wedge\tau_N}\int_{\mathrm{Z}}\|\gamma(s,\mathbf{u}(s),z)\|_{\mathbb{H}}^2\mathcal{N}(\mathrm{d}s,\mathrm{d}z)\\
&\le\|\mathbf{u}_0\|_{\mathbb{H}}^2+\varepsilon\int_0^{t\wedge\tau_N}\|\mathbf{u}(s)\|_{\mathbb{H}}^2\mathrm{d}s+\frac{1}{\varepsilon}\int_0^{t\wedge\tau_N}\|\mathbf{N}\pi_s\|_{\mathbb{H}}^2\mathrm{d}s+K\int_0^{t\wedge\tau_N}\big(1+\|\mathbf{u}(s)\|_{\mathbb{H}}^2\big)\mathrm{d}s\\
&+2\int_0^{t\wedge\tau_N}(\sigma(s,\mathbf{u}(s))\mathrm{d}\mathrm{W}(s),\mathbf{u}(s))_{\mathbb{H}}\\
&+\int_0^{t\wedge\tau_N}\int_{\mathrm{Z}}\big[2(\gamma(s-,\mathbf{u}(s-),z),\mathbf{u}(s-))_{\mathbb{H}}+\|\gamma(s,\mathbf{u}(s),z)\|_{\mathbb{H}}^2\big]\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}z),
\end{aligned}
\tag{3.6}
$$

for $\varepsilon>0$, where we used the fact that $(\mathbf{B}(\mathbf{u}),\mathbf{u})_{\mathbb{H}}=\mathrm{b}(\mathbf{u},\mathbf{u},\mathbf{u})=0$. Note that $\|\mathbf{A}^{1/2}\mathbf{u}\|_{\mathbb{H}}^2\ge\lambda_1\|\mathbf{u}\|_{\mathbb{H}}^2$ for all $\mathbf{u}\in\mathbb{V}$ and $\|\mathbf{A}^{1/2}\mathbf{u}\|_{\mathbb{H}}$ is equivalent to $\|\mathbf{u}\|_{\mathbb{V}}$. Thus by making use of these facts in (3.6), we obtain

$$
\begin{aligned}
&\|\mathbf{u}(t\wedge\tau_N)\|_{\mathbb{H}}^2+\left(2\nu-\frac{K}{\lambda_1}-\frac{\varepsilon}{\lambda_1}\right)\int_0^{t\wedge\tau_N}\|\mathbf{u}(s)\|_{\mathbb{V}}^2\mathrm{d}s\\
&\le\|\mathbf{u}_0\|_{\mathbb{H}}^2+\frac{2}{\varepsilon}\int_0^t\int_{\mathbb{U}}\ell^2(\mathbf{v})\pi_s(\mathrm{d}\mathbf{v})\mathrm{d}s+\left(\frac{2C^2}{\varepsilon}+K\right)t\\
&\quad+2\int_0^{t\wedge\tau_N}(\sigma(s,\mathbf{u}(s))\mathrm{d}\mathrm{W}(s),\mathbf{u}(s))_{\mathbb{H}}\\
&\quad+\int_0^{t\wedge\tau_N}\int_{\mathrm{Z}}\big[2(\gamma(s-,\mathbf{u}(s-),z),\mathbf{u}(s-))_{\mathbb{H}}+\|\gamma(s,\mathbf{u}(s),z)\|_{\mathbb{H}}^2\big]\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}z),
\end{aligned}
\tag{3.7}
$$

where C is a constant appearing in (H.4). Let us take expectation in (3.7) and use the fact that the last two terms in the right hand side of the inequality (3.7) are local martingales with zero expectation to find

