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A SINGLE SERVER QUEUE WITH FUZZY SERVICE TIME DISTRIBUTION FUNCTION

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ABSRACT: A single server fuzzy queue is analyzed using a general approach based on Zadeh's extension principle. Probability generating function for the number of customers is derived in Fuzzy environment for this queuing model, whose arrival process is Poisson and service time distribution is a fuzzy function. Particular cases are deduced and a numerical study is also carried out.

1. Introduction

Queueing models have wide application in manufacturing system, telecommunication, and service organizations, where in different types of customers are serviced by different type of severs. In the traditional queueing theory, the inter-arrival time and service time are required to follow certain pre-assigned probability distributions. However in many practical situations the arrival patterns and service times can be more realistically described by linguistic expressions like fast, moderate or slow rather than by a probability distribution. This gives the scope of studying queues in the context of fuzzy set theory. Fuzzy queues can be effectively applied in fields like manufacturing system, telecommunication, and data processing. Li and Lee (1989) have derived analytical results for two fuzzy queueing systems based on Zadeh's (1978) extension principle using the possibility concept. Chanas *et al*., (1988) and Lee and Nagi (1992) proposed the α-cut and two variable simulation approaches for analyzing Fuzzy queues. Buckley *et al.* (2006) have analyzed the fuzzy markov chains.

In this paper we present a new methodology for analyzing fuzzy queues using cumulative possibility distributions.

2. The Model

Consider a single server queuing system in which the interarrival time is negative exponential with mean arrival rate λ and the service time is a fuzzy random variable taking values over a set A with finite expected value γ. Thus the service times are governed by a general fuzzy possibility distribution function $F(.)$ specified by $F(\cdot) = \int_F \alpha(\theta)/F_\theta(\cdot);\ \theta \in A$ where $\{F_\theta(x) : \theta \in A\}$ denotes a family of fuzzy cumulative distribution functions indexed by the parameter θ and $\alpha : A \to [0.1]$ be the corresponding membership function. We further assume that the service discipline is FCFS. We denote such a queueing system by the notation $M|FG|1$ queue.

Let $\tilde{p}_i$ denote the probability that there are i arrivals during the service time $\tilde{S}$. $\tilde{P}_i$ is a fuzzy function induced by $F(\cdot)$. Since the arrival process is Poisson with parameter λ, the fuzzy probability function can be defined by

$$\mu_{\tilde{p}_i}(x) = \sup_{t \in R^+} \left\{ G(t) \middle| x = \frac{e^{-\lambda t}(\lambda t)^i}{i!} \right\}$$

This fuzzy queueing system can be considered as a member in a set Q of $M|G|1$ queueing systems. In other words $M|FG|1$ queue is a fuzzy queue whose original is an $M|G|1$ queue.

3. THE ANALYSIS

Let $X(\tilde{S})$ denote the imbedded fuzzy stochastic process representing the number of customers in the system. We consider the basic stochastic process $\{X_n : n \geq 0\}$ which is a Markov chain induced by the Markov chain related to the $M|G|1$ queue. The transition probability matrix of the Markov chain is

$$P = (p_{ij}) \quad \text{where} \quad p_{ij} = \Pr\{x_{n+1} = j | x_n = i\}$$

and

$$p_{ij} = \begin{cases} 0 & i \geq 1 \quad j < i-1 \\ p_{j-i+1} & i \geq 1 \quad j \geq i-1 \\ p_j & i = 0 \quad j \geq 0 \end{cases}$$

The membership function of p_{ij} is given by

$$\mu_{p_{ij}}(x) = \sup_{t \in R^+} \left\{ F(t) \middle| x = \int_0^\infty \frac{e^{-\lambda t}(\lambda t)^j}{j!} \, dF(t) \right\}$$

Let $\{\pi_i\}$ be the steady state probability for this model, with membership function $\mu_{\pi_i}(x) = \sup_{t \in R^+} \{F(t) | x = \pi_i\}$.

