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SOME RESULTS ON CERTAIN VOLTERRA INTEGRAL AND INTEGRO-DIFFERENTIAL FUNCTIONAL EQUATIONS WITH FINITE DELAY

Haribhau L. Tidke

Abstract: In this paper, we establish the existence, uniqueness and other properties of solutions of certain Volterra integral and integrodifferential functional equations with finite delay. The results are obtained by using the well-known Banach fixed point theorem coupled with Bielecki type norm and the integral inequalities with explicit estimates.

Keywords: Existence, Uniqueness, Volterra integral and integrodifferential functional equations, Finite delay, Fixed point theorem, Integral inequalities

Mathematics Subject Classification: 26D10, 34A12, 34G20, 37C25, 45D05, 97I70

1. INTRODUCTION

Consider the Volterra integral and integrodifferential functional equations with finite delay of the forms:

$$x(t) = g(t) + \int_a^t f\big(t, s, x(s), x(s-1)\big)ds, \tag{1}$$

and

$$x'(t) = F(t, x(t), \int_a^t k(t, s, x(s))ds, x(s-1)), \tag{2}$$

$$x(t-1) = \phi(t)(a \le t < 1), x(a) = x_0, \tag{3}$$

for $-\infty < a \le t < +\infty$, where x,g,f,k,F are real vectors with n components and ' denotes the derivative. Let $\mathbb{R}^n$ de notes the real n-dimensional Euclidean space with appropriate norm denoted by |˙| and $\mathbb{R}^n$the set of real numbers. Let $I = [a, +\infty), \mathbb{R}_+ = [0, +\infty)$ be the given subsets of R and assume that $k \in C(I^2 \times \mathbb{R}^n, \mathbb{R}^n)$ for $a \le s \le t < +\infty$, $f \in C(I^2 \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n)$, $F \in C(I \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n)$ and $g \in C(I, \mathbb{R}^n)$. The function ϕ is continuous for which $\lim_{t\to 1-0}\phi(t)$ exists.

The sufficient literature exists dealing with the special and even more general versions of equations (1) and (2)–(3) by using different techniques (see [3]-[5], [8]-[19]) and the reference given therein. Owing to the importance of equations of the these forms arising in many physical problems, the simple , unified and concise treatment for these equations are required.

The purpose of this paper is to study the existence, uniqueness and other properties of solutions of equations (1) and (2)–(3) under various assumptions on the functions f, F, k and g. The main tools employed in the analysis are based on the applications of the Banach fixed point theorem (see [3, 4]) coupled with Bielecki type norm (see [2, 4]) and the integral inequalities with explicit estimates given in [6] and [7].

2 Existence and Uniqueness

Our methods involve Banach's fixed–point theorem and we now introduce the appropriate metric space setting. Let $\beta > 0$ be a constant and consider the space of continuous functions $C(I, \mathbb{R}^n)$ such that $\sup_{t\in I} \frac{|x(t)|}{e^{\beta(t-a)}} < \infty$ and denote this special space by $C_\beta(I, \mathbb{R}^n)$. We couple the linear space $C_\beta(I, \mathbb{R}^n)$ with suitable metric, namely

$$d_\beta^\infty(x, y) = \sup_{t\in I} \frac{|x(t)-y(t)|}{e^{\beta(t-a)}},$$

with a norm defined by

$$|x|_\beta^\infty = \sup_{t\in I} \frac{|x(t)|}{e^{\beta(t-a)}}.$$

The above definitions of d_β^∞ and $|\cdot|_\beta^\infty$ are the variants of Bielecki's metric and norm [2].

The following Lemma proved in [4] deals with some important properties of d_β^∞ and $|\cdot|_\beta^\infty$.

Lemma 1 If $\beta > 0$ is a constant, then:

i. d_β^∞ is a metric,
ii. $|\cdot|_\beta^\infty$ is a norm,
iii. $(C_\beta(I, \mathbb{R}^n), |\cdot|_\beta^\infty)$ is a Banach space,
iv. $(C_\beta(I, \mathbb{R}^n), |\cdot|_\beta^\infty)$ is a complete metric space.

2.1 Solution to Integral Equation:

We are now ready to present the main result concerning the existence and uniqueness of solutions of equation (1).

Theorem 1 Let β>0, M≥0 be constants. Suppose that the functions f and g in equation (1) satisfy the conditions

$$|f(t, s, x, y) - f(t, s, \bar{x}, \bar{y})| \leq M[|x - \bar{x}| + |y - \bar{y}|], \tag{4}$$

and

$$d_1 = \sup_{t\in I} \frac{1}{e^{\beta(t-a)}} \{|\int_a^t f(t, s, 0,0)ds| + |g(t)|\} < \infty. \tag{5}$$

If $\frac{2M}{\beta} < 1$,, then the integral equation (1) has a unique solution $x \in C_\beta(I, \mathbb{R}^n)$.