$$
\begin{aligned}
&\mathbb{E}^{\mathbb{P}}\big[\|\mathbf{u}(t\wedge\tau_N)\|_{\mathbb{H}}^2\big]+\left(2\nu-\frac{K}{\lambda_1}-\frac{\varepsilon}{\lambda_1}\right)\mathbb{E}^{\mathbb{P}}\left(\int_0^{t\wedge\tau_N}\|\mathbf{u}(s)\|_{\mathbb{V}}^2\mathrm{d}s\right)\\
&\le\mathbb{E}\big[\|\mathbf{u}_0\|_{\mathbb{H}}^2\big]+\frac{2}{\varepsilon}\mathbb{E}^{\Pi}\left[\int_0^t\int_{\mathbb{U}}\ell^2(\mathbf{v})\pi_s(\mathrm{d}\mathbf{v})\mathrm{d}s\right]+\left(\frac{2C^2}{\varepsilon}+K\right)t.
\end{aligned}
\tag{3.8}
$$

By using (3.4) and $K<2\nu\lambda_1$, we get

$$
\frac{1}{t}\mathbb{E}^{\mathbb{P}}\left(\int_0^{t\wedge\tau_N}\|\mathbf{u}(s)\|_{\mathbb{V}}^2\mathrm{d}s\right)\le\widetilde{K},\ \text{for all } t>T_0,
\tag{3.9}
$$

where $\widetilde{K}$ is a constant independent of t. Clearly, τ_N tends to infinity, a.s., as $N\to\infty$ and hence $t\wedge\tau_N\to t$ as $N\to\infty$. Thus by using the Markov's inequality, we have

$$
\begin{aligned}
\lim_{R\to\infty}\sup_{T>T_0}\left\{\frac{1}{T}\int_0^T\mathbb{P}\big(\|\mathbf{u}(t)\|_{\mathbb{V}}>R\big)\mathrm{d}t\right\}&\le\lim_{R\to\infty}\sup_{T>T_0}\frac{1}{R^2}\mathbb{E}^{\mathbb{P}}\left[\frac{1}{T}\int_0^T\|\mathbf{u}(t)\|_{\mathbb{V}}^2\mathrm{d}t\right]\\
&=0.
\end{aligned}
\tag{3.10}
$$

Hence the estimate in (3.10) implies the existence of an invariant measure for the system (2.6) with support in $\mathbb{V}$ by a result of Chow and Khasminskii (see [4], Theorem 1.5.15, [2]). □

We make the following assumptions to establish the uniqueness of invariant measures for the system (2.6) (see [14, 10]).

Hypothesis 3.3. *There exist positive constants K, L such that, for some $\alpha \in [1/4, 1/2)$,*

(A.1) $\sigma : \mathbb{H} \to \mathbb{H}$ *is a bounded linear operator with* $\mathrm{R}(\sigma Q^{1/2})$ *dense in* $\mathrm{D}\big(\mathbf{A}^{\frac{1}{4}+\frac{\alpha}{2}}\big)$, *and*

$$\mathrm{D}\big(\mathbf{A}^{2\alpha}\big) \subset \mathrm{R}\Big(\sigma Q^{1/2}\Big) \subset \mathrm{D}\Big(\mathbf{A}^{\frac{1}{4}+\frac{\alpha}{2}+\varepsilon}\Big)$$

for some $\varepsilon > 0$, where $\mathrm{R}(\cdot)$ denotes the range of the operator.

(A.2) $\displaystyle\int_{\mathrm{Z}} \|\mathbf{A}^{\alpha}\boldsymbol{\gamma}(t,\mathbf{u},z)\|^2_{\mathbb{H}}\lambda(\mathrm{d}z) \leq K\big(1+\|\mathbf{A}^{\alpha}\mathbf{u}\|^2_{\mathbb{H}}\big),$

(A.3) $\displaystyle\int_{\mathrm{Z}} \|\mathbf{A}^{\alpha}(\boldsymbol{\gamma}(t,\mathbf{u}_1,z)-\boldsymbol{\gamma}(t,\mathbf{u}_2,z)\|^2_{\mathbb{H}}\lambda(\mathrm{d}z) \leq L\|\mathbf{A}^{\alpha}(\mathbf{u}_1-\mathbf{u}_2)\|^2_{\mathbb{H}},$

