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FUZZY REAL SESQUILINEAR FORM AND ITS PROPERTIES

ABSTRACT: *Defininition of fuzzy real sesquilinear form is introduced. A decomposition theorem from a fuzzy real sesquilinear form into a family of sesquilinear forms and other decomposition theorem from a family of sesquilinear forms into a fuzzy real sesquilinear form are established.*

Keywords: *Fuzzy real sesquilinear form, equipotent fuzzy real sesquilinear form, decomposition theorem.*

1. INTRODUCTION

The notion of fuzzy norm on a linear space was first introduced by Katsaras [10] in 1984. Felbin [7], Cheng & Mordeson [5], Bag & Samanta [2] etc. defined fuzzy normed linear spaces in different approaches. Sklar [1] introduced the idea of real probabilistic inner product space. R. Biswas [4], A.M.El-Abyed & H.M.Hamouly [6], J.K.Kohli & R.Kumar [11], Pinaki Mazumder & S.K.Samanta [12], A.Hasankhani, A.Nazari, M.Saheli [9], M.Goudarzi & S.M.Vaezpour [8], S. Vijayabalaji [15], Mukherjee and Bag [13] introduced the concept of fuzzy inner product space in different aspects and many results have been developed in such spaces. Sesquilinear form plays an important role in functional analysis to develop operator theory. In this paper we define fuzzy real sesquilinear form. A decomposition theorem from a fuzzy real sesquilinear form into a family of sesquilinear forms is established.

The organization of the paper is as follows:

Section 2 comprises some preliminary results. Section 3 provides the definition of fuzzy real sesquilinear form and some examples are given. In Section 4 the decomposition theorem of a fuzzy real sesquilinear form into a family of sesquilinear forms is established.

2. PRELIMINARIES

Definition 2.1. [8] A fuzzy inner product space (FIP-space) is a triplet $(X, F, *)$, where X is a real vector space, $*$ is a continuous t-norm, F is a fuzzy set on $X^2 \times R$ and the following conditions hold for every $x, y, z \in X$ and $s, t, r \in R$,

(FI-1) $F(x, x, 0) = 0$ and $F(x, x, t) > 0$, for each $t > 0$;

(FI-2) $F(x, x, t) \neq H(t)$ for some $t \in R$ if and only if $x \neq 0$;

(FI-3) $F(x, y, t) = F(y, x, t)$;

(FI-4) For any real no α,

$$F(\alpha x, y, t) = \begin{cases} F\left(x, y, \frac{t}{\alpha}\right) & if\ \alpha > 0 \\ H(t) & if\ \alpha = 0 \\ 1 - F\left(x, y, \frac{t}{-\alpha}\right) & if\ \alpha < 0 \end{cases}$$

Where $H(t) = \begin{cases} 1 & if\ t > 0 \\ 0 & if\ t \leq 0 \end{cases}$

(FI-5) $\sup_{s+r=t} \left(F(x, z, s) * F(y, z, r)\right) = F(x + y, z, t)$;

(FI-6) $F(x, y, .) : R \to [0, 1]$ is continuous on $R\{0\}$;

(FI-7) $\lim_{t \to +\infty} F(x, y, t) = 1$

Definition 2.2 [14] Let X be a linear space over R (the set of real numbers). Then a fuzzy subset $F : \to X \times X \times R \to [0, 1]$ is called fuzzy real inner product on X if $\forall\ x, y, z \in X$ and $t, c \in R$ the following conditions hold,

(FI-1) $F(x, x, t) = 0\ \forall t < 0$

(FI-2) $(F(x, x, t) = 1\ \forall t > 0)$ iff $x = 0$

(FI-3) $F(x, y, t) = F(y, x, t)$

(FI-4) $F(cx, y, t) = \begin{cases} F\left(x, y, \frac{t}{c}\right) & if\ c > 0 \\ H(t) & if\ c = 0 \\ 1 - F\left(x, y, \frac{t}{c}\right) & if\ c < 0 \end{cases}$

where $H(t) = \begin{cases} 1 \; if \; t > 0 \\ 0 \; if \; t \le 0 \end{cases}$

(FI-5) $F(x+y, z, t+s) \ge \; F(x, z, t) \wedge F(y, z, s)$

(FI-6) $\lim_{t \to +\infty} F(x, y, t) = 1$

The pair (X, F) is said to be a fuzzy real inner product space.

3. FUZZY REAL SESQUILINEAR FORM

Definition 3.1 Let X and Y be two linear spaces over R (the set of real numbers) and $*$ is a continuous t-norm. $h: X \times Y \times R \to [0,1]$ is said to be a real fuzzy sesquilinear form if $\forall\, x, x_1, x_2 \in X$ and $\forall\, y, y_1, y_2 \in Y$ the following conditions hold

(FS-1) $h(x_1 + x_2, y, s+t) \ge h(x_1, y, s) * h(x_2, y, t)$ for $s, t > 0$

(FS-2) $h(x, y_1 + y_2, s+t) \ge h(x, y_1, s) * h(x, y_2, t)$ for $s, t > 0$

(FS-3) For any $c \in R$ and $t \ne 0$

$$h(cx, y, t) = h(x, cy, t) = \begin{cases} h\left(x, y, \frac{t}{c}\right) & if \; c > 0 \\ H(t) & if \; c = 0 \\ 1 - h\left(x, y, \frac{t}{c}\right) & if \; c < 0 \end{cases}$$

Where $H(t) = \begin{cases} 1 & if \; t > 0 \\ 0 & if \; t \le 0 \end{cases}$

Remark 3.2 $h(x, y, .): (0, \infty) \to [0,1]$ and $h(x, y, .): (-\infty, 0) \to [0,1]$ are non-decreasing.

Proof. Let $t_1, t_2 > 0$ and $t_1 > t_2$.

Therefore $t_1 - t_2 > 0$.

For $x \in X$ and $y \in Y$

$h(0 + x, y, t_1 - t_2 + t_2) \ge h(0, y, t_1 - t_2) * h(x, y, t_2)$

$\Rightarrow h(x, y, t_1) \ge 1 * h(x, y, t_2)$ [From (FS-3) and since $H(t_1 - t_2) = 1$]

$$\Rightarrow h(x, y, t_1) \geq h(x, y, t_2).$$

Thus $h(x, y, .): (0, \infty) \rightarrow [0, 1]$ is non-decreasing.

Now let $t_1, t_2 < 0$ and $t_1 > t_2$.

Then $t_1 - t_2 > 0$

$h(0 - x, y, t_1 - t_2 - t_1) \geq h(0, y, t_1 - t_2) * h(-x, y, -t_1)$

$\Rightarrow h(-x, y, -t_2) \geq 1 * h(-x, y, -t_1)$ [From (FS-3) and since $H(t_1 - t_2) = 1$]

$$\Rightarrow h(-x, y, -t_2) \geq h(-x, y, -t_1)$$

$$\Rightarrow 1 - h(x, y, t_2) \geq 1 - h(x, y, t_1)$$

$$\Rightarrow h(x, y, t_1) \geq h(x, y, t_2)$$

Thus $h(x, y, .): (-\infty, 0) \rightarrow [0, 1]$ is non-decreasing.

Example 3.3 Let $g: X \times Y \rightarrow R$ be a real sesquilinear form, where X and Y are real linear spaces. Define $h: X \times Y \times R \rightarrow [0, 1]$ by $h(\alpha x, y, t) = h(x, \alpha y, t)$ $= H(t)$ for $\alpha = 0$ and for $\alpha \neq 0$,

$$h(\alpha x, y, t) = \begin{cases} 1 & if\ t > g(\alpha x, y) \\ \frac{1}{2} & if\ t = g(\alpha x, y) \\ 0 & if\ t < g(\alpha x, y) \end{cases}$$

Then h is a fuzzy real sesquilinear form w.r.t. $*$ t-norm.

Proof. (FS-1) Let s, t be two +ve real numbers.

(Case-I) Let $t > g(x_1, z)$ and $s > g(x_2, z)$.

So $h(x_1, z, t) = 1$, $h(x_2, z, s) = 1$.