Proof. One can easily write the following equivalent formulation of equation (1),

$$x(t) = g(t) + \int_a^t f(t,s,x(s),x(s-1))ds - \int_a^t f(t,s,0,0)ds + \int_a^t f(t,s,0,0)ds, for \quad t \in I. \tag{6}$$

Now we will prove that (6) has a unique solution and therefore, so the equation (1). Let $x \in C_\beta(I, \mathbb{R}^n)$ and define the operator T by

$$(Tx)(t) = g(t) + \int_a^t f(t,s,x(s),x(s-1))ds - \int_a^t f(t,s,0,0)ds + \int_a^t f(t,s,0,0)ds, for \quad t \in I. \tag{7}$$

Next we shall show that T maps $C_\beta(I, \mathbb{R}^n)$ into itself. From (7) and using the hypotheses we have

$$\begin{aligned} |Tx|_\beta^\infty &= \sup_{t\in I} \frac{|(Tx)(t)|}{e^{\beta(t-a)}} \\ &\le \sup_{t\in I} \frac{1}{e^{\beta(t-a)}} \{|g(t)| + \int_a^t |f(t,s,0,0)|ds\} \\ &\quad + \sup_{t\in I} \frac{1}{e^{\beta(t-a)}} \int_a^t |f(t,s,x(s),x(s-1)) - f(t,s,0,0)|ds \\ &\le d_1 + \sup_{t\in I} \frac{1}{e^{\beta(t-a)}} \int_a^t M[|x(s)| + |x(s-1)|]ds \\ &\le d_1 + \frac{M}{\beta}|x|_\beta^\infty + \sup_{t\in I} \frac{M}{e^{\beta(t-a)}} \int_a^t |x(s-1)|ds, for \quad t \in I. \end{aligned} \tag{8}$$

We consider the following two cases.

Case 1: $a \le t < 1$. From (8) and using hypotheses, we have

$$\begin{aligned} |Tx|_\beta^\infty &\le d_1 + \frac{M}{\beta}|x|_\beta^\infty + \sup_{t\in I} \frac{M}{e^{\beta(t-a)}} \int_a^t |\phi(s)|ds, \\ &\le d_1 + \frac{M}{\beta}|x|_\beta^\infty + \sup_{t\in[a,1)} \frac{M}{e^{\beta(t-a)}} \int_a^1 |\phi(s)|ds, \\ &= d_1 + M\int_a^1 |\phi(s)|ds + \frac{M}{\beta}|x|_\beta^\infty, \\ &= d_1 + d_2 + \frac{M}{\beta}|x|_\beta^\infty < \infty, \end{aligned} \tag{9}$$

where

$$d_2 = M\int_a^1 |\phi(s)|ds. \tag{10}$$

Case 2: $1 \le t < \infty$ From (8) and using hypotheses, we have

$$|Tx|_\beta^\infty \le d_1 + \frac{M}{\beta}|x|_\beta^\infty + \sup_{t\in I}\frac{M}{e^{\beta(t-a)}}\{\int_a^1 |\phi(s)|ds + \int_1^t |x(s-1)|ds\},$$
$$= d_1 + d_2 + \frac{M}{\beta}|x|_\beta^\infty + \sup_{t\in I}\frac{M}{e^{\beta(t-a)}}\int_1^t |x(s-1)|ds. \tag{11}$$

By making the change of variable (i. e. $s-1=\sigma$), we obtain

$$\int_1^t |x(s-1)|ds = \int_0^{t-1} |x(\sigma)|d\sigma \le \int_a^t |x(\sigma)|d\sigma. \tag{12}$$

Using (12) in (11), we get

$$|Tx|_\beta^\infty \le d_1 + d_2 + \frac{M}{\beta}|x|_\beta^\infty + \sup_{t\in I}\frac{M}{e^{\beta(t-a)}}\int_a^t |x(\sigma)|d\sigma,$$
$$\le d_1 + d_2 + \frac{M}{\beta}|x|_\beta^\infty + \frac{M}{\beta}|x|_\beta^\infty = d_1 + d_2 + 2\frac{M}{\beta}|x|_\beta^\infty < \infty. \tag{13}$$

This proves that the operator T maps $C(I,\mathbb{R}^n)$ into it self.