(A.4) $\displaystyle\sup_{\|\mathbf{u}\|_{\mathbb{H}}\leq M}\int_{\mathrm{Z}^c_m} \|\mathbf{A}^{\alpha}\boldsymbol{\gamma}(t,\mathbf{u},z)\|^2_{\mathbb{H}}\mathrm{d}z \to 0$ *as $m \to \infty$, for any fixed $M > 0$.*

Example 3.4. *An example of an operator satisfying hypothesis* (A.1) *is* $\mathbf{A}^{-\delta}\mathrm{L}$ *with* L *an isomorphism in* $\mathbb{H}$ *and* $\frac{3}{8} < \delta \leq \frac{1}{2}$.

In order to establish the uniqueness of invariant measure, we first consider the Ornstein-Uhlenbeck process $\mathbf{z}$ that is the solution of

$$\left.\begin{aligned} \mathrm{d}\mathbf{z}(t) + \mathbf{A}\mathbf{z}(t)\mathrm{d}t = \sigma\mathrm{dW}(t),\\ \mathbf{z}(0) = 0. \end{aligned}\right\} \tag{3.11}$$

By an application of the Galerkin approximation and suitable a-priori energy estimates, one can prove the following theorem (see [14, 10]).

Theorem 3.5. *Let $\mathbf{u}_0 \in \mathrm{D}(\mathbf{A}^{\alpha})$, and Hypothesis 3.3 be satisfied. Then there exists a unique solution $\mathbf{u}$ of (2.6) such that*

$$\mathbf{u}(\cdot,\omega) - \mathbf{z}(\cdot,\omega) \in \mathbb{D}(0,T;\mathrm{D}(\mathbf{A}^{\alpha})) \cap \mathbb{L}^{\frac{4}{1-2\alpha}}(0,T;\mathrm{D}(\mathbf{A}^{\frac{1}{4}+\frac{\alpha}{2}})) \cap \mathbb{L}^2(0,T;\mathrm{D}(\mathbf{A}^{\frac{1}{2}+\alpha})),$$

$\mathbb{P}-a.s., \omega \in \Omega$. *This is a Markov process satisfying the Feller property.*

Proof. See Theorem 2.3, [10] for the uncontrolled case. □

By Theorem 3.5, the solution $\mathbf{u}(\cdot)$ of (2.6) belongs to $\mathbb{D}(0,T;\mathrm{D}(\mathbf{A}^{\alpha}))$, $\mathbb{P}-$a.s., for the initial value $\mathbf{u}_0 \in \mathrm{D}(\mathbf{A}^{\alpha})$. Also, by Theorem 2.6, [13], there exists a unique strong solution in $\mathbb{D}(0,T;\mathbb{H})\cap\mathbb{L}^2(0,T;\mathrm{D}(\mathbf{A}^{\frac{1}{2}}))$, $\mathbb{P}-$a.s., for the initial value $\mathbf{u}_0 \in \mathbb{H}$ under the Hypothesis 2.1. By the uniqueness result, we get $\mathbf{u} \in \mathbb{D}(0,T;\mathrm{D}(\mathbf{A}^{\alpha}))$, $\mathbb{P}-$a.s., for $\mathbf{u}_0 \in \mathbb{H}$. Also, by Theorem 3.2, there exists an invariant probability measure, which has support in $\mathrm{D}(\mathbf{A}^{\frac{1}{2}}) \subset \mathrm{D}(\mathbf{A}^{\alpha})$.