Then $t + s > g(x_1, z) + g(x_2, z) = g(x_1 + x_2, z)$

$$\Rightarrow h(x_1 + x_2, z, t + s) = 1 \geq h(x_1, z, t) * h(x_2, z, s).$$

(Case-II) Let $t = g(x_1, z)$ and $s > g(x_2, z)$ or $t > g(x_1, z)$ and $s = g(x_2, z)$ hold.

So $h(x_1, z, t) = \frac{1}{2}$, $h(x_2, z, s) = 1$ or $h(x_1, z, t) = 1$, $h(x_2, z, s) = \frac{1}{2}$.

Then $t + s > g(x_1, z) + g(x_2, z) = g(x_1 + x_2, z)$

$$\Rightarrow h(x_1 + x_2, z, t + s) = 1 > \frac{1}{2} \geq h(x_1, z, t) * h(x_2, z, s).$$

(Case-III) If any one of $t < g(x_1, z)$ or $s < g(x_2, z)$ holds,

then $h(x_1, z, t) = 0$ or $h(x_2, z, s) = 0$

So $h(x_1, z, t) * h(x_2, z, s) = 0$ and clearly

$h(x_1 + x_2, z, t + s) \geq h(x_1, z, t) * h(x_2, z, s)$.

Thus in all the three cases we have, $h(x_1 + x_2, z, t + s) \geq h(x_1, z, t) * h(x_2, z, s)$.

(FS-2) Similarly we can show that, $h(x, y_1 + y_2, t + s) \geq h(x, y_1, t) * h(x, y_2, s)$.

(FS-3) (Case-I) Let $\alpha > 0$.

(Subcase-i) Let $t > 0$ then $\frac{t}{\alpha} > 0$.

If $h(\alpha x, y, t) = 1$, then $t > g(\alpha x, y) = \alpha g(x, y)$

$$\Rightarrow \frac{t}{\alpha} > g(x, y)$$

$$\Rightarrow h\left(x, y, \frac{t}{\alpha}\right) = 1$$

If $h(\alpha x, y, t) = \frac{1}{2}$, then $t = g(\alpha x, y)$

$$\Rightarrow \frac{t}{\alpha} = g(x, y)$$

$$\Rightarrow h\left(x, y, \frac{t}{\alpha}\right) = \frac{1}{2}$$

Similarly $h(\alpha x, y, t) = 0 \Rightarrow h\left(x, y, \frac{t}{\alpha}\right) = 0$.

(Subcase-ii) Let $t < 0$ then similarly we get $h(\alpha x, y, t) = h\left(x, y, \frac{t}{\alpha}\right)$.

(Case-II) Let $\alpha = 0$, then by definition $h(\alpha x, y, t) = \mathrm{H(t)}$.

(Case-III) Let $\alpha < 0$, then $\alpha = -\mathrm{p}$ (say) for some $\mathrm{p} > 0$.

(Subcase-i) Let $t > 0$ then $\frac{t}{\alpha} < 0$.

If $h(\alpha x, y, t) = 1$, then $t > g(\alpha x, y)$

$$\Rightarrow t > -p\, g(x, y)$$

$$\Rightarrow \frac{t}{p} > -g(x, y)$$

$$\Rightarrow \frac{t}{-p} < g(x, y)$$

$$\Rightarrow \frac{t}{\alpha} < g(x, y)$$

$$\Rightarrow h\left(x, y, \frac{t}{\alpha}\right) = 0.$$

Therefore $h(\alpha x, y, t) = 1 - h\left(x, y, \frac{t}{\alpha}\right)$.

If $h(\alpha x, y, t) = \frac{1}{2}$, then $t = g(\alpha x, y)$

$$\Rightarrow \frac{t}{\alpha} = g(x, y)$$

$$\Rightarrow h\left(x, y, \frac{t}{\alpha}\right) = \frac{1}{2}.$$

Thus $h(\alpha x, y, t) = \frac{1}{2} = 1 - \frac{1}{2} = 1 - h\left(x, y, \frac{t}{\alpha}\right)$.

Similarly if $h(\alpha x, y, t) = 0$, then $t < g(\alpha x, y)$

$$\Rightarrow t < -p\, g(x, y)$$

$$\Rightarrow \frac{t}{p} < -g(x, y)$$

$$\Rightarrow \frac{t}{\alpha} > g(x, y)$$

$$\Rightarrow h\left(x, y, \frac{t}{\alpha}\right) = 1.$$

Therefore $h(\alpha x, y, t) = 1 - h\left(x, y, \frac{t}{\alpha}\right)$.

(Subcase-ii) Let $t < 0$ then $\frac{t}{\alpha} > 0$.

If $h(\alpha x, y, t) = 1$, then $t > g(\alpha x, y)$

$$\Rightarrow \frac{t}{\alpha} < g(x,y)$$

$$\Rightarrow h\left(x,y,\frac{t}{\alpha}\right) = 0.$$

Therefore $h(\alpha x,y,t) = 1 - h\left(x,y,\frac{t}{\alpha}\right)$.

If $h(\alpha x,y,t) = \frac{1}{2}$, then $t = \alpha|g(\alpha x,y)|$

$$\Rightarrow \frac{t}{\alpha} = |g(x,y)|$$

$$\Rightarrow h\left(x,y,\frac{t}{\alpha}\right) = \frac{1}{2}$$

$$\Rightarrow h(\alpha x,y,t) = 1 - h\left(x,y,\frac{t}{\alpha}\right).$$

If $h(\alpha x,y,t) = 0$, then $t < g(\alpha x,y)$

$$\Rightarrow \frac{t}{\alpha} > g(x,y)$$

$$\Rightarrow h\left(x,y,\frac{t}{\alpha}\right) = 1.$$

$$\Rightarrow h(\alpha x,y,t) = 1 - h\left(x,y,\frac{t}{\alpha}\right).$$

Also we can show that $h(\alpha x,y,t) = h(x,\alpha y,t)$.

Thus in all the cases we have, for any $\alpha \in R$ and $t \neq 0$

$$h(\alpha x,y,t) = h(x,\alpha y,t) = \begin{cases} h\left(x,y,\dfrac{t}{\alpha}\right) & if\ \alpha > 0 \\ H(t) & if\ \alpha = 0 \\ 1 - h\left(x,y,\dfrac{t}{\alpha}\right) & if\ \alpha < 0 \end{cases}$$

Hence h is a fuzzy real sesquilinear form.

Example 3.4 Let (X,F) be a fuzzy real inner product space.

Define $h(x,y,t) = F(x,y,t)\ \forall t \in R$. Then h is a fuzzy real sesquilinear form on $X \times X \times R$.

4. DECOMPOSITION THEOREM

In this Section a decomposition theorem from a fuzzy real sesquilinear form into a family of sesquilinear forms is established.

Theorem 4.1 Let X and Y be two real linear spaces and $h: X \times Y \times R \to [0,1]$be a fuzzy real sesquilinear form and choose 'min' as $*$ t-norm.

Define for $\alpha \in (0,1)$

$h_\alpha^+(x,y) = \wedge\{t > 0 : h(x,y,t) \geq \alpha\}$,

$h_\alpha^-(x,y) = \vee\{t < 0 : h(x,y,t) \leq 1-\alpha\}$ and

$h_\alpha(x,y) = h_\alpha^+(x,y) + h_\alpha^-(x,y)$

Then $\{h_\alpha(.,.) : \alpha \in (0,1)\}$ forms a family of sesquilinear form on $X \times Y$and we are called α-sesquilinear forms of h.

Proof. (I) Let $\alpha \in (0,1), \forall\, x, x_1, x_2 \in X$ and $\forall\, y, z \in Y$.

We have, $h_\alpha^+(x,y) = \wedge\{t > 0 : h(x,y,t) \geq \alpha\}$,

$h_\alpha^-(x,y) = \vee\{t < 0 : h(x,y,t) \leq 1-\alpha\}$

Let $\epsilon > 0$ be given.