Now we verify that the operator T is a contraction map. Let $x, y \in C(I,\mathbb{R}^n)$. From (7) and using the hypotheses, we have

$$d_\beta^\infty(Tx,Ty) = \sup_{t\in I}\frac{|(Tx)(t)-(Ty)(t)|}{e^{\beta(t-a)}}$$
$$\le \sup_{t\in I}\frac{1}{e^{\beta(t-a)}}\int_a^t M[|x(s)-y(s)| + |x(s-1)-y(s-1)|]ds$$
$$\le \frac{M}{\beta}|x-y|_\beta^\infty + \sup_{t\in I}\frac{M}{e^{\beta(t-a)}}\int_a^t [|x(s-1)-y(s-1)|]ds$$
$$\le \frac{M}{\beta}d_\beta^\infty(x,y) + \sup_{t\in I}\frac{M}{e^{\beta(t-a)}}\int_a^t [|x(s-1)-y(s-1)|]ds. \tag{14}$$

We consider the following two cases.

$$d_\beta^\infty(Tx,Ty) \le \frac{M}{\beta}d_\beta^\infty(x,y) + \sup_{t\in I}\frac{M}{e^{\beta(t-a)}}\int_a^t [|\phi(s)-\phi(s)|]ds$$
$$\le \frac{M}{\beta}d_\beta^\infty(x,y). \tag{15}$$

Case 2: $\le t < \infty$. From (14) and using hypotheses with change of variable, we obtain

$$d_\beta^\infty(Tx,Ty) \le 2\frac{M}{\beta}d_\beta^\infty(x,y). \tag{16}$$

Since $2\frac{M}{\beta} < 1$, it follows from the Banach fixed point theorem that T has a unique fixed point in $C(I,\mathbb{R}^n)$, , which is the required solution of equation (1). The proof is complete.

2.2 Solution to Integro-differential Equation:

The result concerning the solutions of equation (2)–(3).

Theorem 2 Let $\beta > 0,\ M \geq 0,\ L > 0$ be constants. Suppose that the functions F and k in equation (2) satisfy the conditions

$$|F(t,x,y,z) - F(t,\bar{x},\bar{y},\bar{z})| \leq M[|x-\bar{x}| + |y-\bar{y}| + |z-\bar{z}|],$$
$$|k(t,s,x) - k(t,s,\bar{x})| \leq L|x-\bar{x}|, \tag{18}$$

and $$d_3 = |x_0| + \sup_{t\in I} \frac{1}{e^{\beta(t-a)}}\{|\int_a^t F(t,0,\int_a^s k(s,\sigma,0)d\sigma,0)ds|\} < \infty.$$

If $M\frac{(2+L)}{\beta} < 1,$ then the integral equation (2)–(3) has a unique solution $x \in C_\beta(I,\mathbb{R}^n)$.

Proof. Let $x \in C(I,\mathbb{R}^n)$, and define the operator T by

$$\begin{aligned}(Tx)(t) = x_0 + \int_a^t F(t,s,x(s),\int_a^s k(s,\sigma,x(\sigma))d\sigma, x(s-1))ds \\ - \int_a^t F(t,0,\int_a^s k(s,\sigma,0)d\sigma,0)ds \\ + \int_a^t F(t,0,\int_a^s k(s,\sigma,0)d\sigma,0)ds, for \quad t \in I.\end{aligned} \tag{19}$$

The T maps $C_\beta(I,\mathbb{R}^n)$ into itself and is a contraction map, one can be completed by closely looking at the proof of Theorem 1 given above with corresponding modifications. Here we omit the details.

3 Explicit estimates on the solutions via inequalities

In this section, we obtain estimates on the solutions and study the continuous dependence on initial data and on the functions involved therein to the equations (1) and (2)–(3) under some suitable assumptions.

We need the following versions of the inequalities given in ([6], p. 20), see also ([7], p. 11, Remark 1.2.1), and ([7], p. 29). We shall state them here for completeness.

Lemma 2 Let $u(t) \in C(I,\mathbb{R}_+),\ \ r(t,\sigma), \frac{\partial}{\partial t} r(t,\sigma) \in C(D,\mathbb{R}_+),$ where $D = \{(t,\sigma) \in I^2 : a \leq \sigma \leq t < \infty\}$ and $c \geq 0$ is a constant.

If $u(t) \leq c + \int_a^t r(t,\sigma)u(\sigma)d\sigma, for\ \ t \in I, then\ \ u(t)$
$\leq c\ exp\{\int_a^t A(s)ds\}, for\ \ t \in$
where A(t) $= r(t,t) + \int_a^t \frac{\partial}{\partial t} r(t,\tau)d\tau.$

Lemma 3 Let $u(t), p(t) \in C(I,\mathbb{R}_+),\ \ r(t,\sigma), \frac{\partial}{\partial t} r(t,\sigma) \in C(D,\mathbb{R}_+),$ where $D = \{(t,\sigma) \in I^2 : a \leq \sigma \leq t < \infty\}$ and $c \geq 0$ is a constant. If

$$u(t) \le c + \int_a^t p(s)[u(s) + \int_a^s r(s,\sigma)u(\sigma)d\sigma],$$

for $t \in I$,then

$$u(t) \le c[1 + \int_t^a p(s)\exp(\int_a^s [p(\sigma) + A(\sigma)]d\sigma)ds],$$

for $t \in I$, where A(t) is as in Lemma 2.

3.1 Uniqueness of solutions without the existence part:

The following theorem shows the uniqueness of solutions to (1) without the existence part.

