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SURVEY ON GEOMETRY OF STATISTICAL SUBMANIFOLDS

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Abstract: We survey main results of geometry of submanifolds of statistical manifolds. We discuss recent work on submanifolds of holomorphic statistical manifolds. The notion of statistical structure on Sasakian manifolds is also briefly discussed.

Keywords: Statistical manifolds, holomorphic submanifolds, invariant Sasakian manifolds.

1. INTRODUCTION

Statistical manifolds introduced by Amari (1985) have been actively studied in recent times as it finds many applications in the field of information geometry, hessian geometry, statistical inference, neural networks, document classification, face recognition, image analysis, clustering, control systems and many more. Statistical manifolds may be considered as manifolds consisting of certain probability density functions. They are geometrically formulated as Riemannian manifolds with a certain affine connection. Their complex version named holomorphic statistical manifolds are studied by T. Kurose [13]. It is natural for geometers to try to build the submanifold theory and the complex manifold theory of statistical manifolds. Recently Furuhata [5], Hasegawa [7] and [10] has done work on CR statistical submanifolds of holomorphic statistical manifolds.

In this article, we collect the recent work done in the field of geometry of statistical submanifolds. The paper is organized as: in Section 2, we give elementary theory of statistical manifolds and in Section 3, we discuss the geometry of statistical submanifolds. In Section 4, the basic definitions and the properties of holomorphic statistical submanifolds are given. In Section 5, we give some results on statistical real hypersurfaces. Finally in Section 6, we discuss the notion of Sasakian statistical structure on odd dimensional hypersurfaces.

2. STATISTICAL MANIFOLDS

This section is fully devoted to several fundamental notions, formulas and definitions related to the theory of statistical manifolds. Let $\bar{M}$ be an m-dimensional manifold, $\bar{\nabla}$ an affine connection and $\bar{g}$ a Riemannian metric on $\bar{M}$. We denote by $T_x\bar{M}$ the tangent space at a point $x \in \bar{M}$ and by $T\bar{M}$ the tangent bundle over $\bar{M}$. By $\Gamma(E)$, we denote the set of all C^∞ sections of a vector bundle $E \to \bar{M}$ so $\Gamma(T\bar{M}^{p,q})$ means the set of tensors fields of type (p, q) on $\bar{M}$.

Definition 2.1: $(\bar{M},\overline{\nabla},\bar{g})$ is called a statistical manifold if: (a) $\overline{\nabla}$ is of torsion free and (b) $(\overline{\nabla}_X\bar{g})(Y,Z)=(\overline{\nabla}_Y\bar{g})(X,Z)$, for $X,Y,Z\in\Gamma(T\bar{M})$. The pair $(\overline{\nabla},\bar{g})$ is called a statistical structure on $\bar{M}$.

Definition 2.2: The affine connection $\overline{\nabla}'$ of $\bar{M}$ is called the dual connection of $\overline{\nabla}$ with respect to $\bar{g}$ if $X\bar{g}(Y,Z)=\bar{g}(\overline{\nabla}_XY,Z)+\bar{g}(Y,\overline{\nabla}'_XZ)$, for $X,Y,Z\in\Gamma(T\bar{M})$. Trivially, we have $(\overline{\nabla}')'=\overline{\nabla}$.

Let $\overline{\nabla}^g$ be the Levi-Civita connection of $\bar{g}$. By definition, the pair $(\overline{\nabla}^g,\bar{g})$ is a statistical structure which is called a Riemannian statistical structure or trivial statistical structure. Using the above definitions, we have the following propositions:

Proposition 2.3: For a statistical manifold $(\bar{M},\overline{\nabla},\overline{\nabla}',\bar{g})$ we have $[X,Y]=[X,Y]'$ where $[X,Y]=\overline{\nabla}_XY-\overline{\nabla}_YX$ and $[X,Y]'=\overline{\nabla}'_XY-\overline{\nabla}'_YX$ as both $\overline{\nabla}$ and $\overline{\nabla}'$ are of torsion free.

Proposition 2.4: $(\bar{M},\overline{\nabla},\bar{g})$ is a statistical manifold if and only if $(\bar{M},\overline{\nabla}',\bar{g})$ is a statistical manifold.

Definition 2.5: A statistical manifold $(\bar{M},\overline{\nabla},\bar{g})$ is said to be of constant curvature $k\in\mathcal{R}$ if $\bar{R}(X,Y)Z=k\{\bar{g}(Y,Z)X-\bar{g}(X,Z)Y\}$ holds, where $\bar{R}(X,Y)Z=\overline{\nabla}_X\overline{\nabla}_YZ-\overline{\nabla}_Y\overline{\nabla}_XZ-\overline{\nabla}_{[X,Y]}Z$. A statistical structure $(\overline{\nabla},\bar{g})$ of constant curvature 0 is called a Hessian structure.