Let $(\mathrm{P}_t)_{t\geq 0}$ be the Markov semigroup associated to the system (2.6) in the space $\mathrm{C}_b(\mathrm{D}(\mathbf{A}^{\alpha}))$. It is well known that the *irreducibility* and *strong Feller property* imply the equivalence of the measures $\mathrm{P}(t,\mathbf{x},\cdot)$ (see Theorem 4.1, [14]). The irreducibility property (see Theorem 3.7, [10] for the uncontrolled case) of the solution of the controlled stochastic Navier-Stokes system (2.6) and the strong Feller property (see Theorem 3.4, [10] for the uncontrolled case) of the semigroup $(\mathrm{P}_t)_{t\geq 0}$ imply the following:

(P.1) For all $t > 0$, all $\mathbf{x}, \mathbf{y} \in \mathrm{D}(\mathbf{A}^\alpha)$, there exists an $\widetilde{M} > 0$ such that for all $\rho > 0$,

$$\mathrm{P}(t, \mathbf{x}, \mathrm{J}(\mathbf{y}, \rho, \widetilde{M})) = \mathbb{P}\Big\{\mathbf{u}(t; \mathbf{x}) \in \mathrm{J}(\mathbf{y}, \rho, \widetilde{M})\Big\} > 0,$$

where $\mathrm{J}(\mathbf{y}, \rho, \widetilde{M}) = \Big\{\mathbf{w} \in \mathrm{D}(\mathbf{A}^\alpha) : \|\mathbf{A}^\alpha(\mathbf{w} - \mathbf{y})\|_{\mathbb{H}} < \rho, \|\mathbf{A}^\alpha \mathbf{w}\|_{\mathbb{H}} \leq \widetilde{M}\Big\}$.

(P.2) For all $\Gamma \in \mathscr{B}(\mathrm{D}(\mathbf{A}^\alpha))$ (the Borel $\sigma-$algebra of $\mathrm{D}(\mathbf{A}^\alpha)$), all $t > 0$, and all $\mathbf{x}_n, \mathbf{x} \in \mathrm{D}(\mathbf{A}^\alpha)$ such that $\mathbf{x}_n \to \mathbf{x}$ in $\mathbb{H}$ and $\|\mathbf{A}^\alpha \mathbf{x}_n\|_{\mathbb{H}} \leq C$ for some constant $C > 0$, then we have

$$\mathrm{P}(t, \mathbf{x}_n, \Gamma) \to \mathrm{P}(t, \mathbf{x}, \Gamma).$$

In other words, $(\mathrm{P}_t)_{t \geq 0}$ can be extended to a continuous operator from $\mathrm{B}_b(\mathrm{D}(\mathbf{A}^\alpha))$ to $\mathrm{C}_b(\mathrm{D}(\mathbf{A}^\alpha))$, where $\mathrm{B}_b(\mathrm{D}(\mathbf{A}^\alpha))$ is the space of bounded measurable functions on $\mathrm{D}(\mathbf{A}^\alpha)$.

Properties (P.1) and (P.2) imply the irreducibility and the strong Feller property in $\mathrm{D}(\mathbf{A}^\alpha)$ and hence the transition probabilities $\mathrm{P}(t, \mathbf{x}, \cdot)$, $t > 0$, $\mathbf{x} \in \mathbb{H}$, associated with the controlled stochastic Navier-Stokes system (2.6), are equivalent on $\mathrm{D}(\mathbf{A}^\alpha)$, for all $t > 0$ and $\mathbf{u}(0) = \mathbf{x} \in \mathrm{D}(\mathbf{A}^\alpha)$ (see Theorem 4.1, [14]). By Lemma 4.1, [14], they are equivalent on $\mathbb{H}$ for all $t > 0$ and $\mathbf{x} \in \mathbb{H}$. The uniqueness of invariant measures follows from this and hence we have the following theorem:

Theorem 3.6. *Under the Hypothesis 2.1, Hypothesis 3.3 and (3.4), there exists a unique invariant measure for the controlled stochastic Navier-Stokes system (2.6) for each control measure* Π.