Theorem 3 Suppose that the function f in equation (1) satisfies the condition

$$|f(t,s,x,y) - f(t,s,\bar{x},\bar{y})| \le p(t,s)[|x-\bar{x}| + |y-\bar{y}|], \quad (20)$$

where $p(t,\sigma), \frac{\partial}{\partial t}p(t,\sigma) \in C(D,\mathbb{R}_+)$, where D is as in Lemma 2. Then the problem (1) has at most one solution $x \in C_\beta(I,\mathbb{R}^n)$.

Proof. Let x and y be two solutions of (1) and $u(t) = |x(t) - y(t),\ t \in I$. Then we have

$$\begin{aligned} u(t) &\le \int_a^t |f(t,s,x(s),x(s-1)) - f(t,s,y(s),y(s-1))|ds \quad (21)\\ &\le \int_a^t p(t,s)[u(s) + |x(s-1) - y(s-1)|]ds. \end{aligned}$$

We consider the following two cases.

Case 1: $a \le t < 1$. From (21) and using hypotheses, we have

$$u(t) \le \int_a^t p(t,s)[u(s) + |\phi(s) - \phi(s)|]ds \le \int_a^t p(t,s)u(s)ds. \quad (22)$$

Now, a suitable application of Lemma 2 to (22) (with c=0 and $r(t,s) = p(t,s)$) yields

$$|x(t) - y(t)| \le 0. \quad (23)$$

Case 2: $1 \le t < \infty$. From (21) and using hypotheses with change of variable $(s - 1 = \sigma)$, we obtain

$$\begin{aligned} u(t) &\le \int_a^t p(t,s)u(s)ds + \int_a^1 p(t,s)|\phi(s) - \phi(s)|ds\\ &\quad + \int_1^t p(t,s)|x(s-1) - y(s-1)|ds \quad (24)\\ &\le \int_a^t [p(t,s) + p(t,s+1)]u(s)ds. \end{aligned}$$

Now, a suitable application of Lemma 2 to (24) (with c=0 and $r(t,s) = p(t,s) + p(t,s+1)$) yields

$$|x(t) - y(t)| \leq 0. \tag{25}$$

From (23) and (25), we have $x(t) = y(t)$ for $t \in I$.. Thus there is at most one solution to (1). This completes the proof.

The following theorem shows the uniqueness of solutions to (2)–(3) without the existence part.

Theorem 4 Suppose that the function F,k in equation (2)–(3) satisfy the conditions

$$|F(t,x,y,z) - F(t,\bar{x},\bar{y},\bar{z})| \leq q(t)[|x-\bar{x}| + |y-\bar{y}| + |z-\bar{z}|], \tag{26}$$

$$|k(t,s,x) - k(t,s,\bar{x})| \leq r(t,s)|x-\bar{x}|, \tag{27}$$

where $q \in C(I,\mathbb{R}_+), r(t,\sigma), \frac{\partial}{\partial t} r(t,\sigma) \in C(D,\mathbb{R}_+)$, where D is as in Lemma 2. Then the problem (2)–(3) has at most one solution $x \in C_\beta(I,\mathbb{R}^n)$.

Proof. Let x and y be two solutions of (2)–(3) and $u(t) = |x(t) - y(t), \ t \in I$. The proof can be completed by closely looking at the proof of Theorem 3 given above with suitable modifications and application of Lemma 3. Therefore, here we omit the details.

3.2 Estimate on solutions:

The following theorem concerning the estimate on solution of (1).

Theorem 5 Suppose that the function f in equation (1) satisfies the condition (20). *If* $x(t), \ t \in I,$ is any solution of equation (1), then

$$|x(t)| \leq [d_3 + \int_a^1 p(1,s)|\phi(s)|ds]\exp(\int_a^t B(s)ds), \tag{28}$$

for $a \leq t < 1$ and

$$|x(t)| \leq [d_3 + \int_a^1 p(1,s)|\phi(s)|ds]\exp(\int_a^t C(s)ds), \tag{29}$$

for $1 \leq t < \infty$, where B(t) and C(t) follow the definition of A(t) as in Lemma 2 and

$$d_3 = \sup_{t\in I}| \int_a^t f(t,s,0,0)ds + g(t)| < \infty.$$

Proof. By using the fact that the solution x(t) of equation (1) satisfies the equivalent equation (6) and the hypotheses, we have

$$\begin{aligned} |x(t)| &\leq |g(t) + \int_a^t f(t,s,0,0)ds| \\ &\quad + \int_a^t |f(t,s,x(s),x(s-1)) - f(t,s,0,0)|ds \\ &\leq d_3 + \int_a^t p(t,s)[|x(s)| + |x(s-1)|]ds. \end{aligned} \tag{30}$$

We consider the following two cases.