Define the generating functions $\pi(s) = \sum_{i=0}^{\infty} \pi_i s^i$ and $P(s) = \sum_{i=0}^{\infty} p_i s^i$

By applying Zadeh's extension principle the probability generating function of $\{\pi_i\}$ can be obtained interms of membership function as

$$\mu_{\pi(s)}(K) = \sup_{\substack{x, y \in R^+ \\ \frac{x}{y} < 1 \\ |z| \le 1}} \left\{ \min \left\{ \mu_\lambda(x), \mu_\gamma(y), \mu_s(z) \,\middle|\, \frac{K = \left(1 - \frac{x}{y}\right)(1-\gamma) \int_0^\infty e^{-\lambda(1-z)t}\, dF(t)}{\int_0^\infty e^{-\lambda(1-z)t}\, dF(t) - z} \right\} \right\}$$

The mean queue length (L) and mean waiting time (W) for this system interms of their membership functions are

$$\mu_L(K) = \sup_{\substack{x, y \in R^+ \\ \frac{x}{y} < 1}} \left\{ \min \left\{ \mu_\lambda(x), \mu_\gamma(y), \mu_\sigma(z) \,\middle|\, K = \frac{x}{y} + \frac{\left(\frac{x}{y}\right)^2 + x^2 z^2}{2\left(1 - \frac{x}{y}\right)} \right\} \right\}$$

and

$$\mu_W(K) = \sup_{\substack{x, y \in R^+ \\ \frac{x}{y} < 1}} \left\{ \min \left\{ \mu_\lambda(x), \mu_\gamma(y), \mu_\sigma(z) \,\middle|\, K = \frac{1}{y} + \frac{\left(\frac{x}{y}\right)^2 + x^2 z^2}{2\left(1 - \frac{x}{y}\right)} \right\} \right\}$$

4. PARTICULAR CASES

(i) If $F(t) = d$ for all t, in the crisp case, the above model concides with $M|D|1$ queue. The service time S is approximately known and is represented by the possibility distribution $\pi(t) = \pi_s(t)$.

The membership function corresponding to p_i, $\pi(s)$, π_n, L and W for this case are given by

$$\mu_{\tilde{P}_i}(x) = \sup_{t \in R^+} \left\{ \mu_s(t) \,\middle|\, x = \frac{e^{-\lambda t}(\lambda t)^i}{i!} \right\}$$

$$\mu_{\pi(s)}(K) = \sup_{\substack{x, y \in R^+ \\ \frac{x}{y} < 1}} \left\{ \min \left\{ \mu_\lambda(x), \mu_\gamma(y), \mu_s(z) \,\middle|\, K = \frac{(1-\lambda t)(1-z)}{(1 - z e^{-xt(1-z)})} \right\} \right\}$$

$$\mu_{\pi_0}(x) = \sup_{\substack{t \in R^+ \\ t < \frac{1}{\lambda}}} \{ \mu_s(t) \mid x = 1 - \lambda t \} \qquad \text{for all} \qquad \lambda \in R^+$$

$$\mu_{\pi_1}(x) = \sup_{\substack{t \in R^+ \\ t < \frac{1}{\lambda}}} \{\mu_s(t) \mid x = (1-\lambda t)(e^{\lambda t} - 1)\} \quad \text{for all} \quad \lambda \in R^+$$

$$\mu_{\pi_n}(x) = \sup_{\substack{t \in R^+ \\ t < \frac{1}{\lambda}}} \left\{ \mu_s(t) \mid x = (1-\lambda t) \sum_{k=1}^{n} (-1)^{n-k} \left[\frac{(k\lambda t)^{n-k}}{(n-k)!} + \frac{(k\lambda t)^{n-k-1}}{(n-k-1)!} \right] e^{k\lambda t} \right\} \quad \text{for all} \quad n \geq 2$$

$$\mu_L(x) = \sup_{\substack{t \in R^+ \\ t < \frac{1}{\lambda}}} \left\{ \mu_s(t) \mid x = \frac{\lambda t(2-\lambda t)}{2(1-\lambda t)} \right\}$$

$$\mu_W(x) = \sup_{\substack{t \in R^+ \\ t < \frac{1}{\lambda}}} \left\{ \mu_s(t) \mid x = \frac{t(2-\lambda t)}{2(1-\lambda t)} \right\}$$

(ii) If we assume that the service time $\tilde{S}$ is approximately known and is represented by a possibility distribution $\pi(t) = \mu_{\tilde{S}}(t)$ which restricts its possible values then the results of our model coincide with that obtained by Li and Lee (1989) for their $M|F|1$ fuzzy queue.