Since, there exists a unique stationary measure for the semigroup $(\mathrm{P}_t)_{t \geq 0}$, it is *ergodic* (see Theorem 9, [8], Theorem 3.2.6, [5]). Let μ be unique invariant measure for the the semigroup $(\mathrm{P}_t)_{t \geq 0}$ associated to the system (2.6). By Dynkin's formula, we have

$$\mathbb{E}^{\mathbb{P}}_{\mathbf{x}}[f(\mathbf{u}(t))] = \mathbb{E}^{\mathbb{P}}_{\mathbf{x}}[f(\mathbf{x})] + \mathbb{E}^{\mathbb{P}}_{\mathbf{x}}\left[\int_0^t (\mathscr{L} f)(\mathbf{u}(s))\mathrm{d}s\right], \text{ for all } f \in \mathrm{D}(\mathscr{L}), \tag{3.12}$$

where $\mathbf{u}(0) = \mathbf{x}$. Thus, from (3.12), we obtain

$$\langle \mathrm{P}^*_t \mu, f\rangle = \langle \mathrm{P}^*_0 \mu, f\rangle + \int_0^t \langle \mathrm{P}^*_s \mu, \mathscr{L} f\rangle \mathrm{d}s. \tag{3.13}$$

Since μ is an invariant measure, $\mathrm{P}^*_t \mu = \mu$, for all $t \geq 0$, and hence we have

$$\langle \mu, \mathscr{L} f\rangle = 0, \text{ for all } f \in \mathrm{D}(\mathscr{L}). \tag{3.14}$$

4. Optimal Ergodic Control

Let us now prove the existence of an ergodic control which is optimal in the class of stationary measures for the system (2.6). The existence of an optimal and ergodic probability measure result discussed here is general and it holds in both two and three dimensions, and also for degenerate noise, as the existence of a unique invariant measure is known for these cases.

Theorem 4.1. *Under the Hypothesis 2.1, Hypothesis 3.3, (2.13) and (3.4), there exists an ergodic control which is optimal within the class of stationary measures for the controlled stochastic Navier-Stokes system (2.6).*

Proof. Let us define

$$\Gamma := \{\mu \in \mathcal{P}(\mathbb{V}) \times \mathcal{P}(\mathcal{P}(\mathbb{U})) : \mathcal{L}(\mathbf{u}(t), \pi(t)) = \mu, \text{ for all } t > 0\}, \tag{4.1}$$

where $\mathcal{L}(\mathbf{u}(t), \pi(t))$ is the law of $(\mathbf{u}(t), \pi(t))$, for all $t \geq 0$. The set Γ is closed by the strong Feller property (P.2). Let us now consider the set

$$\Gamma_R := \left\{\mu \in \mathcal{P}(\mathbb{V}) \times \mathcal{P}(\mathcal{P}(\mathbb{U})) : \int \mathcal{K}(\mathbf{u}, \mathbf{v}) \mathrm{d}\mu \leq R\right\}, \tag{4.2}$$

for some large $R > 0$. Since $\int \mathcal{K}(\mathbf{u}, \mathbf{v}) \mathrm{d}\mu \leq R$, using (1.5), we have

$$\int \left[\|\mathbf{u}\|_{\mathbb{V}}^2 + \ell^4(\mathbf{v})\right] \mathrm{d}\mu \leq R. \tag{4.3}$$

Let us use (4.3) in the energy estimate (2.14) (for $k = 2$) to obtain

$$\int \|\mathbf{u}\|_{\mathbb{H}}^2 \|\mathbf{u}\|_{\mathbb{V}}^2 \mathrm{d}\mu \leq C(R). \tag{4.4}$$