Case 1: $a \le t < 1$.. From (30) and using hypotheses, we have

$$|x(t)| \le d_3 + \int_a^1 p(1,s)|\phi(s)|ds + \int_a^t p(t,s)|x(s)|ds. \tag{31}$$

Now, a suitable application of Lemma 2 to (31) yields

$$|x(t)| \le [d_3 + \int_a^1 p(1,s)|\phi(s)|ds]\exp(\int_a^t B(s)ds). \tag{32}$$

Case 2: $1 \le t < \infty$.. From (30) and using hypotheses with change of variable (s-1=σ), we obtain

$$\begin{aligned} |x(t)| &\le d_3 + \int_a^1 p(1,s)|\phi(s)|ds + \int_a^t p(t,s)|x(s)|ds \\ &\quad + \int_1^t p(t,s)|x(s-1)|ds \\ &\le d_3 + \int_a^1 p(1,s)|\phi(s)|ds \\ &\quad + \int_a^t [p(t,s) + p(t,s+1)]|x(s)|ds. \end{aligned} \tag{33}$$

Now, a suitable application of Lemma 2 to (33) yields

$$|x(t)| \le [d_3 + \int_a^1 p(1,s)|\phi(s)|ds]\exp(\int_a^t C(s)ds). \tag{34}$$

This completes the proof.

Next, we obtain the estimate on solution of the problem (2)–(3).

Theorem 6 Suppose that the function F,k in equation (2)–(3) satisfy the conditions (26) and (27) respectively. If $x(t),\ t \in I,$ is any solution of the problem (2)–(3), then

$$|x(t)| \le [d_4 + \int_a^1 q(s)|\phi(s)|ds][1 + \int_a^t q(s)\exp(\int_a^s [q(\sigma) + B(\sigma)]d\sigma)ds], \tag{35}$$

for $a \le t < 1$ and

$$|x(t)| \le [d_4 + \int_a^1 q(s)|\phi(s)|ds][1 + \int_a^t (q(s) + q(s+1))\exp(\int_a^s [q(\sigma) + q(\sigma+1) + C(\sigma)]d\sigma)ds], \tag{36}$$

for $1 \le t < \infty$, where B(t) and C(t) follow the definition of A(t) as in Lemma 2 and

$$d_4 = \sup_{t \in I} | \int_a^t F(s, \int_a^s k(s,\sigma,0)d\sigma, 0)ds + x_0| < \infty.$$

Proof. Let x be any solution of (2)–(3). One can easily prove this result looking at the proof of Theorem 5 with corresponding modifications and application of Lemma 3. Thus, we omit the details.

4 Continuous Dependence

In this section we shall deal with the continuous dependence of solutions of equations (1) and (2)–(3) on the initial data, functions involved therein and also on parameters.

4.1 Dependence on initial data

We shall deal with the continuous dependence of solutions of equations (2)–(3) on initial data.

Theorem 7 Suppose the hypotheses of Theorem 4 are hold. Let x and y be the solutions of the problem (2)–(3) with the initial conditions

$$x(t-1)=\phi(t)(a\le t<1), x(a)=x_0, \tag{37}$$

$$y(t-1)=\psi(t)(a\le t<1), y(a)=y_0, \tag{38}$$

respectively. Then

$$|x(t)-y(t)|\le[|x_0-y_0|+\int_a^1 q(s)|\phi(s)-\psi(s)|ds][1+\int_a^t q(s)\exp(\int_a^s[q(\sigma)+B(\sigma)]d\sigma)ds], \tag{39}$$

for $a\le t<1$ and

$$|x(t)-y(t)|\le[|x_0-y_0|+\int_a^1 q(s)|\phi(s)-\psi(s)|ds]$$
$$\times[1+\int_a^t (q(s)+q(s+1))\exp(\int_a^s[q(\sigma)+q(\sigma+1)+C(\sigma)]d\sigma)ds], \tag{40}$$

for $1\le t<\infty$, where B(t) and C(t) follow the definition of A(t) as in Lemma 2.

Proof. By using the fact that x(t) and y(t) are solutions of equations (2)–(3) and $u(t)=|x(t)-y(t)|$ for $t\in I$. Then by the hypotheses, we have

$$|x(t)-y(t)|\le|x_0-y_0|+\int_a^t|F(s,x(s),\int_a^s k(s,\sigma,x(\sigma))d\sigma,x(s-1))$$

$$-F(s,y(s),\int_a^s k(s,\sigma,y(\sigma))d\sigma,y(s-1))|ds$$
$$\le|x_0-y_0|+\int_a^t q(s)[u(s)+\int_a^s r(s,\sigma)u(\sigma)d\sigma]ds \tag{41}$$
$$+\int_a^t q(s)|x(s-1)-y(s-1)|ds.$$

We consider the following two cases.