Now $h(x_1 + x_2,\ y, h_\alpha^+(x_1,y) +\ h_\alpha^+(x_2,y) + \epsilon)$

$\geq\ h(x_1,\ y,\ h_\alpha^+(x_1,y) + \frac{\epsilon}{2}) \wedge h(x_2, y, h_\alpha^+(x_2,y) + \frac{\epsilon}{2})$

$\geq (\alpha) \wedge (\alpha) = \alpha.$

Therefore $\wedge\{t > 0 : h(x_1 + x_2\,, y, t) \geq \alpha\} \leq h_\alpha^+(x_1,y) +\ h_\alpha^+(x_2,y) + \epsilon$

$\Rightarrow h_\alpha^+(x_1 + x_2\,, y) \leq h_\alpha^+(x_1,y) +\ h_\alpha^+(x_2,y) + \epsilon$

Since ϵ is arbitrary, so

$h_\alpha^+(x_1 + x_2\,, y) \leq h_\alpha^+(x_1,y) +\ h_\alpha^+(x_2,y).$ (4.1.1)

Let $A = 1 - [(1 -\ h(x_1,\ z,\ h_\alpha^+(x_1,z) - \frac{\epsilon}{2})) \wedge (1 -\ h(x_2,\ z,\ h_\alpha^+(x_2,z) - \frac{\epsilon}{2}))]$

Then $A = 1 - [\ h(-x_1,\ z, -h_\alpha^+(x_1,z) + \frac{\epsilon}{2}) \wedge h(-x_2,\ z, -\ h_\alpha^+(x_2,z) + \frac{\epsilon}{2})]$

$\geq 1 -\ h(-x_1 - x_2,\ z, -h_\alpha^+(x_1,z) - h_\alpha^+(x_2,z) +\ \epsilon)$

$=\ h(x_1 + x_2,\ z, h_\alpha^+(x_1,z) + h_\alpha^+(x_2,z) -\ \epsilon)$

Therefore $A \geq h(x_1 + x_2, z, h_\alpha^+(x_1, z) + h_\alpha^+(x_2, z) - \epsilon)$ (4.1.2)

Now $h(x_1, z, h_\alpha^+(x_1, z) - \frac{\epsilon}{2}) < \alpha, h(x_2, z, h_\alpha^+(x_2, z) - \frac{\epsilon}{2}) < \alpha$

$\Rightarrow 1 - h(x_1, z, h_\alpha^+(x_1, z) - \frac{\epsilon}{2}) > 1 - \alpha,\ 1 - h(x_2, z, h_\alpha^+(x_2, z) - \frac{\epsilon}{2}) > 1 - \alpha$

$\Rightarrow [1 - h(x_1, z, h_\alpha^+(x_1, z) - \frac{\epsilon}{2})] \wedge [1 - h(x_2, z, h_\alpha^+(x_2, z) - \frac{\epsilon}{2})] > 1 - \alpha$

$\Rightarrow A = 1 - [(1 - h(x_1, z, h_\alpha^+(x_1, z) - \frac{\epsilon}{2})) \wedge (1 - h(x_2, z, h_\alpha^+(x_2, z) - \frac{\epsilon}{2}))] < \alpha.$

From (4.1.2) we have $h(x_1 + x_2, z, h_\alpha^+(x_1, z) + h_\alpha^+(x_2, z) - \epsilon) \leq A < \alpha$

$\Rightarrow h_\alpha^+(x_1 + x_2, z) \geq h_\alpha^+(x_1, z) + h_\alpha^+(x_2, z) - \epsilon$ (since h is non-decreasing)

Since $\epsilon > 0$ is arbitrary, we have

$$h_\alpha^+(x_1 + x_2, z) \geq h_\alpha^+(x_1, z) + h_\alpha^+(x_2, z). \quad (4.1.3)$$

From (4.1.1) and (4.1.3) we have

$$h_\alpha^+(x_1 + x_2, z) = h_\alpha^+(x_1, z) + h_\alpha^+(x_2, z). \quad (4.1.4)$$

Now $h(x_1 + x_2, z, h_\alpha^-(x_1, z) + h_\alpha^-(x_2, z) + \epsilon)$

$\geq h(x_1, z, h_\alpha^-(x_1, z) + \frac{\epsilon}{2}) \wedge h(x_2, z, h_\alpha^-(x_2, z) + \frac{\epsilon}{2})$

$> (1 - \alpha) \wedge (1 - \alpha) = 1 - \alpha$

Therefore $\vee\{t < 0 : h(x_1 + x_2, z, t) \leq 1 - \alpha\} \leq h_\alpha^-(x_1, z) + h_\alpha^-(x_2, z) + \epsilon.$

$\Rightarrow h_\alpha^-(x_1 + x_2, z) \leq h_\alpha^-(x_1, z) + h_\alpha^-(x_2, z) + \epsilon.$

Since $\epsilon > 0$ is arbitrary,

$$h_\alpha^-(x_1 + x_2, z) \leq h_\alpha^-(x_1, z) + h_\alpha^-(x_2, z). \quad (4.1.5)$$

Now if

$$B = 1 - [(1 - h(x_1, z, h_\alpha^-(x_1, z) - \frac{\epsilon}{2})) \wedge (1 - h(x_2, z, h_\alpha^-(x_2, z) - \frac{\epsilon}{2}))]$$

$$= 1 - [h(-x_1, z, -h_\alpha^-(x_1, z) + \frac{\epsilon}{2}) \wedge h(-x_2, z, -h_\alpha^-(x_2, z) + \frac{\epsilon}{2})]$$

$$\geq 1 - h(-x_1 - x_2, z, -h_\alpha^-(x_1, z) - h_\alpha^-(x_2, z) + \epsilon)$$

$$= h(x_1 + x_2, z, h_\alpha^-(x_1, z) + h_\alpha^-(x_2, z) - \epsilon)$$

Therefore $B \geq h(x_1 + x_2, z, h_\alpha^-(x_1, z) + h_\alpha^-(x_2, z) - \epsilon)$ (4.1.6)

Now $h(x_1, z, h_\alpha^-(x_1, z) - \frac{\epsilon}{2}) \leq 1 - \alpha, h(x_2, z, h_\alpha^-(x_2, z) - \frac{\epsilon}{2}) \leq 1 - \alpha$

$\Rightarrow 1 - h(x_1, z, h_\alpha^-(x_1, z) - \frac{\epsilon}{2}) \geq \alpha, \; 1 - h(x_2, z, h_\alpha^-(x_2, z) - \frac{\epsilon}{2}) \geq \alpha$

$\Rightarrow [1 - h(x_1, z, h_\alpha^-(x_1, z) - \frac{\epsilon}{2})] \wedge [1 - h(x_2, z, h_\alpha^-(x_2, z) - \frac{\epsilon}{2})] \geq \alpha$

$\Rightarrow B = 1 - [(1 - h(x_1, z, h_\alpha^-(x_1, z) - \frac{\epsilon}{2})) \wedge (1 - h(x_2, z, h_\alpha^-(x_2, z) - \frac{\epsilon}{2}))] \leq 1 - \alpha.$

From (4.1.6) we have $h(x_1 + x_2, z, h_\alpha^-(x_1, z) + h_\alpha^-(x_2, z) - \epsilon) \leq B \leq 1 - \alpha$

$\Rightarrow h_\alpha^-(x_1 + x_2, z) \geq h_\alpha^-(x_1, z) + h_\alpha^-(x_2, z) - \epsilon$.

Since $\epsilon > 0$ is arbitrary, we have

$$h_\alpha^-(x_1 + x_2, z) \geq h_\alpha^-(x_1, z) + h_\alpha^-(x_2, z). \tag{4.1.7}$$

From (4.1.5) and (4.1.7) we have

$$h_\alpha^-(x_1 + x_2, z) = h_\alpha^-(x_1, z) + h_\alpha^-(x_2, z). \tag{4.1.8}$$

Therefore from (4.1.4) and (4.1.8) we get

$h_\alpha(x_1 + x_2, z) = h_\alpha^+(x_1 + x_2, z) + h_\alpha^-(x_1 + x_2, z)$

$= h_\alpha^+(x_1, z) + h_\alpha^+(x_2, z) + h_\alpha^-(x_1, z) + h_\alpha^-(x_2, z)$

$= h_\alpha(x_1, z) + h_\alpha(x_2, z) \quad \forall \alpha \in (0, 1).$

(II) Similarly (as (I)) we can show that,

$$h_\alpha(x, y + z) = h_\alpha(x, y) + h_\alpha(x, z) \quad \forall \alpha \in (0, 1).$$

(III) (a) Let $c > 0$ and $\alpha \in (0, 1)$.