5. NUMERICAL STUDY

A special tune-up station has been established at the end of an automotive assembly line to make adjustment on those vehicles which cannot meet federal exhaust gas emission standards. Failures are completely random and hence justify the Poisson arrival assumption with $\lambda = 0.1$ vehicle per minute. Each arrival is serviced by an adjustment requiring approximately 5 minutes which can be expressed by a trapezoidal furzy number with membership function $t \in R^+$.

$$\mu_s(t) = \begin{cases} 0 & \text{if} \quad t \leq 0 \\ \frac{t}{2} & \text{if} \quad 0 \leq t \leq 2 \\ 1 & \text{if} \quad 2 \leq t \leq 4 \\ (6-t)/2 & \text{if} \quad 4 \leq t \leq 6 \\ 0 & \text{if} \quad t \leq 6 \end{cases}$$

In attempting to evaluate storage space requirements, management needs to know

(a) the mean number of vehicles in the station

(b) the expected sojourn time per vehicle

We have

$$\mu_{\tilde{\pi}_0}(x) = \sup_{\substack{t \in R^+ \\ t < \frac{1}{\lambda}}} \{\mu_{\tilde{S}}(t) \mid x = 1 - \lambda t\}$$

Table 1

x	$\mu_{\tilde{\pi}_0}(x)$
0.4	0
0.5	0.5
0.6	1
0.7	1
0.8	1
0.9	0.5
1.0	0

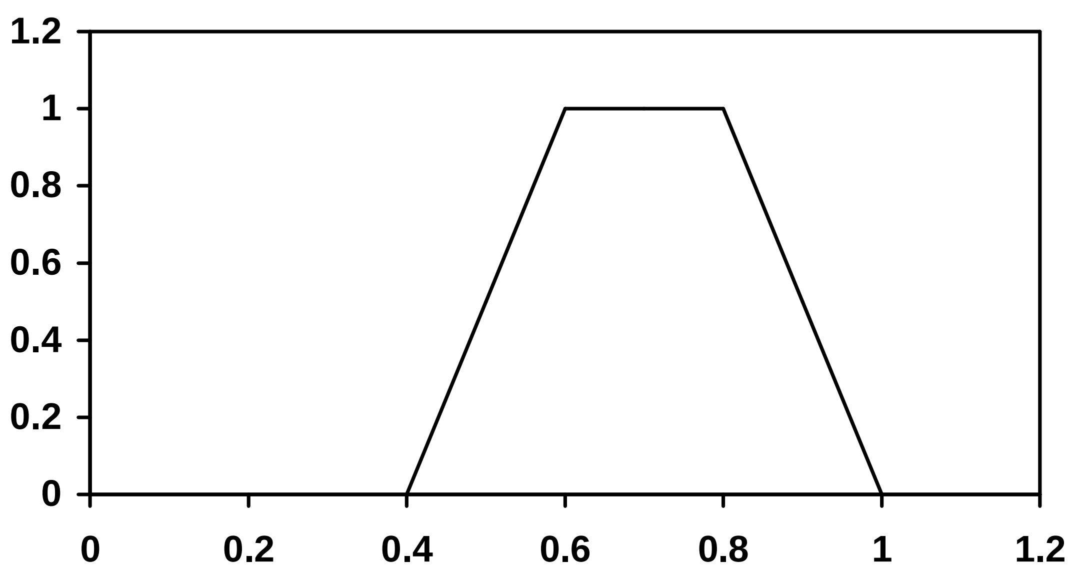

Figure 1

$$\mu_{\tilde{\pi}_1}(x) = \sup_{\substack{t \in R^+ \\ t < \frac{1}{\lambda}}} \{\mu_{\tilde{S}}(t) \,|\, x = (1 - \lambda t)\,(e^{\lambda t} - 1)\}$$