Case 1: $a\le t<1$. From (41) and using hypotheses, we have

$$|x(t)-y(t)|\le|x_0-y_0|+\int_a^1 q(s)|\phi(s)-\psi(s)|ds$$
$$+\int_a^t q(s)[u(s)+\int_a^s r(s,\sigma)u(\sigma)d\sigma]ds. \tag{42}$$

Now, a suitable application of Lemma 3 to (42) yields

$$|x(t)-y(t)| \le [|x_0-y_0| + \int_a^1 q(s)|\phi(s)-\psi(s)|ds] \times [1+\int_a^t q(s)\exp(\int_a^s [q(\sigma)+B(\sigma)]d\sigma)ds]. \tag{43}$$

Case 2: $1 \le t < \infty$. From (41) and using hypotheses with change of variable (s-1=σ), we obtain

$$|x(t)-y(t)| \le |x_0-y_0| + \int_a^1 q(s)|\phi(s)-\psi(s)|ds + \int_a^t [q(s)+q(s+1)][u(s)+\int_a^s r(s,\sigma)u(\sigma)d\sigma]ds. \tag{44}$$

Now, a suitable application of Lemma 3 to (44) yields

$$|x(t)-y(t)| \le [|x_0-y_0| + \int_a^1 q(s)|\phi(s)-\psi(s)|ds] \times [1+\int_a^t [q(s)+q(s+1)]\exp(\int_a^s [q(\sigma)+q(\sigma+1)+C(\sigma)]d\sigma)ds]. \tag{45}$$

This completes the proof.

4.2 Dependence on the functions involved therein:

Consider the equations (1) and (2)–(3) and the corresponding equations

$$y(t) = \bar{g}(t) + \int_a^t \bar{f}(t,s,y(s),y(s-1))ds, \tag{46}$$

and

$$y'(t) = \bar{F}(t,y(t),\int_a^t k(t,s,y(s))ds, y(s-1)), \tag{47}$$

$$y(t-1) = \psi(t)(a \le t < 1), y(a) = y_0, \tag{48}$$

for $-\infty < a \le t < +\infty$, where $y, \bar{g}, \bar{f}, \bar{k}$, $\bar{F}$ and ψ are defined as in (1) and (2)–(3).

The following theorem deal with the continuous dependence of solutions of equation (1) on the functions involved from right side.

Theorem 8 Suppose that the function f in equation (1) satisfies the condition (20). Furthermore suppose that

$$|g(t)-\bar{g}(t)| + \int_a^1 p(1,s)|\phi(s)-\psi(s)|ds + \int_a^1 |f(t,s,y(s),\psi(s)) - \bar{f}(t,s,y(s),\psi(s))|ds \le \varepsilon_1, \tag{49}$$

$$\int_1^t |f(t,s,y(s),y(s-1)) - \bar{f}(t,s,y(s),y(s-1))|ds \le \varepsilon_2, \tag{50}$$

where f and $\bar{f}$ are the functions involved in equations (1) and (46), $\varepsilon_1, \varepsilon_2 > 0$ are arbitrary small constant and y(t) is a solution of equation (46). Then the solution x(t), $t \in I$. of equation (1) depends continuously on the functions involved on the right hand side of equation (1).

Proof. Let $u(t) = |x(t)-y(t)|$, $t \in I$. Using the facts that x(t) and y(t) are the solutions of equations (1) and (46) and the hypotheses we have

We consider the following two cases.

Case 1: $a \leq t < 1$. With application of Lemma 2 and using hypotheses, we have

$$u(t) \leq \varepsilon_1 \exp(\int_a^t B(s)ds). \tag{51}$$

Case 2: $1 \leq t < \infty$. With application of Lemma 2 and using hypotheses with change of variable (s-1=σ), we obtain

$$u(t) \leq (\varepsilon_1 + \varepsilon_2)\exp(\int_a^t C(s)ds). \tag{52}$$

This completes the proof.

Next theorem deal with the continuous dependence of solutions of equations (2)–(3) on the functions involved from right side.

Theorem 9 Suppose that the functions F,k in equation (2) satisfies the condition (26) and (27). Furthermore suppose that

$$\begin{aligned} &|x_0 - y_0| + \int_a^1 q(s)|\phi(s) - \psi(s)|ds \\ &\quad + \int_a^1 |F(s, y(s), \int_a^s k(s,\sigma,y(\sigma)d\sigma), \psi(s)) - \bar{F}(s, y(s), \int_a^s k(s,\sigma,y(\sigma)d\sigma), \psi(s))|ds \leq \varepsilon_1, \end{aligned} \tag{53}$$

$$\int_1^t |F(s, y(s), \int_a^s k(s,\sigma,y(\sigma)d\sigma), y(s-1)) - \bar{F}(s, y(s), \int_a^s k(s,\sigma,y(\sigma)d\sigma), y(s-1))|ds \leq \varepsilon_2, \tag{54}$$

where f and $\bar{\text{f}}$ are the functions involved in equations (2)–(3) and (47), $\varepsilon_1, \varepsilon_2 > 0$ are arbitrary small constant and y(t) is a solution of equation (47). Then the solution x(t), $t \in I$, of equation (2)–(3) depends continuously on the functions involved on the right hand side of equation (2)–(3).