Using (4.3), and the continuity and uniform integrability of the martingales which appear in the martingale problem formulation implies the tightness of Γ_R (see Theorem 8, [32]). Making use of (4.3) and the tightness of Γ_R, one can extract a minimizing sequence (see Lemma 5, [32]) of tight measures $\mu^n \in \Gamma_R$ in the weak topology such that

$$\beta := \lim_{n \to \infty} \int \mathscr{K} \pi \mathrm{d}\mu^n = \inf_{\Gamma} \left\{\int \mathscr{K} \pi \mathrm{d}\mu\right\}, \tag{4.5}$$

where $\mathscr{K}\pi = \int_{\mathbb{U}} \mathcal{K}(\mathbf{u}, \mathbf{v}) \pi(\mathrm{d}\mathbf{v})$ and

$$\langle \mu^n, f\rangle \to \langle \mu, f\rangle \text{ for all } f \in \mathrm{C}_b(\mathbb{H} \times \mathcal{P}(\mathbb{U})), \tag{4.6}$$

and μ is supported in $\mathbb{V} \times \mathcal{P}(\mathbb{U})$ (see Lemma 2.3.3, [38]). Since μ^n is an invariant measure, from (3.14), we know that

$$\langle \mu^n, \mathscr{L} f\rangle = 0, \text{ for all } f \in \mathrm{D}(\mathscr{L}), \tag{4.7}$$

where $\mathscr{L}$ is given in (2.8). Now, we want to show that

$$\langle \mu^n, \mathscr{L} f\rangle = 0 \to \langle \mu, \mathscr{L} f\rangle = 0, \text{ for all } f \in \mathrm{D}(\mathscr{L}). \tag{4.8}$$

In order to obtain (4.8), we need to prove (see Lemma 6.3.1, [38], Lemma 15, [32])

$$\left\langle \mu^n, |\mathscr{L} f|^{1+\varepsilon} \right\rangle \leq C, \tag{4.9}$$

for some $\varepsilon > 0$. Let us take the tame function given in Example 2.3 and bound $\left\langle \mu^n, |\mathscr{L} f|^2 \right\rangle$ using (4.3) and (4.4). Using the fact that $\phi(\cdot) : \mathbb{R}^m \to \mathbb{R}$ is a smooth function with compact support in $\mathbb{R}^m$ and $e_k \in \mathrm{D}(\mathbf{A})$, for $k = 1, \cdots, m$, Hypothesis 2.1, Hypothesis 3.3, and Remark 2.2.2, [35], we obtain

$$\int \left| \left\langle \mathbf{A}\mathbf{u}, \frac{\partial f}{\partial \mathbf{u}} \right\rangle \right|^2 \mathrm{d}\mu^n \leq C \int \|\mathbf{u}\|_{\mathbb{H}}^2 \mathrm{d}\mu^n \leq \frac{C}{\lambda_1} \int \|\mathbf{u}\|_{\mathbb{V}}^2 \mathrm{d}\mu^n \leq C(R), \tag{4.10}$$

$$\int\left|\left\langle \mathbf{B}(\mathbf{u}),\frac{\partial f}{\partial \mathbf{u}}\right\rangle\right|^2 \mathrm{d}\mu^n \le C\int \|\mathbf{u}\|_{\mathbb{H}}^2\|\mathbf{u}\|_{\mathbb{V}}^2\mathrm{d}\mu^n \le C(R), \tag{4.11}$$

$$\int\left|\left\langle \mathbf{N}\pi,\frac{\partial f}{\partial \mathbf{u}}\right\rangle\right|^2 \mathrm{d}\mu^n \le C\int \|\mathbf{N}\pi\|_{\mathbb{H}}^2\mathrm{d}\mu^n \le C\int\big\langle \ell^2(\mathbf{v})+C,\pi\big\rangle\mathrm{d}\mu^n \le C(R), \tag{4.12}$$

$$\int\left|\mathrm{Tr}\left(\sigma Q\sigma^*\frac{\partial^2 f}{\partial \mathbf{u}^2}\right)\right|^2 \mathrm{d}\mu^n \le C\,\mathrm{Tr}(\sigma Q\sigma^*)^2, \tag{4.13}$$