For a statistical manifold $(\bar{M},\overline{\nabla},\bar{g})$ for $X,Y,Z\in\Gamma(T\bar{M})$, we can obtain

$$\bar{g}(\bar{R}(X,Y)Z,W)=-\bar{g}(\bar{R}'(X,Y)W,Z),$$
$$\bar{g}(\bar{R}(X,Y)Z,W)=-\bar{g}(\bar{R}(Y,X)Z,W),$$
$$\bar{g}(\bar{R}'(X,Y)Z,W)=-\bar{g}(\bar{R}'(Y,X)Z,W),$$

Proposition 2.6: A statistical manifold $(\bar{M},\overline{\nabla},\bar{g})$ is of constant curvature k if and only if $(\bar{M},\overline{\nabla}',\bar{g})$ is of constant curvature k. In particular if $(\overline{\nabla},\bar{g})$ is Hessian structure so is $(\overline{\nabla}',\bar{g})$.

Proposition 2.7: For a statistical manifold $(\bar{M},\overline{\nabla},\bar{g})$ as $\overline{\nabla}$ and $\overline{\nabla}'$ are of torsion free, we have the first Bianch's identity:

$$\bar{g}(\bar{R}(X,Y)Z,W)+\bar{g}(\bar{R}(Y,Z)X,W)+\bar{g}(\bar{R}(Z,X)Y,W)=0,$$
$$\bar{g}(\bar{R}'(X,Y)Z,W)+\bar{g}(\bar{R}'(Y,Z)X,W)+\bar{g}(\bar{R}'(Z,X)Y,W)=0,$$

for $X,Y,Z\in\Gamma(T\bar{M})$.

Furuhata [5] for a statistical structure $(\overline{\nabla},\bar{g})$ defined the difference tensor field $K\in\Gamma(T\bar{M}^{(1,2)})$ as $K(X,Y)=\overline{\nabla}_XY-\overline{\nabla}^g_XY$ satisfying $K(X,Y)=K(Y,X)$ and $\bar{g}(K(X,Y),Z)=\bar{g}(Y,K(X,Z))$. The following structure equations for $(\bar{M},\overline{\nabla},\overline{\nabla}',\bar{g})$ can be obtained easily:

$$\begin{aligned}\bar{R}(X,Y)Z=\ &\bar{R}^g(X,Y)Z+\left(\overline{\nabla}^g_XK\right)(Y,Z)-\left(\overline{\nabla}^g_YK\right)(Z,X)\\&+K\big(X,K(Y,Z)\big)-K\big(Y,K(Z,X)\big),\end{aligned}$$

$$\begin{aligned}\bar{R}'(X,Y)Z=\ &\bar{R}^g(X,Y)Z-\left(\overline{\nabla}^g_XK\right)(Y,Z)+\left(\overline{\nabla}^g_YK\right)(Z,X)\\&+K\big(X,K(Y,Z)\big)-K\big(Y,K(Z,X)\big),\end{aligned}$$

$$\bar{R}^g(X,Y)Z = \bar{R}(X,Y)Z - (\bar{\nabla}_X K)(Y,Z) + (\bar{\nabla}_Y K)(Z,X) + K\big(X, K(Y,Z)\big) - K\big(Y, K(Z,X)\big),$$

$$(\bar{\nabla}_X K)(Y,Z) - (\bar{\nabla}_Y K)(Z,X) = 2\big\{K\big(X,K(Y,Z)\big) - K\big(Y,K(Z,X)\big)\big\} + \frac{I}{2}\{\bar{R}(X,Y)Z - \bar{R}'(X,Y)Z\},$$

where, $\bar{R}^g(X,Y)Z = \bar{\nabla}^g_X \bar{\nabla}^g_Y Z - \bar{\nabla}^g_Y \bar{\nabla}^g_X Z - \bar{\nabla}^g_{[X,Y]} Z$.

For a Hessian structure $(\bar{\nabla}, \bar{g})$, we have

$$\bar{R}^g(X,Y)Z = -K\big(X,K(Y,Z)\big) - K\big(Y,K(Z,X)\big) = -\frac{1}{2}\{(\bar{\nabla}_X K)(Y,Z) - (\bar{\nabla}_Y K)(Z,X)\},$$

Definition 2.8 [7]: For a statistical manifold $(\bar{M}, \bar{\nabla}, \bar{g})$ we define the statistical curvature tensor field as $\bar{S}(X,Y)Z = \frac{1}{2}\{\bar{R}(X,Y)Z + \bar{R}'(X,Y)Z\}$ and $\bar{S}(X,Y,Z,W) = \bar{g}(\bar{S}(X,Y)Z, W)$, any for $X,Y,Z,W \in \Gamma(T\bar{M})$