Proof. Let $u(t) = |x(t) - y(t)|,\ t \in I.$ Using the facts that x(t) and y(t) are the solutions of equations (2)–(3) and (47) and the hypotheses we have

We consider the following two cases.

Case 1: $a \leq t < 1$.With application of Lemma 3 and using hypotheses, we have

$$u(t) \leq \varepsilon_1[1 + \int_a^t q(s)\exp(\int_a^s [q(\sigma) + B(\sigma)]d\sigma)ds]. \tag{55}$$

Case 2: $1 \leq t < \infty$. With application of Lemma 3 and using hypotheses with change of variable (s-1=σ), we obtain

$$u(t) \leq (\varepsilon_1 + \varepsilon_2)[1 + \int_a^t (q(s) + q(s+1))\exp(\int_a^s [q(\sigma) + q(\sigma+1) + C(\sigma)]d\sigma)ds]. \tag{56}$$

This completes the proof.

4.3 Dependence on the parameters

We next consider the following systems of Volterra integral equations and integrodifferential equations

$$x(t) = g(t) + \int_a^t f(t, s, x(s), (s-1), \mu_1)ds, \tag{57}$$

$$x(t) = g(t) + \int_a^t f(t, s, x(s), x(s-1), \mu_2)ds, \tag{58}$$

and

$$x'(t) = F(t, x(t), \int_a^t k(t, s, x(s))ds, x(s-1), \mu_1), \tag{59}$$

$$x(t-1) = \phi(t)(a \leq t < 1), x(a) = x_0, \tag{60}$$

$$x'(t) = F(t, x(t), \int_a^t k(t, s, x(s))ds, x(s-1), \mu_2), \tag{61}$$

$$x(t-1) = \psi(t)(a \leq t < 1), x(a) = y_0, \tag{62}$$

for $-\infty < a \leq t < +\infty,$ where μ_1,μ_2 are parameters and $x, \bar{g}, f, k,\ F$ and ψ are defined as in (1) and (2)–(3).

We set forth some hypotheses that will be used in our subsequent discussion

(H1) The function f in equations (57) and (58) satisfies the condition

$$|f(t, s, x, y, \mu_1) - f(t, s, \bar{x}, \bar{y}, \mu_1)| \leq p(t, s)[|x - \bar{x}| + |y - \bar{y}|], \tag{63}$$

$$|f(t, s, x, y, \mu_1) - f(t, s, x, y, \mu_2)| \leq \bar{p}(t, s)|\mu_1 - \mu_2|, \tag{64}$$

where $\bar{p}(t, s), p(t, s), \frac{\partial}{\partial t} p(t, s), \frac{\partial}{\partial t} \bar{p}(t, s) \in C(D, \mathbb{R}_+),$ where D is as in Lemma 2

(H_2) The function F in equation and (59) and (61) satisfy the conditions

$$|F(t, x, y, z, \mu_1) - F(t, \bar{x}, \bar{y}, \bar{z}, \mu_1)| \leq q(t)[|x - \bar{x}| + |y - \bar{y}| + |z - \bar{z}|], \tag{65}$$

$$|F(t, x, y, z, \mu_1) - F(t, x, y, z, \mu_2)| \leq \bar{q}(t)|\mu_1 - \mu_2|, \tag{66}$$

where $q, \bar{q} \in C(I, \mathbb{R}_+).$

(H_3) The function k in equation (59)and (61) satisfy the condition

$$|k(t, s, x) - k(t, s, \bar{x})| \leq r(t, s)|x - \bar{x}|, \tag{67}$$

where $r(t, \sigma), \frac{\partial}{\partial t} r(t, \sigma) \in C(D, \mathbb{R}_+);\ D$ is as in Lemma 2.

The following theorem shows the dependency of solutions of equations (57), (58) on parameters.

Theorem 10 Assume that hypothesis (H_1) holds. Let x and y be the solutions of (57) and (58) respectively. Then

$$|x(t)-y(t)| \le [\int_a^1 \bar{p}(1,s)|\mu_1-\mu_2|ds + \int_a^1 p(1,s)|\phi(s)-\psi(s)|ds]\exp(\int_a^t B(s)ds), \tag{68}$$

for $a \le t < 1$, where

$$B(t) = p(t,t) + \int_a^t \frac{\partial}{\partial t} p(t,\tau)d\tau$$

and

$$|x(t)-y(t)| \le [\int_a^1 \bar{p}(1,s)|\mu_1-\mu_2|ds + \int_a^1 p(1,s)|\phi(s)-\psi(s)|ds]\exp(\int_a^t C(s)ds), \tag{69}$$

for $1 \le t < \infty$, where

$$C(t) = p(t,t) + p(t,t+1) + \int_a^t \frac{\partial}{\partial t}[p(t,\tau)+p(t,\tau+1)]d\tau.$$