Then $h_\alpha(cx, y) = h_\alpha^+(cx, y) + h_\alpha^-(cx, y)$

$= \wedge\{t > 0 : h(cx, y, t) \geq \alpha\} + \vee\{t < 0 : h(cx, y, t) \leq 1 - \alpha\}$

$= \wedge\left\{t > 0 : h\left(x, y, \frac{t}{c}\right) \geq \alpha\right\} + \vee\left\{t < 0 : h\left(x, y, \frac{t}{c}\right) \leq 1 - \alpha\right\}$

$= \wedge\{cs > 0 : h(x, y, s) \geq \alpha\} + \vee\{cs < 0 : h(x, y, s) \leq 1 - \alpha\}$ (where $s = \frac{t}{c}$)

$= c\wedge\{s > 0 : h(x, y, s) \geq \alpha\} + c\vee\{s < 0 : h(x, y, s) \leq 1 - \alpha\}$

$= c\big(\, h_\alpha^+(x,y) + h_\alpha^-(x,y)\big) = ch_\alpha(x,\ y).$

(b) Let $c = 0$. Then $h(cx,y,t)$= H(t) $(t \neq 0)$

Therefore $h_\alpha^+(cx,y) = \wedge\{t>0 : H(t) \geq \alpha\} = \wedge\{t > 0 : 1 \geq \alpha\} = 0$ and $h_\alpha^-(cx,y) = \vee\{t<0 : H(t) \leq 1-\alpha\} = \vee\{t < 0 : 0 \leq 1 - \alpha\} = 0.$

Thus $h_\alpha(cx,\ y) = h_\alpha^+(cx,y) + h_\alpha^-(cx,y) = 0 + 0 = 0 = ch_\alpha(x,\ y).$

(c) Let $c < 0$.

So $h_\alpha^+(cx,y) = \wedge\{t > 0 : h(cx,y,t) \geq \alpha\}$

$$= \wedge\left\{t > 0 : 1 - h\left(x,y,\tfrac{t}{c}\right) \geq \alpha\right\}$$

Let $\frac{t}{c} = s$. Then $t > 0 \Rightarrow s < 0$.

So $h_\alpha^+(cx,y) = \wedge\{cs > 0 : 1 - h(x,y,s) \geq \alpha\}$

$$= c\vee\{s < 0 : 1 - h(x,y,s) \geq \alpha\}$$

$$= c\vee\{s < 0 : h(x,y,s) \leq 1 - \alpha\}$$

$$= ch_\alpha^-(x,y). \tag{4.1.9}$$

Now $h_\alpha^-(cx,y) = \vee\{t < 0 : h(cx,y,t) \leq 1 - \alpha\}$

$$= \vee\left\{t < 0 : 1 - h\left(x,y,\tfrac{t}{c}\right) \leq 1 - \alpha\right\}.$$

Let $\frac{t}{c} = s$. Then $t < 0 \Rightarrow s > 0$.

So $h_\alpha^-(cx,y) = \vee\{cs < 0 : 1 - h(x,y,s) \leq 1 - \alpha\}$

$$= c\wedge\{s > 0 : 1 - h(x,y,s) \leq 1 - \alpha\}$$

$$= c\wedge\{s > 0 : h(x,y,s) \geq \alpha\}$$

$$= ch_\alpha^+(x,y). \tag{4.1.10}$$

Therefore from (4.1.9) and (4.1.10) we have,

$$h_\alpha(cx,\ y) = h_\alpha^+(cx,y) + h_\alpha^-(cx,y)$$

$$= ch_\alpha^-(x,y) + ch_\alpha^+(x,y)$$

$$= c\big(\, h_\alpha^+(x,y) + h_\alpha^-(x,y)\big)$$

$= ch_\alpha(x,\ y).$

Thus for any real number c, $h_\alpha(cx,\ y) = ch_\alpha(x,\ y)$.

(IV) Similarly we can show that, $h_\alpha(x,\ cy) = ch_\alpha(x,\ y)$ for any scalar c, $\forall\ \alpha \in (0,1)$.

Thus from (I)-(IV) it follows that, $h_\alpha(x,\ y)$ is a sesquilinear form and $\{h_\alpha(.,.) : \alpha \in (0,1)\}$ forms a family of sesquilinear form on $X \times Y$.

Theorem 4.2: $\{h_\alpha^+(.,.) : \alpha \in (0,1)\}$ and $\{h_\alpha^-(.,.) : \alpha \in (0,1)\}$ are ascending and descending functions respectively.

Proof. Let $\alpha_1 > \alpha_2$.

So $\{t > 0 : h(x,y,t) \geq \alpha_1\} \subset \{t > 0 : h(x,y,t) \geq \alpha_2\}$

$\Rightarrow \wedge\{t > 0 : h(x,y,t) \geq \alpha_1\} \geq \wedge\{t > 0 : h(x,y,t) \geq \alpha_2\}$

$\Rightarrow h_{\alpha_1}^+(x,\ y) \geq\ h_{\alpha_2}^+(x,\ y)$.

Therefore $\{h_\alpha^+(.,.) : \alpha \in (0,1)\}$ is an ascending family of functions on $X \times Y$.

Similarly if $\alpha_1 > \alpha_2$ then $1 - \alpha_1 < 1 - \alpha_2$.

So $\{t < 0 : h(x,y,t) \leq 1 - \alpha_1\} \subset \{t < 0 : h(x,y,t) \leq 1 - \alpha_2\}$

$\Rightarrow\ \vee\{t < 0 : h(x,y,t) \leq 1 - \alpha_1\} \leq \vee\{t < 0 : h(x,y,t) \leq 1 - \alpha_2\}$

$\Rightarrow h_{\alpha_1}^-(x,\ y) \leq\ h_{\alpha_2}^-(x,\ y)$.

Therefore $\{h_\alpha^-(.,.) : \alpha \in (0,1)\}$ is a descending family of functions on $X \times Y$.

Theorem 4.3: Let $\{h_\alpha(.,.) : \alpha \in (0,1)\}$ be a family of sesquilinear form on $X \times Y$, where X and Y are real linear spaces. We difine a function $h' : X \times Y \times R \to [0,1]$ as

$$h'(x,y,t) = \begin{cases} \vee\{\alpha \in (0,1): h_\alpha(x,y)\ \leq t\} & if\ t > 0 \\ 0 & if\ t = 0 \\ \wedge\ \{\alpha \in (0,1): h_{1-\alpha}(x,y)\ \geq t\} & if\ t < 0 \end{cases}$$

Then h' is a fuzzy real sesquilinear form on $X \times Y \times R$.

Proof. Let $x, x_1, x_2 \in X$ and $y, z \in Y$.

(FS-1) Let $t > 0, s > 0$ then $t + s > 0$.

Let $h'(x_1, z, t) \wedge h'(x_2, z, s) > \alpha$.

Therefore $h'(x_1, z, t) > \alpha$ and $h'(x_2, z, s) > \alpha$

$$\Rightarrow h_\alpha(x_1, z) \leq t \text{ and } h_\alpha(x_2, z) \leq s$$

$$\Rightarrow h_\alpha(x_1 + x_2, z) = h_\alpha(x_1, z) + h_\alpha(x_2, z) \leq t + s$$

$$\Rightarrow h'(x_1 + x_2, z, t + s) \geq \alpha$$

$$\Rightarrow h'(x_1 + x_2, z, t + s) \geq h'(x_1, z, t) \wedge h'(x_2, z, s).$$

(FS-2) Similarly we can show that,

$$h'(x, y + z, t + s) \geq h'(x, y, t) \wedge h'(x, z, s).$$

(FS-3) (a) Let $c > 0$.

If $t > 0$ then $h'(cx, y, t) = \vee\{\alpha \in (0,1): h_\alpha(cx, y) \leq t\}$

$$= \vee\left\{\alpha \in (0,1);\ h_\alpha(x, y) \leq \frac{t}{c}\right\}$$

$$= h'\left(x, y, \frac{t}{c}\right) \text{ (since } \tfrac{\mathrm{t}}{\mathrm{c}} > 0\text{)}.$$

Similarly if t<0 then $h'(cx, y, t) = h'\left(x, y, \frac{t}{c}\right)$.