Proposition 2.9: Let $(\bar{M}, \bar{\nabla}, \bar{g})$ be a statistical manifold. The tensor field $\bar{S} \in \Gamma(T\bar{M}^{(0,4)})$ satisfies

$$\bar{S}(X,Y,Z,W) = -\bar{S}(Y,X,Z,W),$$

$$\bar{S}(X,Y,Z,W) = -\bar{S}(X,Y,W,Z),$$

$$\bar{S}(X,Y,Z,W) + \bar{S}(Y,Z,X,W) + \bar{S}(Z,X,Y,W) = 0,$$

$$\bar{S}(X,Y,Z,W) = \bar{S}(Z,W,X,Y).$$

Definition 2.10 [7]: Let $(\bar{M}, \bar{\nabla}, \bar{g})$ be a statistical manifold. For $x \in \bar{M}$ and a two dimensional subspace $\Pi = span_R(v,w)$ of $T_x\bar{M}$

$$\frac{\bar{S}_x\big((v,w)w, v\big)}{\bar{g}(v,v)\bar{g}(w,w) - \big(\bar{g}(v,w)\big)^2},$$

is called the sectional curvature of $(\bar{M}, \bar{\nabla}, \bar{g})$ for section Π. A statistical manifold $(\bar{M}, \bar{\nabla}, \bar{g})$ is said to be of constant sectional curvature $k \in \mathcal{R}$ if it is constant for x and section Π.

Definition 2.11 [7]: The sectional curvature of a statistical manifold $(\bar{M}, \bar{\nabla}, \bar{g})$ is constant k if and only if $\bar{S}(X,Y)Z = k\{\bar{g}(Y,Z)X - \bar{g}(X,Z)Y\}$, for $X,Y,Z \in \Gamma(T\bar{M})$.

Remark 2.12 [7]: If $\bar{\nabla}$ is the Levi-Civita connection of $\bar{g}$, the definition of sectional curvature coincides with the standard one.

An isometric immersion preserving attached connection is called a statistical immersion. Furuhata [5] studied such immersions, in particular, elementary properties of hypersurfaces in statistical manifolds of constant curvature.

3. STATISTICAL SUBMANIFOLDS

For geometers, it is quite interesting and natural to study the geometry of submanifolds of statistical manifolds. In this section, we have tried to collect the basic definitions and results related to the theory of statistical submanifolds. Let (M, g) be a submanifold of statistical manifold $(\bar{M}, \bar{g})$ with the induced metric g. Let ∇ be an affine connection on M defined by $\nabla_X Y = (\bar{\nabla}_X Y)^T$ where $(\ )^T$ denotes the orthogonal projection of ( ) on the tangent space of M with respect to $\bar{g}$, that is $g(\nabla_X Y, Z) = \bar{g}(\bar{\nabla}_X Y, Z)$ for $X, Y, Z \in \Gamma(TM)$. Then (M, ∇, g) becomes a statistical manifold with the induced statistical structure (∇, g).

Definition 3.1: (M, ∇, g) is a statistical submanifold in $(\bar{M}, \bar{\nabla}, \bar{g})$ if (∇, g) is the induced statistical structure on M.

Let $\Gamma(TM^{\perp})$ be the normal bundle of M in $\bar{M}$. We define the second fundamental form of M for $\bar{\nabla}$ by $h(X, Y) = (\bar{\nabla}_X Y)^{\perp}$ for $X, Y \in \Gamma(TM)$. We define the shape operator and the normal connection $\nabla^{\perp}$ for $\bar{\nabla}$, respectively by $A_V X = -(\bar{\nabla}_X V)^T$ and $\nabla_X^{\perp} V = (\bar{\nabla}_X V)^{\perp}$ for $V \in \Gamma(TM^{\perp})$ and $X \in \Gamma(TM)$. Similar to the Riemannian submanifold theory, we can write the Gauss and Weingarton formulas as $\bar{\nabla}_X Y = \nabla_X Y + h(X, Y)$, $\bar{\nabla}_X V = -A_V X + \nabla_X^{\perp} V$, for $X, Y \in \Gamma(TM)$ and $V \in \Gamma(TM^{\perp})$. Similarly for the induced connection ∇' corresponding to the dual connection $\bar{\nabla}'$ of $\bar{M}$, we can write $\bar{\nabla}'_X Y = \nabla'_X Y + h'(X, Y)$, $\bar{\nabla}'_X V = -A'_V X + \nabla'^{\perp}_X V$. Since $\bar{\nabla}$ is of torsion free therefore $h(X, Y) = h(Y, X)$. By using these equations, we have the following proposition:

Proposition 3.2 [11]: If (M, ∇, g) be a statistical submanifold of $(\bar{M}, \bar{\nabla}, \bar{g})$ with the induced connections ∇ , ∇'on M corresponding to $\bar{\nabla}$ and its dual connection $\bar{\nabla}'$ on $\bar{M}$, respectively. Then

(a) ∇' is also a dual connection on M to ∇ that is $Xg(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla'_X Z)$.

(b) (M, ∇', g) is also a statistical submanifold of $(\bar{M}, \bar{\nabla}', \bar{g})$ with the induced statistical structure (∇', g).