Table 2

x	$\mu_{\tilde{\pi}_1}(x)$
0	0
0.1	0.5
0.2	1
0.3	1
0.4	1
0.5	0.5
0.6	0

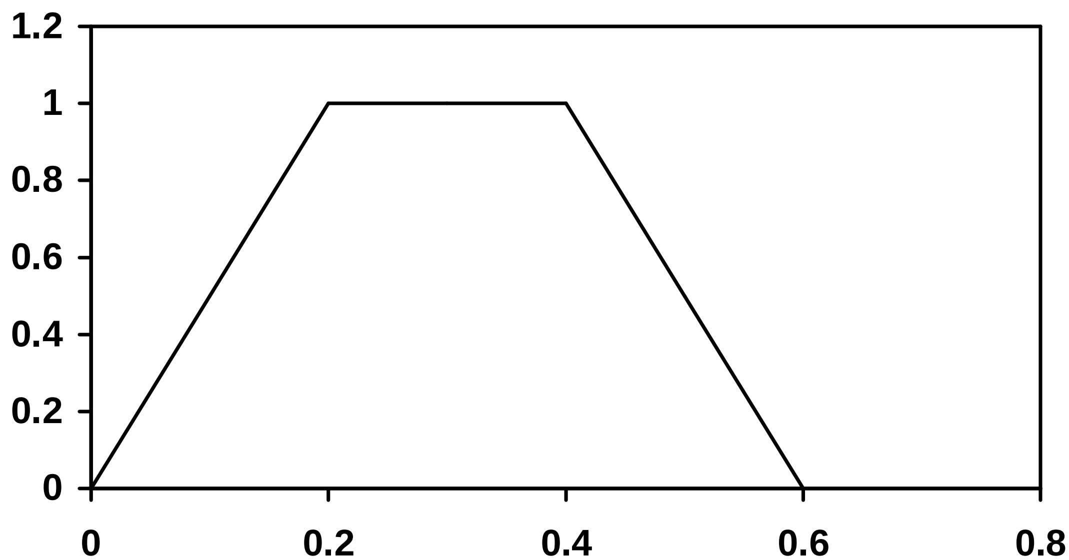

Figure 2

$$\mu_{\tilde{\pi}_2}(x) = \sup_{\substack{t \in R^+ \\ t < \frac{1}{\lambda}}} \{\mu_{\tilde{S}}(t) \,|\, x = (1 - \lambda t)\,(e^{2\lambda t} - e^{\lambda t} - \lambda t e^{\lambda t})\}$$

Table 3

x	$\mu_{\tilde{\pi}_1}(x)$
0	0
0.1	1
0.2	1
0.3	0.91
0.4	0.68
0.5	0.5
0.6	0.35
0.7	0.22
0.8	0.1
0.9	0