$$\begin{aligned}&\int\left|\int_{\mathrm{Z}}\left\{f(\mathbf{u}+\gamma(t,\mathbf{u},z))-f(\mathbf{u})-\left\langle\gamma(t,\mathbf{u},z),\frac{\partial f}{\partial \mathbf{u}}\right\rangle\right\}\lambda(\mathrm{d}z)\right|^2\mathrm{d}\mu^n\\ &\le C\int\int_{\mathrm{Z}}\|\gamma(t,\mathbf{u},z)\|_{\mathbb{H}}^4\lambda(\mathrm{d}z)\mathrm{d}\mu^n \le K\int\big(1+\|\mathbf{u}\|_{\mathbb{H}}^4\big)\mathrm{d}\mu^n \le C(R).\end{aligned} \tag{4.14}$$

The estimates (4.10)-(4.14) easily implies

$$\Big\langle \mu^n, |\mathscr{L}f|^2\Big\rangle \le C(R). \tag{4.15}$$

Using Lemma 6.3.1, [38] (see also Lemma 15, [32]), we have

$$\langle\mu^n,\mathscr{L}f\rangle \to \langle\mu^n,\mathscr{L}f\rangle, \tag{4.16}$$

and hence

$$\langle\mu,\mathscr{L}f\rangle = 0, \text{ for all } f\in \mathrm{D}(\mathscr{L}). \tag{4.17}$$

Since, the running cost $\mathcal{K}(\cdot,\cdot):\mathbb{V}\times\mathbb{U}\to[0,\infty]$ is positive and lower semicontinuous, by Theorem III.55, page 71-III, [9], the map

$$\mu\mapsto\int\mathscr{K}\pi\mathrm{d}\mu$$

is lower semicontinuous (see Lemma 10, [32]). Hence, we have

$$\int\mathscr{K}\pi\mathrm{d}\mu \le \liminf_{n\to\infty}\int\mathscr{K}\pi\mathrm{d}\mu^n < \infty, \tag{4.18}$$

whenever $\mu^n\to\mu$ in the weak topology. Since $\mu^n\in\Gamma$ is a minimizing sequence, we have

$$\liminf_{n\to\infty}\int\mathscr{K}\pi\mathrm{d}\mu^n = \lim_{n\to\infty}\int\mathscr{K}\pi\mathrm{d}\mu^n = \beta. \tag{4.19}$$

Using (4.18) and (4.19), one easily gets

$$\beta = \int\mathscr{K}\pi\mathrm{d}\mu,$$

and μ is an optimal and ergodic probability measure. Hence the cost functional

$$\limsup_{t\to\infty}\frac{1}{t}\int_0^t\mathbb{E}^{\mathbb{P}}\left(\int_{\mathbb{U}}\mathcal{K}(\mathbf{u}(s),\mathbf{v})\pi_s(\mathrm{d}\mathbf{v})\right)\mathrm{d}s, \tag{4.20}$$

achieves a minimum $\langle\mathscr{K}\pi,\mu\rangle$ for $\mu=\mathcal{L}(\mathbf{u}(t),\pi(t))$, for all $t\ge0$. □

Remark 4.2. As the ergodicity results for the 3D stochastic Navier-Stokes equations are known (see [6, 7, 25]) and the estimate (4.11) is true in 3D (see Remark 2.2.2, [35]), the Theorem 4.1 holds in three dimensions also. Note that the Theorem 4.1 works if the noise is degenerate as the ergodicity results are known for this case also (see [27, 17, 29]).

Remark 4.3. Since the abstract functional setting for a class of nonlinear controlled stochastic hydrodynamic models perturbed by Lévy noise, namely *2D magnetohydrodynamic(MHD) equation*, *2D Boussinesq model for the Bénard convection*, *2D magnetic Bénard problem*, *3D Leray α-model for Navier-Stokes equation*, *Shell models of turbulence* are same as that of *2D Navier-Stokes equation*, the optimal ergodic control results obtained in this paper applies to these models also.

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