(c) $\nabla^{\perp}$ and $\nabla'^{\perp}$ are also dual connections with respect to the induced metric $g^{\perp}$ on $\Gamma(TM^{\perp})$ that is $Xg^{\perp}(U, V) = g^{\perp}(\nabla_X^{\perp} U, V) + g^{\perp}(U, \nabla'^{\perp}_X V)$ for $U, V \in \Gamma(TM^{\perp})$ and $X \in \Gamma(TM)$.

(d) For $X, Y \in \Gamma(TM)$ and $V \in \Gamma(TM^{\perp})$

$$g(A_V X, Y) = g(h'(X, Y), V),$$

$$g(A'_V X, Y) = g(h(X, Y), V)$$

which are different from the results obtained during the development of Riemannian submanifold theory.

The corresponding Gauss, Codazzi and Ricci equations to the Riemannian submanifold theory are given by the following result.

Proposition 3.3 [3, 15]: Let $\overline{\nabla}$ be a statistical connection on $\overline{M}$ and ∇ be the induced connection on M. Let $\bar{R}$ and R be the Riemannian curvature tensors of $\overline{\nabla}$ and ∇ respectively. Then

$$\bar{g}(\bar{R}(X,Y)Z,W) = g(R(X,Y)Z,W) + \bar{g}\big(h(X,Z),h'(Y,W)\big) - \bar{g}\big(h'(X,W),h(Y,Z)\big),$$

$$(\bar{R}(X,Y)Z)^{\perp} = \nabla_X^{\perp}h(Y,Z) - h(\nabla_X Y,Z) - h(Y,\nabla_X Z) - \{\nabla_Y^{\perp}h(X,Z) - h(\nabla_Y X,Z) - h(X,\nabla_Y Z)\},$$

$$\bar{g}(R^{\perp}(X,Y)U,V) = \bar{g}(\bar{R}(X,Y)U,V) + g([A'_U,A_V]X,Y),$$

where, $R^{\perp}$ is the Riemannian curvature tensor on $\Gamma(TM^{\perp})$, $U,V \in \Gamma(TM^{\perp})$ and $[A'_U,A_V] = A'_U A_V - A_V A'_U$.

For the equations of Gauss, Codazzi and Ricci with respect to the dual connection $\overline{\nabla}'$ on $\overline{M}$, we have

Proposition 3.4 [3]: Let $\overline{\nabla}'$ be a dual connection on $\overline{M}$ and ∇' be the induced connection on M. Let $\bar{R}'$ *and* R$'$ be the Riemannian curvature tensors for $\overline{\nabla}'$ and ∇' respectively. Then

$$\bar{g}(\bar{R}'(X,Y)Z,W) = g(R'(X,Y)Z,W) + \bar{g}\big(h'(X,Z),h(Y,W)\big) - \bar{g}\big(h(X,W),h'(Y,Z)\big),$$

$$(\bar{R}'(X,Y)Z)^{\perp} = \nabla_X'^{\perp}h(Y,Z) - h'(\nabla_X Y,Z) - h'(Y,\nabla_X Z) - \big\{\nabla_Y'^{\perp}h(X,Z) - h'(\nabla_Y X,Z) - h'(X,\nabla_Y Z)\big\},$$

$$\bar{g}\big(R'^{\perp}(X,Y)U,V\big) = \bar{g}(\bar{R}'(X,Y)U,V) + g([A_U,A'_V]X,Y),$$

where, $R'^{\perp}$ is the Riemannian curvature tensor on $\Gamma(TM^{\perp})$, $U,V \in \Gamma(TM^{\perp})$ and $[A_U,A'_V] = A_U A'_V - A'_V A_U$.

Definition 3.5 [7]: Let (M,∇,g) be a statistical submanifold of dimension n in $(\overline{M},\overline{\nabla},\bar{g})$. We define the mean curvature field of M for $\overline{\nabla}$ by

$$H = \frac{1}{n} tr_g h$$

where, tr_g is the trace with respect to g.

Definition 3.6 [4]: M is said to be totally geodesic with respect to $\overline{\nabla}$ if the second fundamental form h of M for $\overline{\nabla}$ vanishes identically. M is said to be totally umbilical with respect to $\overline{\nabla}$ if $h(X,Y) = Hg(X,Y)$ holds. M is said to be totally normally umbilical with respect to $\overline{\nabla}$ if $A_U X = g(H,U)X$ for any $X \in \Gamma(TM)$ and $U \in \Gamma(TM^{\perp})$.

Similarly, we can define M to be totally geodesic, totally umbilical and totally normally umbilical with respect to $\overline{\nabla}'$. Using the above proposition we can easily deduce the following theorem:

Theorem 3.7: Let (M, ∇, g) be a submanifold of a statistical manifold $(\bar{M}, \bar{\nabla}, \bar{g})$. If M is totally geodesic then $\bar{R}(X, Y) = R(X, Y)$, for any $X, Y \in \Gamma(TM)$.