(b) Let $c = 0$.

If $t > 0$ then $h'(cx, y, t) = \vee\{\alpha \in (0,1): h_\alpha(cx, y) \leq t\}$

$= \vee\{\alpha \in (0,1);\ \alpha \in (0,1)\}$ (since $h_\alpha(cx, y) = c\, h_\alpha(x, y) = 0$)

$= 1$.

If $t < 0$ then $h'(cx, y, t) = \wedge\{\alpha \in (0,1): h_{1-\alpha}(cx, y) \geq t\}$

$= \wedge\{\alpha \in (0,1);\ \alpha \in (0,1)\}$ (since $h_{1-\alpha}(cx, y) = c\, h_{1-\alpha}(x, y) = 0$)

$= 0$.

Also if $t = 0$ then $h'(cx, y, t) = 0$

Thus $h'(cx, y, t) = H(t)$.

(c) Let $c < 0$.

If $t > 0$ then $h'(cx, y, t) = \vee\{\alpha \in (0,1)\colon\ h_\alpha(cx, y) \ \leq t\}$

$$= \vee\left\{\alpha \in (0,1);\ h_\alpha(x, y) \geq \frac{t}{c}\right\}.$$

Let $\alpha = 1 - \beta$

Then $h'(cx, y, t) = \vee\left\{(1-\beta) \in (0,1)\colon\ h_{1-\beta}(x, y) \ \geq \frac{t}{c}\right\}$

$$= 1 - \wedge\left\{\beta \in (0,1);\ h_{1-\beta}(x, y) \geq \frac{t}{c}\right\}$$

$$= 1 - h'\left(x, y, \frac{t}{c}\right) \ \ (\text{since } \frac{t}{c} < 0).$$

If $t < 0$ then $h'(cx, y, t) = \wedge\{\alpha \in (0,1)\colon\ h_{1-\alpha}(cx, y) \ \geq t\}$

$$= \wedge\left\{\alpha \in (0,1);\ h_{1-\alpha}(x, y) \leq \frac{t}{c}\right\}$$

Let $\alpha = 1 - \beta$

Then $h'(cx, y, t) = \wedge\left\{(1-\beta) \in (0,1)\colon\ h_\beta(x, y) \ \leq \frac{t}{c}\right\}$

$$= 1 - \vee\left\{\beta \in (0,1);\ h_\beta(x, y) \leq \frac{t}{c}\right\}$$

$$= 1 - h'\left(x, y, \frac{t}{c}\right) \ \ (\text{since } \frac{t}{c} > 0).$$

Also we can show that, $h'(cx, y, t) = h'(x, cy, t)$, for any real number c.

Therefore h' is a fuzzy real sesquilinear form on $X \times Y \times R$.

Definition 4.4: Let X and Y be two linear spaces over R and h be a fuzzy real sesquilinear form on $X \times Y \times R$. For $x \in X, y \in Y$, we define

$$h(x, y, t+) = h_+(x, y, t) = \lim_{s \to t+} h(x, y, s) \text{ and}$$

$$h(x, y, t-) = h_-(x, y, t) = \lim_{s \to t-} h(x, y, s).$$

Theorem 4.5: Let X and Y be two linear spaces over R and h_1, h_2 be two fuzzy real sesquilinear form on $X \times Y \times R$. Then $\forall\, x \in X,\ \forall\, y \in Y, \forall t \in R - \{0\}$,

$h_1(x, y, t+) = h_2(x, y, t+)$ and $h_1(x, y, t-) = h_2(x, y, t-)$ iff

$h_{\alpha}^{1,+}(x,y) = h_{\alpha}^{2,+}(x,y)$ and $h_{\alpha}^{1,-}(x,y) = h_{\alpha}^{2,-}(x,y)\ \ \forall\alpha \in (0,1)$

where $h_{\alpha}^{i,+}(x,y) = \wedge\{t > 0 :\ h_i(x,y,t\ ) \geq \alpha\}$, and

$h_{\alpha}^{i,-}(x,y) = \vee\{t < 0 :\ h_i(x,y,t\ ) \leq 1-\alpha\}$, $i = 1,2$.

Proof. Let $h_{\alpha}^{1,+}(x,y) = h_{\alpha}^{2,+}(x,y)$ and $h_{\alpha}^{1,-}(x,y) = h_{\alpha}^{2,-}(x,y)\ \ \forall\alpha \in (0,1)$

If possible suppose that for some $t = t_0$, $h_1(x,y,t_0+) \neq h_2(x,y,t_0+)$.

Without loss of generality we may assume $h_1(x,y,t_0+) < h_2(x,y,t_0+)$.

If $t_0 > 0$ then for $0 < t_0 < t < t_0 + \varepsilon$ $(\varepsilon > 0)$, $h_1(x,y,t) < h_2(x,y,t)$.

Choose β such that $h_1(x,y,t) < \beta < h_2(x,y,t)$.

Also $h_{\alpha}^{i,+}(x,y) = \wedge\{t > 0 :\ h_i(x,y,t\ ) \geq \alpha\}$, $i = 1,2$.

$h_{\beta}^{2,+}(x,y) \leq t_0$ and $h_{\beta}^{1,+}(x,y) \geq t_0 + \varepsilon$ - a contradiction.

If $t_0 < 0$ then for $t_0 < t < t_0 + \varepsilon < 0$ $(\varepsilon > 0)$, $h_1(x,y,t) < h_2(x,y,t)$.

Choose β such that $h_1(x,y,t) < \beta < h_2(x,y,t)$.

Now $h_{\alpha}^{i,-}(x,y) = \vee\{t > 0 :\ h_i(x,y,t\ ) \leq 1-\alpha\}$, $i = 1,2$.

Therefore $h_{1-\beta}^{2,-}(x,y) \leq t_0$ and $h_{1-\beta}^{1,-}(x,y) \geq t_0 + \varepsilon$ - a contradiction.

Hence $h_1(x,y,t+) = h_2(x,y,t+)\ \forall t \in R-\{0\}$.

Similarly we can prove $h_1(x,y,t-) = h_2(x,y,t-)\ \forall t \in R-\{0\}$.

Conversely suppose $h_1(x,y,t+) = h_2(x,y,t+)$ and $h_1(x,y,t-) = h_2(x,y,t-)$ hold $\forall t \in R-\{0\}$.

If possible let $\exists\ \alpha_0 \in (0,1)$ such that $h_{\alpha_0}^{1,+}(x,y) \neq h_{\alpha_0}^{2,+}(x,y)$ or $h_{\alpha_0}^{1,-}(x,y) \neq h_{\alpha_0}^{2,-}(x,y)$.

Let $h_{\alpha_0}^{1,+}(x,y) \neq h_{\alpha_0}^{2,+}(x,y)$.

Without loss of generality we can choose k_1, k_2, k_3 such that

$h_{\alpha_0}^{1,+}(x,y) > k_1 > k_2 > k_3 > h_{\alpha_0}^{2,+}(x,y)$.

Therefore $h_1(x,y,k_1) < \alpha_0, h_2(x,y,k_3) \geq \alpha_0$.

Thus $\alpha_0 > h_1(x,y,k_1) \geq h_1(x,y,k_2+), h_2(x,y,k_2-) \geq\ h_2(x,y,k_3) \geq \alpha_0$.

Therefore we have $h_1(x,y,k_2+) < \alpha_0 \le h_2(x,y,k_2-) \le h_2(x,y,k_2+)$

$\Rightarrow h_1(x,y,k_2+) < h_2(x,y,k_2+)$ -a contradiction.

Therefore $h_{\alpha_0}^{1,+}(x,y) \neq h_{\alpha_0}^{2,+}(x,y)$, $\forall \alpha \in (0,1), \forall\, x \in X, \forall\, y \in Y$.

Similarly $h_{\alpha_0}^{1,-}(x,y) \neq h_{\alpha_0}^{2,-}(x,y)$, $\forall \alpha \in (0,1), \forall\, x \in X, \forall\, y \in Y$.