Proof. Let x(t) and y(t) be solutions of equations (57) and (58) respectively and let $u(t) = |x(t)\ y(t)|,\ t \in I.$ Then by the hypotheses, we have

$$\begin{aligned} |x(t)-y(t)| &\le \int_a^t |f(t,s,x(s),x(s-1),\mu_1) - f(t,s,y(s),y(s-1),\mu_1)|ds \\ &\quad + \int_a^t |f(t,s,y(s),y(s-1),\mu_1) - f(t,s,y(s),y(s-1),\mu_2)|ds \\ &\le \int_a^t p(t,s)[|x(s)-y(s)| + |x(s-1)-y(s-1)|]ds + \int_a^t \bar{p}(t,s)|\mu_1-\mu_2|ds \\ &\le \int_a^t \bar{p}(t,s)|\mu_1-\mu_2|ds + \int_a^t p(t,s)|x(s-1)-y(s-1)|ds + \int_a^t p(t,s)u(s)ds. \end{aligned} \tag{70}$$

We consider the following two cases.

Case 1: $a \le t < 1.$ From (70) and using hypotheses, we have

$$u(t) \le \int_a^1 \bar{p}(1,s)|\mu_1-\mu_2|ds + \int_a^1 p(1,s)|\phi(s)-\psi(s)|ds + \int_a^t p(t,s)u(s)ds. \tag{71}$$

Now, a suitable application of Lemma 3 to (71) yields

$$u(t) \le [\int_a^1 \bar{p}(1,s)|\mu_1-\mu_2|ds + \int_a^1 p(1,s)|\phi(s)-\psi(s)|ds]\exp(\int_a^t B(s)ds). \tag{72}$$

Case 2: $1 \le t < \infty.$ From (70) and using hypotheses with change of variable (s-1=σ), we obta

$$\begin{aligned} u(t) &\le \int_a^1 \bar{p}(1,s)|\mu_1-\mu_2|ds + \int_a^1 p(1,s)|\phi(s)-\psi(s)|ds + \int_a^t p(t,s)u(s)ds \\ &\quad + \int_1^t p(t,s)|x(s-1)-y(s-1)|ds \\ &\le \int_a^1 \bar{p}(1,s)|\mu_1-\mu_2|ds + \int_a^1 p(1,s)|\phi(s)-\psi(s)|ds + \int_a^t p(t,s)u(s)ds \\ &\quad + \int_1^t p(t,s)u(s-1)ds \\ &\le \int_a^1 \bar{p}(1,s)|\mu_1-\mu_2|ds + \int_a^1 p(1,s)|\phi(s)-\psi(s)|ds + \int_a^t [p(t,s)+p(t,s+1)]u(s)ds. \end{aligned} \tag{73}$$

Now, a suitable application of Lemma 3 to (73) yields

$$u(t) \leq [\int_a^1 \bar{p}(1,s)|\mu_1 - \mu_2|ds + \int_a^1 p(1,s)|\phi(s) - \psi(s)|ds]\exp(\int_a^t C(s)ds). \quad (74)$$

This completes the proof.

The following theorem shows the dependency of solutions of equations (59) and (61)on parameters.

Theorem 11 Assume that hypotheses (H_2)-(H_3) hold. Let x and y be the solutions of (59)–(60) and (61)–(62) respectively. Then

$$|x(t) - y(t)| \leq [|x_0 - y_0| + \int_a^1 \bar{q}(s)|\mu_1 - \mu_2|ds + \int_a^1 q(s)|\phi(s) - \psi(s)|ds] \times [1 + \int_a^t q(s)\exp(\int_a^s [q(\sigma) + A(\sigma)]d\sigma)ds], \quad (75)$$

for $a \leq t < 1$ and

$$|x(t) - y(t)| \leq [|x_0 - y_0| + \int_a^1 \bar{q}(s)|\mu_1 - \mu_2|ds + \int_a^1 q(s)|\phi(s) - \psi(s)|ds] \times [1 + \int_a^t [q(s) + q(s+1)]\exp(\int_a^s [q(\sigma) + q(\sigma+1) + A(\sigma)]d\sigma)ds], \quad (76)$$

for $1 \leq t < \infty$, where $A(t) = r(t,t) + \int_a^t \frac{\partial}{\partial t} r(t,\tau)d\tau$.

Proof. The proof of this theorem can be completed by following the proof of Theorem 10 with suitable modifications. We omit the details here.

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Dr. Haribhau L. Tidke
Department of Mathematics,
School of Mathematical Sciences,
Kavayitri Bahinabai Chaudhari
North Maharashtra University, Jalgaon-425 001, India
E-mail : Wtharibhau@gmail.com