Submanifold theory for statistical manifolds was initially developed by Vos [15] with applications to Bartlett correction for which he had given an invariant expression using curvature that described how a statistical submanifold is curving in a supermanifold. We will discuss the recent work done on theory of statistical submanifolds by Furuhata [5], Milijevic [10], Aydin [3], Furuhata and Hasegawa [7] and Siddiqui et al [4]. Aydin [3] studied the behavior of submanifolds in statistical manifolds of constant curvature and investigated curvature properties of such submanifolds. Some main results for submanifolds with any codimension and hypersurfaces of statistical manifolds of constant curvature developed in Aydin [3] includes the following:

Proposition 3.8 [3]: Let M be an n-dimensional submanifold of an m-dimensional statistical manifold $\bar{M}(k)$ of constant curvature $k \in \mathcal{R}$. Assume that the imbedding curvature tensor H and H' satisfy $h(X, Y) = g(X, Y)H$ and $h'(X, Y) = g(X, Y)H'$, for any $X, Y \in \Gamma(TM)$. Then M is also a statistical manifold of constant curvature $k + g(H, H')$, where $g(H, H')$ is constant.

Definition 3.9 [3]: Let $\bar{M}$ be an m-dimensional statistical manifold. Then the Ricci tensor $\bar{Q}$ of type (0,2) is defined by $\bar{Q}(Y, Z) = trace\{X \to \bar{R}(X, Y)Z\}$, where $\bar{R}$ is the curvature tensor field of the affine connection $\bar{\nabla}$ on $\bar{M}$.

Theorem 3.10 [3]: Let $\bar{M}(k)$ be an m-dimensional statistical manifold of constant curvature $k \in \mathcal{R}$ and M an n-dimensional statistical submanifold of $\bar{M}(k)$. Also, let $\{e_1 \dots\dots\dots\dots\dots e_n\}$ and $\{f_1, \dots\dots\dots\dots\dots f_{m-n}\}$ be orthonormal tangent and normal frames, respectively on M. Then the induced Ricci tensor Q and the Ricci tensor Q' of M satisfy

$$Q(X, Y) = k(n-1)g(X, Y) + \sum_{i=1}^{m-n} \{g(A_{f_i}X, Y)trA'_{f_i} - g(A_{f_i}Y, A'_{f_i}X)\},$$

$$Q'(X, Y) = k(n-1)g(X, Y) + \sum_{i=1}^{m-n} \{g(A'_{f_i}X, Y)trA_{f_i} - g(A_{f_i}X, A'_{f_i}Y)\},$$

where, A_{f_i} and A'_{f_i} are linear transformations defined as in paragraph after Definition 3.1.

Definition 3.11: Let ∇ be a torsion free affine connection on a Riemannian manifold M that admits a parallel volume element Ω such that $\nabla\Omega = 0$, then (∇, Ω) is called an equiaffine structure on M.

Lemma 3.12 [3]: Let $\bar{M}(k)$ be an m-dimensional statistical manifold of constant curvature $k \in \mathcal{R}$ and M an n-dimensional submanifold of $\bar{M}(k)$. Assume that the affine connection ∇ of M is equiaffine then $\sum_{i=1}^{m-n}[A_{f_i}, A'_{f_i}] = 0$ where $[A_{f_i}, A'_{f_i}] = A_{f_i}A'_{f_i} - A'_{f_i}A_{f_i}$.

Proposition 3.13 [3]: Let $\bar{M}(k)$ be an m-dimensional statistical manifold of constant curvature $k \in \mathcal{R}$ and M an n-dimensional statistical submanifold of $\bar{M}(k)$. Then

$$2r \geq n(n-1)k + n^2\bar{g}(H,H') - \|h\|\|h'\|,$$

where, r is the scalar curvature of (M,∇,g), that is $r = \sum_{1\leq i,j\leq n} g(R(e_i,e_j)e_j,e_i)$.

Proposition 3.14 [3]: Let M be a statistical hypersurface of an $(n+1)$-dimensional statistical manifold $\bar{M}(k)$ of constant curvature $k \in \mathcal{R}$. We have

$$2r \geq n(n-1)k + n^2\|H\|\|H'\| - \|h\|\|h'\|,$$

where, r is the scalar curvature of M.

Proposition 3.15 [--3]: Let M be a statistical hypersurface of an $(n+1)$-dimensional statistical manifold $\bar{M}(k)$. For each $X \in T_x(M)$ we have

$$Ric(X) = (n-1)k + n\bar{g}(h'(X,X),H) - \sum_{i=1}^{n} h_{i1}h'_{i1},$$

$$Ric(X) = (n-1)k + n\bar{g}(h(X,X),H') - \sum_{i=1}^{n} h_{i1}h'_{i1},$$

where, $h_{i1}h'_{i1} = \bar{g}(h(X,e_i),h'(X,e_j))$.