Definition 4.6: Let X and Y be two linear spaces over R and h_1, h_2 be two fuzzy real sesquilinear forms on $X \times Y \times R$. Then h_1 and $h_2$are said to be equipotent if

$h_1(x,y,t-) = h_2(x,y,t-)$ and $h_1(x,y,t+) = h_2(x,y,t+)$ $\forall t \in R - \{0\}$ and $\forall\, x \in X, \forall\, y \in Y$.

Theorem 4.7: Let X and Y be two linear spaces over R and h be a fuzzy real sesquilinear form on $X \times Y \times R$ and $\{h_\alpha(.,.) : \alpha \in (0,1)\}$ denotes the families of α-sesquilinear form of h where

$h_\alpha(x,y) = h_\alpha^+(x,y) + h_\alpha^-(x,y)$. Define h'': $X \times Y \times R \to [0,1]$ as

$$h''(x,y,t) = \begin{cases} \vee\{\alpha \in (0,1): h_\alpha^+(x,y) \le t\} & if\ t > 0 \\ 0 & if\ t = 0 \\ \wedge\ \{\alpha \in (0,1): h_{1-\alpha}^-(x,y) \ge t\} & if\ t < 0 \end{cases}$$

Then h'' is a fuzzy real sesquilinear form on $X \times Y \times R$ and h, h'' are equipotent.

Proof. Let $x, x_1, x_2 \in X$ and $y, z \in Y$.

(FS-1) Let $t > 0, s > 0$ then $t + s > 0$.

Let $h''(x_1,z,t) \wedge h''(x_2,z,s) > \alpha$ where $\alpha \in (0,1)$.

Therefore $h''(x_1,z,t) > \alpha$ and $h''(x_2,z,s) > \alpha$

$\Rightarrow h_\alpha^+(x_1, z) \le t$ and $h_\alpha^+(x_2, z) \le s$

$\Rightarrow h_\alpha^+(x_1+x_2,z) = h_\alpha^+(x_1, z) + h_\alpha^+(x_2, z) \le t + s$ (From (I) of Theorem 4.1)

$\Rightarrow h''(x_1+x_2,z,t+s) \ge \alpha$

$\Rightarrow h''(x_1+x_2,z,t+s) \ge h''(x_1,z,t) \wedge h''(x_2,z,s)$.

(FS-2) Similarly we can show that,

$$h''(x, y + z, s + t) \geq h''(x, y, s) \wedge h''(x, z, t), \text{ for } s, t > 0.$$

(FS-3) (a) Let $c > 0$.

If $t > 0$ then $h''(cx, y, t) = \vee\{\alpha \in (0,1):\ h_\alpha^+(cx, y) \ \leq t\}$

Now $h_\alpha^+(cx, y) = \wedge\{t > 0 : h(cx, y, t) \geq \alpha\}$

$$= \wedge\left\{t > 0 : h\left(x, y, \frac{t}{c}\right) \geq \alpha\right\}$$

$$= \wedge\{cs > 0 : h(x, y, s) \geq \alpha\} \ \ (\text{where } \ s = \tfrac{t}{c})$$

$$= c\wedge\{s > 0 : h(x, y, s) \geq \alpha\}$$

$$= ch_\alpha^+(x, y).$$

Therefore $h''(cx, y, t) = \vee\{\alpha \in (0,1):\ ch_\alpha^+(x, y) \ \leq t\}$

$$= \vee\{\alpha \in (0,1):\ h_\alpha^+(x, y) \ \leq \tfrac{t}{c}\}$$

$$= h''(x, y, \tfrac{t}{c}) \ \ (\text{since } \tfrac{t}{c} > 0)$$

Similarly if $t < 0$ then $h''(cx, y, t) = h''(x, y, \frac{t}{c})$.

(b) Let $c = 0$.

If $t > 0$ then $h''(cx, y, t) = \vee\{\alpha \in (0,1):\ h_\alpha^+(cx, y) \ \leq t\}$

Now $h_\alpha^+(cx, y) = \wedge\{t > 0 : h(cx, y, t) \geq \alpha\} = 0$ (since $h(cx, y, t) = H(t)$).

Therefore $h''(cx, y, t) = \vee\{\alpha \in (0,1)\,;\ \alpha \in (0,1)\} = 1$.

If $t = 0$ then $h''(cx, y, t) = 0$.

If $t < 0$ then $h''(cx, y, t) = \wedge\{\alpha \in (0,1):\ h_{1-\alpha}^-(cx, y) \ \geq t\}$.

Now $h_{1-\alpha}^-(cx, y) = \vee\{t < 0 : h(cx, y, t) \leq \alpha\} = 0$ (since $h(cx, y, t) = H(t)$).

Therefore $h''(cx, y, t) = \wedge\{\alpha \in (0,1)\,;\ \alpha \in (0,1)\} = 0$.

Thus $h''(cx, y, t) = H(t)$ when $c = 0$.

(c) Let $c < 0$.

If $t > 0$ then $h''(cx, y, t) = \vee\{\alpha \in (0,1) \colon h_\alpha^+(cx, y) \le t\}$

Now $h_\alpha^+(cx, y) = \wedge\{t > 0 : h(cx, y, t) \ge \alpha\}$

$$= \wedge\left\{t > 0 : 1 - h\left(x, y, \frac{t}{c}\right) \ge \alpha\right\}$$

$$= \wedge\left\{t > 0 : h\left(x, y, \frac{t}{c}\right) \le 1 - \alpha\right\}$$

$$= \wedge\{cs > 0 : h(x, y, s) \le 1 - \alpha\} \text{ (where } s = \tfrac{t}{c})$$

$$= c\vee\{s < 0 : h(x, y, s) \le 1 - \alpha\}$$

$$= ch_\alpha^-(x, y).$$

Therefore $h''(cx, y, t) = \vee\{\alpha \in (0,1) \colon ch_\alpha^-(x, y) \le t\}$

$$= \vee\{\alpha \in (0,1) \colon h_\alpha^-(x, y) \ge \frac{t}{c}\}$$

Let $\alpha = 1 - \beta$

Then $h''(cx, y, t) = \vee\{(1-\beta) \in (0,1) \colon h_{1-\beta}^-(cx, y) \ge \frac{t}{c}\}$

$$= 1 - \wedge\left\{\beta \in (0,1) \colon h_{1-\beta}^-(x, y) \ge \frac{t}{c}\right\}$$

$$= 1 - h''\left(x, y, \frac{t}{c}\right) \text{ (since } \tfrac{t}{c} < 0).$$

Similarly if $t < 0$ then $h''(cx, y, t) = 1 - h''(x, y, \frac{t}{c})$.

Also we can show that $h''(cx, y, t) = h''(x, cy, t)\ \forall\, c \in R$.

Therefore h'' is a fuzzy real sesquilinear form on $X \times Y \times R$.

Now we shall show that h, h'' are equipotent.

If possible suppose that for some $t = t_0 \in R, h(x, y, t_0 -) \ne h''(x, y, t_0 -)$.

Without loss of generality we may suppose $h(x, y, t_0 -) < h''(x, y, t_0 -)$.

Choose β such that $h(x, y, t_0 -) < \beta < h''(x, y, t_0 -)$.

(Case-I) If $t_0 > 0$ then for $0 < t_0 - \varepsilon < t < t_0$ $(\varepsilon > 0)$.

$h(x, y, t) < \beta < h''(x, y, t)$.

Now for $t_0 - \varepsilon < t < t_0$, $h(x, y, t) < \beta \Rightarrow h_\beta^+(x, y) \geq t_0$.

But $h''(x, y, t) > \beta \Rightarrow h_\beta^+(x, y) \leq t \;\; \forall \; t \in (t_0 - \varepsilon, t_0)$.

Thus we arrive at a contradiction.

Therefore $h(x, y, t-) = h''(x, y, t-) \;\forall t > 0$.

(Case-II) If $t_0 < 0$ then for $t_0 - \varepsilon < t < t_0 < 0$ $(\varepsilon > 0)$.

Choose β such that, $h(x, y, t) < \beta < h''(x, y, t)$.

Now for $t_0 - \varepsilon < t < t_0$, $h(x, y, t) < \beta \Rightarrow h_{1-\beta}^-(x, y) \geq t_0$.