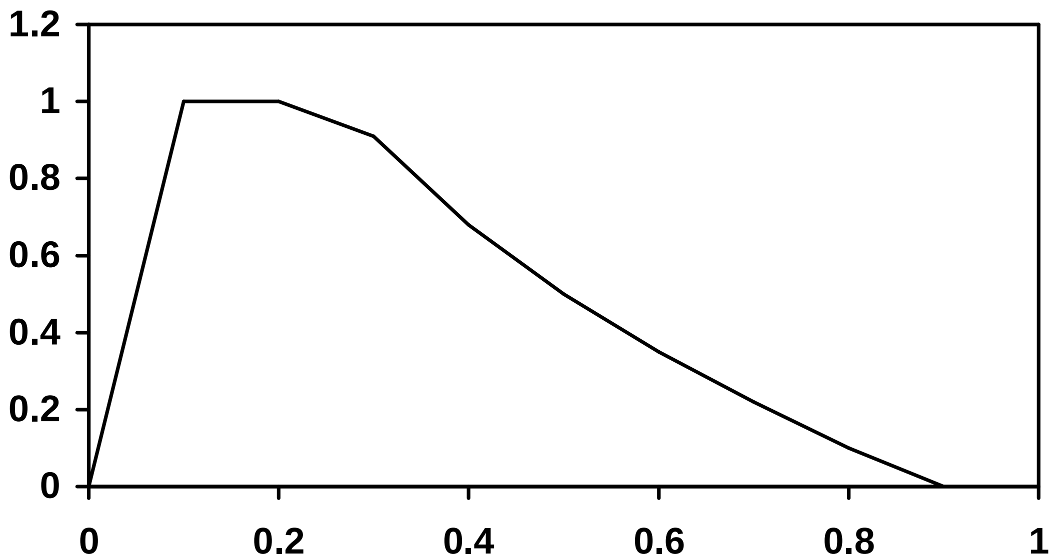

Figure 3

$$\mu_{\tilde{L}}(x) = \sup_{\substack{t \in R^+ \\ t < \frac{1}{\lambda}}} \left\{ \mu_{\tilde{S}}(t) \mid x = \frac{\lambda t(2-\lambda t)}{2(1-\lambda t)} \right\}$$

Table 4

x	$\mu_{\tilde{L}}(x)$
0	0
0.1	0.48
0.2	0.9
0.3	1
0.4	1
0.5	1
0.6	0.83
0.7	0.6
0.8	0.4
1.0	0.07
1.1	0

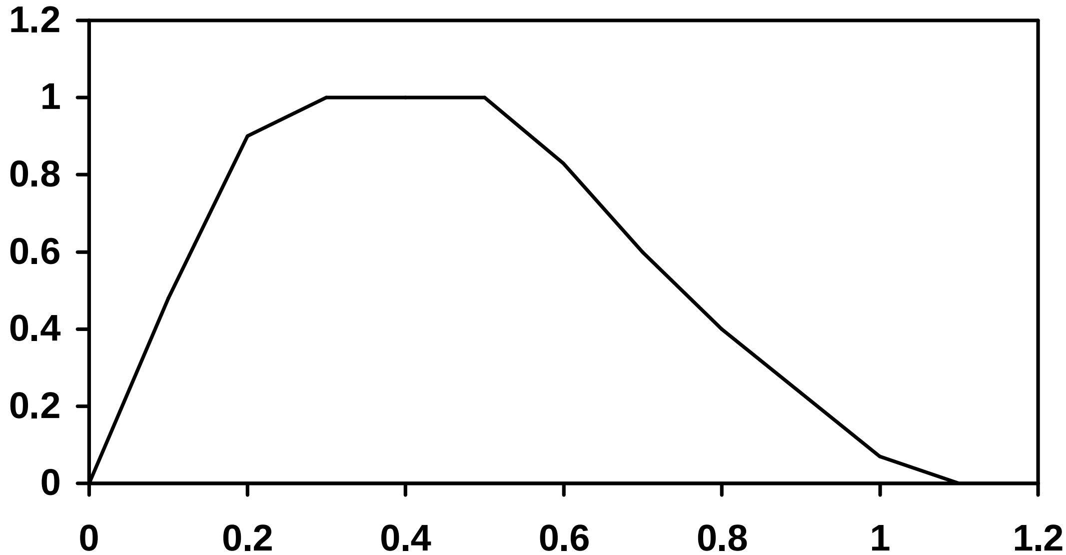

Figure 4

$$\mu_{\tilde{W}}(x) = \sup_{\substack{t \in R^+ \\ t < \frac{1}{\lambda}}} \left\{ \mu_{\tilde{S}}(t) \,\middle|\, x = \frac{t(2 - \lambda t)}{2(1 - \lambda t)} \right\}$$

Table 5

x	$\mu_{\tilde{W}}(x)$
0	0
1	0.48
2	0.9
3	1
4	1
5	1
6	0.83
7	0.6
8	0.4
10	0.07
11	0

Figure 5

Figures 5.1, 5.2 and 5.3 give the numerical solutions for the steady state probabilities while figures 5.4 and 5.5 give the numerical solutions for the mean queue length and the waiting time in the system respectively.

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