Theorem 3.16 [3]: Let M be an n-dimensional statistical submanifold of an m-dimensional statistical manifold $\bar{M}(k)$. For each $X \in T_x(M)$, we have

$$Ric(X) \geq 2Ric^g(X) - \frac{n^2}{8}\bar{g}(H,H) - \frac{n^2}{8}\bar{g}(H',H') + (n-1)k - 2(n-1)max\bar{R}^0(X\wedge .)$$

where, $Ric^g(X)$ is Ricci tensor with respect to the Levi-Civita connection $\bar{\nabla}^g$ on $\bar{M}$. In particular, M is a minimal submanifold.

4. HOLOMORPHIC STATISTICAL SUBMANIFOLDS

Holomorphic statistical manifolds were initially studied by Kurose [13] and later on Furuhata [4], [5], [7], [10] and many more developed the theory of submanifolds of holomorphic statistical manifolds. A holomorphic statistical manifolds is considered as a special Kahler manifold with a certain connection.

Definition 4.1 [7]: Let $\bar{M}$ be an almost complex manifold with almost complex structure $J \in \Gamma(T\bar{M}^{(1,1)})$ and an affine connection $\bar{\nabla}$ of $\bar{M}$. A quadruple $(\bar{M},\bar{\nabla},\bar{g},J)$ is called a holomorphic statistical manifold if: (a) $(\bar{M},\bar{\nabla},\bar{g})$ is a statistical manifold and (b) $\omega(X,Y) = \bar{g}(X,JY)$ is a $\bar{\nabla}$-parallel 2-form on $\bar{M}$.

The skew-symmetry of ω, that is $\omega(X,Y) = -\omega(Y,X)$ implies that $(\bar{g},J)$ is an almost Hermitian structure and the condition $\bar{\nabla}\omega = 0$ implies that ω is closed since $\bar{\nabla}$ is of torsion free. It may be noted that a holomorphic statistical manifold $(\bar{M},\bar{\nabla},\bar{g},J)$ is a special Kahler manifold if $\bar{\nabla}$ is flat, that is

$$(\bar{\nabla}_X J)Y = (\bar{\nabla}_Y J)X.$$

Lemma 4.2 [11]: Let $(\bar{M}, \bar{g}, J)$ be a Kahler manifold. If we define a connection $\bar{\nabla}$ as $\bar{\nabla}= \bar{\nabla}^g + K$, where K is a (1,2)-tensor field satisfying the next three conditions:

$$K(X,Y) = K(Y,X),$$

$$\bar{g}(K(X,Y),Z) = \bar{g}\big(Y, K(X,Z)\big),$$

$$K(X,JY) = -JK(X,Y),$$

for $X,Y,Z \in \Gamma(T\bar{M})$ then $(\bar{M}, \bar{\nabla}, \bar{g}, J)$ is a holomorphic statistical manifold.

Lemma 4.3 [5]: The following hold for a holomorphic statistical manifold $(\bar{M}, \bar{\nabla}, \bar{g}, J)$:

$$\bar{\nabla}^J = \bar{\nabla}', where\ \bar{\nabla}^J_X = J^{-1}\bar{\nabla}_X(JY),$$

$$\bar{\nabla}_X(JY) = J\bar{\nabla}'_X Y,$$

$$\bar{R}(X,Y)JZ = J\bar{R}'(X,Y)Z$$

Definition 4.4 [5]: A holomorphic statistical manifold $(\bar{M}, \bar{\nabla}, \bar{g}, J)$ is said to be of constant holomorphic curvature $k \in \mathcal{R}$ if

$$\bar{R}(X,Y)Z = \frac{k}{4}\{\bar{g}(Y,Z)X - \bar{g}(X,Z)Y + \bar{g}(JY,Z)JX - \bar{g}(JX,Z)JY + 2\bar{g}(X,JY)JZ\},$$

for any $X,Y,Z \in \Gamma(T\bar{M})$. A holomorphic statistical manifold $\bar{M}$ of constant holomorphic curvature is called a holomorphic statistical space form.

In this section, we discuss the statistical submanifold theory in holomorphic statistical manifolds. Let $(\bar{M}, \bar{g}, J)$ be a Kahler manifold and M a submanifold in $\bar{M}$. For $X \in \Gamma(TM)$ and $V \in \Gamma(TM^{\perp})$, define

$$JX = PX + FX, JV = tV + fV$$

where, $PX = (JX)^T, FX = (JX)^{\perp}, tV = (JV)^T, fV = (JV)^{\perp}$.

It is easy to obtain

$$g(PX,Y) = -g(X,PY),$$

$$g(fU,V) = -g(U,fV),$$

$$g(FX,V) = -g(X,tV),$$

$$P^2 = -Id_{TM} - tF, FP + fF = 0,$$

$$Pt + tf = 0, f^2 = -Id_{TM} - Ft,$$

for, $X,Y \in \Gamma(TM)$ and $U,V \in \Gamma(TM^{\perp})$.