But $h''(x, y, t) > \beta \Rightarrow h_{1-\beta}^-(x, y) \leq t \quad \forall \; t \in (t_0 - \varepsilon, t_0)$ - a contradiction.

Therefore $h(x, y, t-) = h''(x, y, t-) \;\forall t < 0$.

Similarly we can prove $h(x, y, t+) = h''(x, y, t+) \;\forall t \in R - \{0\}$.

Hence h and h'' are equipotent.

Now a question may arise whether h and h'' are equal or not.

To discuss the issue we assume,

(FS-4) For a fuzzy real sesquilinear form h on $X \times Y \times R$ where X and Y are two linear spaces over R,

$$\lim_{t \to +\infty} h(x, y, t) = 1$$

(FS-5) For $x \neq 0 \neq y$, $h(x, y, .): (0, \infty) \to [0, 1]$and $h(x, y, .): (-\infty, 0) \to [0, 1]$ are both upper and lower semicontinuous respectively and $h(x, y, 0) = 0$.

Theorem 4.8: Let $X$and Y be two linear spaces over R and h be a fuzzy real sesquilinear form on $X \times Y \times R$ satisfying (FS-5).

Let for $\alpha \in (0, 1)$, $h_\alpha^+(x, y) = \wedge\{t > 0 : \; h(x, y, t) \geq \alpha\}$,

$h_\alpha^-(x, y) = \vee\{t < 0 : \; h(x, y, t) \leq 1 - \alpha\}$ and

$h_\alpha(x, y) = h_\alpha^+(x, y) + h_\alpha^-(x, y)$.

Define a function $h'': X \times Y \times R \to [0,1]$ as

$$h''(x,y,t) = \begin{cases} \vee\{\alpha \in (0,1): h_\alpha^+(x,y) \le t\} & if\ t > 0 \\ 0 & if\ t = 0 \\ \wedge\{\alpha \in (0,1): h_{1-\alpha}^-(x,y) \ge t\} & if\ t < 0 \end{cases}$$

Then (i) $\{h_\alpha(.,.): \alpha \in (0,1)\}$ is a family of real sesquilinear forms on $X \times Y$.

(ii) h'' is a fuzzy real sesquilinear form on $X \times Y \times R$.

(iii) $h = h''$

To prove this theorem we first prove the following lemma.

Lemma 4.9: Let h be a fuzzy real sesquilinear form on $X \times Y \times R$ satisfying (FS-4) where X and Y are two real linear spaces. Let $x_0(\neq 0) \in X, y_0(\neq 0) \in$Y and for $\alpha \in (0,1)$,

$h_\alpha^+(x_0,y_0) = \wedge\{t > 0 : h(x_0,y_0,t) \ge \alpha\}$,

$h_\alpha^-(x_0,y_0) = \vee\{t < 0 : h(x_0,y_0,t) \le 1-\alpha\}$. Then

(1) if $h(x,y,.): (0,\infty) \to [0,1]$ and $h(x,y,.): (-\infty,0) \to [0,1]$ are both upper and lower semicontinuous respectively and

(a) If for $t_0 > 0$, $h(x_0,y_0,t_0) = \alpha_0 \in (0,1)$ then $h\left(x_0,y_0,h_{\alpha_0}^+(x_0,y_0)\right) = \alpha_0$

(b) If for $t_0 < 0$, $h(x_0,y_0,t_0) = \alpha_0 \in (0,1)$ then $h\left(x_0,y_0,h_{1-\alpha_0}^-(x_0,y_0)\right)$ $= \alpha_0$

(2) if $h(x,y,.): (0,\infty) \to [0,1]$ and $h(x,y,.): (-\infty,0) \to [0,1]$ are both upper and lower semicontinuous respectively, then for any $\alpha \in (0,1)$, $h\big(x_0,y_0,h_\alpha^+(x_0,y_0)\big) = \alpha$ and $h\big(x_0,y_0,h_{1-\alpha}^-(x_0,y_0)\big) = \alpha$.

Proof.

(1) (a) Note that $h_{\alpha_0}^+(x_0,y_0) = \wedge\{t > 0 : h(x_0,y_0,t) \ge \alpha_0\}$. Let $t_0 > 0$.

Since $h(x_0,y_0,t_0) = \alpha_0$, we get $h_{\alpha_0}^+(x_0,y_0) \le t_0$.

Since , $h(x_0,y_0,.): (0,\infty) \to [0,1]$ is nondecreasing, we have $\alpha_0 = h(x_0,y_0,t_0) \ge h\left(x_0,y_0,h_{\alpha_0}^+(x_0,y_0)\right)$

$\Rightarrow\ h\left(x_0, y_0, h^+_{\alpha_0}(x_0, y_0)\right) \leq \alpha_0$.

If possible suppose that $h\left(x_0, y_0, h^+_{\alpha_0}(x_0, y_0)\right) < \alpha_0$.

Then by the upper semicontinuity of $h(x_0, y_0, .): (0, \infty) \to [0, 1]$,

$\exists t' >\ h^+_{\alpha_0}(x_0, y_0)$ such that $h(x_0, y_0, t') < \alpha_0$.

Then $h^+_{\alpha_0}(x_0, y_0) = \wedge\{t > 0 : h(x_0, y_0, t) \geq \alpha_0\} \geq t' > h^+_{\alpha_0}(x_0, y_0)$ - a contradiction.

Therefore $h\left(x_0, y_0, h^+_{\alpha_0}(x_0, y_0)\right) = \alpha_0$.

(b) Let $t_0 < 0$ then $h^-_{1-\alpha_0}(x_0, y_0) = \vee\{t < 0 : h(x_0, y_0, t) \leq \alpha_0\}$.

Since $h(x_0, y_0, t_0) = \alpha_0$, we get $h^-_{1-\alpha_0}(x_0, y_0) \geq t_0$.

Since , $h(x_0, y_0, .): (-\infty, 0) \to [0, 1]$ is nondecreasing,

we have $\alpha_0 =\ h(x_0, y_0, t_0) \leq\ h\left(x_0, y_0, h^-_{1-\alpha_0}(x_0, y_0)\right)$

$$\Rightarrow\ h\left(x_0, y_0, h^-_{1-\alpha_0}(x_0, y_0)\right) \geq \alpha_0.$$

If possible suppose that $h\left(x_0, y_0, h^-_{1-\alpha_0}(x_0, y_0)\right) > \alpha_0$.

Then by the lower semicontinuity of $h(x_0, y_0, .): (-\infty, 0) \to [0, 1]$,

$\exists t' <\ h^-_{1-\alpha_0}(x_0, y_0)$ such that $h(x_0, y_0, t') > \alpha_0$.

Then $h^-_{1-\alpha_0}(x_0, y_0) = \vee\{t < 0 : h(x_0, y_0, t) \leq \alpha_0\} \leq t' < h^-_{1-\alpha_0}(x_0, y_0)$ - a contradiction.

Therefore $h\left(x_0, y_0, h^-_{1-\alpha_0}(x_0, y_0)\right) = \alpha_0$.

(2) Since h satisfies (FS-4) we have $\forall\, x \in X, \forall\, y \in Y$

$$\lim_{t \to +\infty} h(x, y, t) = 1$$

$$\Rightarrow \lim_{t \to +\infty} h(-x, y, t) = 1$$

$$\Rightarrow \lim_{t \to -\infty} h(x, y, t) = 0$$

Since $h(x, y, .): (0, \infty) \to [0, 1]$ and $h(x, y, .): (-\infty, 0) \to [0, 1]$ are both upper and lower semicontinuous respectively, so for each $\alpha \in (0, 1), \exists\, t \in R - \{0\}$ such that $h(x_0, y_0, t) = \alpha$

Then by (1), the proof follows.

Now we prove the main Theorem 4.8

Proof. (i) and (ii) are followed from Theorem 4.1 and Theorem 4.3. For (iii) if $t_0 = 0$ then $h(x, y, t_0) = 0 = h''(x, y, t_0)$ (from definition). So consider the following cases for $t_0 \neq 0$.

(Case-I) Either $x_0 = 0$ or $y_0 = 0$.

Let $t_0 > 0$ then $h(x_0, y_0, t_0) = 1$ (since $h(0, y_0, t_0) = H(t)$).