Definition 4.5 [4]: A statistical submanifold M is called a CR-statistical submanifold in holomorphic statistical manifold $\bar{M}$ of dimension $2m \geq 4$ if M is CR- submanifold in $\bar{M}$, that is, there exists a differentiable distribution $D: x \to D_x \in T_xM$ on M satisfying the following conditions: (a) D is holomorphic $JD_x = D_x \subset T_xM$ for each $x \in M$ and (b) the complementary orthogonal distribution $D^\perp : x \to D_x^\perp \subset T_xM$ is totally real $JD_x^\perp \subset T_xM^\perp$ for each $x \in M$.

Let N be a subbundle of $TM^\perp$ defined as $N_x = \{V \in T_xM^\perp : V \perp JD_x^\perp\}$ for each $x \in M$. Accordingly, we have $T\overline{M} = TM \oplus TM^\perp = \{D \oplus D^\perp\} \oplus \{JD^\perp \oplus N\}$ and $FP = 0, fF = 0, tF = 0, Pt = 0, P^3 = -P, f^3 = -f$.

Definition 4.6 [4]: A statistical submanifold M of a holomorphic statistical manifold $\overline{M}$ is called holomorphic ($F = 0\ and\ t = 0$) if the almost complex structure J of $\overline{M}$ carries each tangent space of M into itself whereas it is said to be totally real($P = 0$) if the almost complex structure J of $\overline{M}$ carries each tangent space of M into its corresponding normal space.

If $JD^\perp = TM^\perp$ and $D \neq 0$ then M is called a generic submanifold ($f = 0$). If $D^\perp = TM$ and $JD^\perp = TM^\perp$ then M is called a Lagrangian submanifold ($P = 0\ and\ f = 0$). If $D \neq 0$ and $D^\perp \neq 0$ then M is said to be proper CR-submanifold of $\overline{M}$.

Furuhata [5] proved that if a hypersurface in a statistical manifold of constant curvature carries a holomorphic statistical structure of constant holomorphic curvature then the hypersurface is a special Kahler manifold.

Theorem 4.7 [5]: Let $(\overline{M}, \overline{\nabla}, \bar{g})$ be a $(2n + 1)$-dimensional statistical manifold of constant curvature $\bar{k}$ with $n \geq 2$ and (M, ∇, g) a holomorphic statistical submanifold of constant curvature k. If there exists a statistical immersion $f : M \to \overline{M}$ of codimension 1, then the curvature k vanishes.

Lemma 4.8 [7]: Let (M, ∇, g) be a statistical submanifold in $\overline{M}$. Then

$$\nabla_X(PY) - A_{FY}X = P\nabla'_XY + th'(X, Y),$$

$$h(X, PY) + \nabla_X^\perp FY = fh'(X, Y) + F\nabla'_XY,$$

$$\nabla_X(tV) - A_{tV}X = -PA'_VX + t\nabla'^\perp_XV,$$

$$h(X, tV) + \nabla_X^\perp(tV) = -FA'_VX + f\nabla_X^\perp V,$$

for, $X, Y \in \Gamma(TM)$ and $V \in \Gamma(TM^\perp)$.

Remark 4.9 [7]: Let (M, ∇, g) be a holomorphic statistical submanifold in $\overline{M}$. Using above lemma, we have

$$\nabla_X(JY) = J\nabla'_XY,$$

$$h(X, JY) = Jh'(X, Y),$$

$$A_{JV}X = JA'_VX,$$

$$\nabla_X^\perp(JV) = J\nabla'^\perp_XV,$$

for, $X, Y \in \Gamma(TM)$ and $V \in \Gamma(TM^\perp)$. Accordingly, the mean curvature vector fields for $\overline{\nabla}$ and $\overline{\nabla}'$ vanish.

Theorem 4.10 [10]: Let M be a statistical submanifold of a holomoprphic statistical manifold $\bar{M}$. Then $\nabla_X^{\perp}FY = F(\nabla'_X Y)$ holds if and only if $A'_V PY = -A_{fV}Y$ for $X, Y \in \Gamma(TM)$ and $V \in \Gamma(TM^{\perp})$.

Theorem 4.11 [10]: Let M be a statistical submanifold of holomorphic statistical manifold $\bar{M}$. If $\nabla_X^{\perp}FY = F\nabla'_X Y$ holds then the curvature tensor R' and the normal curvature tensor $R^{\perp}$ satisfy $FR'(X,Y)Z = R'(X,Y)FZ$,for any $X, Y, Z \in \Gamma(TM)$.

Theorem 4.12 [7]: Let (M, ∇, g) be a CR-statistical submanifold in $\bar{M}$. Then $A_{JV}W = A_{JW}V$ and $A'_{JV}W = A'_{JW}V$ for $V, W \in D^{\perp}$.