Now $h_\alpha^+(x_0, y_0) = \wedge\{t > 0 : h(x_0, y_0, t) \geq \alpha\}$

$= \wedge\{t > 0 : t > 0\} = 0.$

Therefore $h''(x_0, y_0, t_0) = \vee\{\alpha \in (0, 1): h_\alpha^+(x_0, y_0) \leq t_0\}$

$= \vee\{\alpha \in (0, 1): \alpha \in (0, 1)\} = 1.$

Thus $h(x_0, y_0, t_0) = h''(x_0, y_0, t_0)$ if $t_0 > 0$.

If $t_0 < 0$ then $h(x_0, y_0, t_0) = 0$.

Now $h_{1-\alpha}^-(x_0, y_0) = \vee\{t < 0 : h(x_0, y_0, t) \leq \alpha\} = 0$ (since $h(x_0, y_0, t) = H(t)$ if either $x_0 = 0$ or $y_0 = 0$).

Therefore $h''(x_0, y_0, t_0) = \wedge\{\alpha \in (0, 1): h_{1-\alpha}^-(x_0, y_0) \geq t_0\}$

$= \wedge\{\alpha \in (0, 1): \alpha \in (0, 1)\} = 0.$

Thus $h(x_0, y_0, t_0) = h''(x_0, y_0, t_0)$ if $t_0 < 0$.

(Case-II) If $x_0 \neq 0$, $y_0 \neq 0$ and $h(x_0, y_0, t_0) = 0$.

Let $t_0 > 0$ then $h_\alpha^+(x_0, y_0) = \wedge\{t > 0 : h(x_0, y_0, t) \geq \alpha\}$.

Since $h(x_0, y_0, t_0) = 0 < \alpha$,

so $h_\alpha^+(x_0, y_0) > t_0$.

Therefore $h''(x_0, y_0, t_0) = \vee\{\alpha \in (0, 1): h_\alpha^+(x_0, y_0) \leq t_0\}$

$= \vee\{\alpha \in (0, 1): \emptyset\} = 0.$

Thus $h(x_0, y_0, t_0) = h''(x_0, y_0, t_0)$.

Let $t_0 < 0$ and $h(x_0, y_0, t_0) = 0 < \alpha \ \forall \alpha \in (0,1)$.

So $h^-_{1-\alpha}(x_0, y_0) \geq t_0 \ \forall \alpha \in (0,1)$.

Therefore $h''(x_0, y_0, t_0) = \wedge\{\alpha \in (0,1): h^-_{1-\alpha}(x_0, y_0) \geq t_0\}$

$$= \wedge\{\alpha \in (0,1): \alpha \in (0,1)\} = 0.$$

Thus $h(x_0, y_0, t_0) = h''(x_0, y_0, t_0)$ if $t_0 > 0$.

(Case-III) When $x_0 \neq 0, y_0 \neq 0$ and $0 < h(x_0, y_0, t_0) < 1$.

Let $h(x_0, y_0, t_0) = \alpha_0$. Then $0 < \alpha_0 < 1$.

If $t_0 > 0$ then $h^+_{\alpha_0}(x_0, y_0) = \wedge\{t > 0 : h(x_0, y_0, t) \geq \alpha_0\}$

$$\Rightarrow h^+_{\alpha_0}(x_0, y_0) \leq t_0.$$

Now $h''(x, y, t)$=$\vee\{\alpha \in (0,1): h^+_\alpha(x, y) \leq t\}$.

So $h''(x_0, y_0, t_0) \geq \alpha_0 = h(x_0, y_0, t_0)$

$$\Rightarrow h''(x_0, y_0, t_0) \geq h(x_0, y_0, t_0). \qquad (4.8.1)$$

Now from Lemma 4.9(1a), we have $h^+_{\alpha_0}(x_0, y_0) = t_0$.

For $1 > \alpha > \alpha_0$, let $h^+_\alpha(x_0, y_0) = t''$. Then $t'' \geq t_0$.

By Lemma 4.9(2), $h(x_0, y_0, t'') = \alpha$.

So $h(x_0, y_0, t'') = \alpha > \alpha_0 = h(x_0, y_0, t_0)$.

Since $h(x_0, y_0, .): (0, \infty) \to [0,1]$ is nondecreasing,

$h(x_0, y_0, t'') > \alpha_0 = h(x_0, y_0, t_0) \Rightarrow t'' > t_0$.

So for $1 > \alpha > \alpha_0$, $h^+_\alpha(x_0, y_0) = t'' > t_0$.

Hence $h''(x_0, y_0, t_0) \leq \alpha_0 = h(x_0, y_0, t_0)$. (4.8.2)

From (4.8.1) and (4.8.2) we have $h''(x_0, y_0, t_0) = h(x_0, y_0, t_0)$

If $t_0 < 0$ then $h^-_{1-\alpha_0}(x_0, y_0) = \vee\{t < 0 : h(x_0, y_0, t) \leq \alpha_0\}$

$$\Rightarrow h^-_{1-\alpha_0}(x_0, y_0) \leq t_0.$$

Now $h''(x, y, t) = \wedge\{\alpha \in (0,1): h^-_{1-\alpha}(x, y) \geq t\}$.

So $h''(x_0, y_0, t_0) \leq \alpha_0 = h(x_0, y_0, t_0)$

$$\Rightarrow h''(x_0, y_0, t_0) \leq h(x_0, y_0, t_0). \qquad (4.8.3)$$

Now from Lemma 4.9(1b), we have $h^-_{1-\alpha_0}(x_0, y_0) = t_0$.

For $\alpha_0 < \alpha < 0$, let $h^-_{1-\alpha}(x_0, y_0) = t''$. Then $t'' \geq t_0$.

By Lemma 4.9(2), $h(x_0, y_0, t'') = \alpha$.

So $h(x_0, y_0, t'') = \alpha > \alpha_0 = h(x_0, y_0, t_0)$.

Since $h(x_0, y_0, .): (-\infty, 0) \to [0, 1]$ is nondecreasing,

$h(x_0, y_0, t'') > \alpha_0 = h(x_0, y_0, t_0) \Rightarrow t'' > t_0$.

So for $\alpha_0 < \alpha < 0$, $h^-_{1-\alpha}(x_0, y_0) = t'' > t_0$.

Hence $h''(x_0, y_0, t_0) \leq \alpha_0 = h(x_0, y_0, t_0)$. (4.8.4)

By (4.8.3) and (4.8.4) we have $h''(x_0, y_0, t_0) = h(x_0, y_0, t_0)$.

(Case-IV) When $x_0 \neq 0$, $y_0 \neq 0$ and $h(x_0, y_0, t_0) = 1$.

If $t_0 > 0$ then $h^+_\alpha(x_0, y_0) = \wedge\{t > 0 : h(x_0, y_0, t) \geq \alpha\}$.

Since $h(x_0, y_0, t_0) = 1 > \alpha \ \forall \alpha \in (0, 1)$, so $h^+_\alpha(x_0, y_0) \leq t_0 \ \ \forall \alpha \in (0, 1)$

Therefore $h''(x_0, y_0, t_0) = \vee\{\alpha \in (0, 1): h^+_\alpha(x_0, y_0) \leq t_0\}$

$$= \vee\{\alpha \in (0, 1): \alpha \in (0, 1)\} = 1.$$

Thus $h(x_0, y_0, t_0) = h''(x_0, y_0, t_0)$.

Let $t_0 < 0$ and $h(x_0, y_0, t_0) = 1 > \alpha \ \forall \alpha \in (0, 1)$.

So $h^-_{1-\alpha}(x_0, y_0) < t_0 \ \forall \alpha \in (0, 1)$.

Therefore $h''(x_0, y_0, t_0) = \wedge\{\alpha \in (0, 1): h^-_{1-\alpha}(x_0, y_0) \geq t_0\}$

$$= \wedge\{\alpha \in (0, 1): \emptyset\} = 1.$$

Hence $h'' = h$.

5. CONCLUSION

We introduce a definition of fuzzy real sesquilinear form. We have established a decomposition theorem from a fuzzy real sesquilinear form into a family of sesquilinear forms. There is a wide scope to develop fuzzy operator theory and fuzzy spectral theory. With this decomposition we can study many results of operator theory and for that purpose this paper will be helpful.

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