Definition 4.13 [11]: Let $(\bar{M}, \bar{\nabla}, \bar{g})$ be a holomorphic statistical manifold and let M be its submanifold. Then M is said to be a totally real submanifold of $\bar{M}$ if $JTM \subset TM^{\perp}$.

Proposition 4.14 [11]: Let $(\bar{M}, \bar{\nabla}, \bar{g}, J)$ be a holomorphic statistical manifold.

Let M be a totally real submanifold of $\bar{M}$. Then

$$A_{JY}X + th'(X,Y) = 0,$$

$$fh'(X,Y) = D_X^{\perp}(JY) - J\nabla'_X Y,$$

$$A_{fV}X = \nabla_X(tV) - t\nabla'^{\perp}_X V,$$

$$-JA'_V X - h(X, tV) = \nabla_X^{\perp}(fV) - f\left(\nabla'^{\perp}_X V\right),$$

and their duals hold.

Let M be a totally real submanifold of $\bar{M}$ with $\nabla_X^{\perp}(fV) = f(\nabla'^{\perp}_X V)$. Then $A'_V = 0$ for $V \in \Gamma(N)$,

$$h(X,Y) = JA'_{JY}X,$$

$$\nabla_X^{\perp}(JY) = J\left(\nabla'^{\perp}_X Y\right),$$

and their duals hold, where N is defined as in Definition 4.5.

Proposition 4.15 [11]: Let $(\bar{M}, \bar{\nabla}, \bar{g}, J)$ be a holomorphic statistical manifold. We define $I(X,Y) = (\bar{\nabla}_X J)Y, I'(X,Y) = (\bar{\nabla}'_X J)Y$, for $X, Y \in \Gamma(T\bar{M})$. Then $I(X,Y) = I(Y,X)$, $I(X,Y) = -I'(X,Y)$ and $I(X,JY) = -JI(X,Y)$.

Theorem 4.16 [10]: Let $(\bar{M}, \bar{\nabla}, \bar{g}, J)$ be a holomorphic statistical manifold and M be a totally real submanifold of $\bar{M}$ with $dim\bar{M} = 2dimM$. Then $I(X,Y) \in \Gamma(TM^{\perp})$ for $X, Y \in \Gamma(TM)$ if and only if $A = A'$.

Some results on totally umbilical CR-statistical submanifolds with respect to $\bar{\nabla}$ and $\bar{\nabla}'$ in holomorphic statistical manifolds with constant holomorphic curvature are obtained in [7], [10] are as follows:

Theorem 4.17 [7]: Let $(\bar{M},\bar{\nabla},\bar{g},J)$ be a holomorphic statistical manifold and (M,∇,g) a generic submanifold in $\bar{M}$ of codimension greater than one. If M is totally umbiolical with respect to $\bar{\nabla}$, then M is totally geodesic with respect to $\bar{\nabla}$.

Theorem 4.18 [7]: Let $(\bar{M},\bar{\nabla},\bar{g},J)$ be a holomorphic statistical manifold and (M,∇,g) a proper CR-submanifold in $\bar{M}$. If M is totally umbilical with respect to $\bar{\nabla}$ and $\bar{\nabla}'$, then the sectional curvature of $\bar{M}$ for a CR-section of M vanishes, where a plane section $X \wedge V$ with $X \in D$ and $V \in D^{\perp}$ is called a CR-section of M.

Theorem 4.19 [10]: Let M be a CR-statistical submanifold in a holomorphic statistical manifold $\bar{M}$. If M is totally umbilical with respect to $\bar{\nabla}$ and $\bar{\nabla}'$ such that $JH' \in \Gamma(N)$, then we have either (a) M is totally geodesic with respect to $\bar{\nabla}'$ or (b) $dim\, D \geq 2$.

Theorem 4.20 [10]: Let M be a CR-statistical submanifold in a holomorphic statistical manifold in a holomorphic statistical manifold $\bar{M}$. If M is totally umbilical with respect to $\bar{\nabla}$ and $\bar{\nabla}'$, then for any $X \in \Gamma(D^{\perp})$ we have : (a) $\nabla_X^{\perp} H \in JD^{\perp}$ or (b) $\nabla_X^{\perp} H = 0$. Similarly, (a)' $\nabla'^{\perp}_X H' \in JD^{\perp}$ or (b)' $\nabla'^{\perp}_X H' = 0$.

Milijevi´c [10] has given the following results for CR-statistical submanifolds in holomorphic statistical space form:

Theorem 4.21 [10]: Let M be a CR-statistical submanifold in a holomorphic statistical space form $\bar{M}(k)$. If M is totally umbilical with respect to $\bar{\nabla}$ and $\bar{\nabla}'$ such that $\nabla_X H = 0$ and $\nabla'_X H' = 0$ for any $X \in \Gamma(D^{\perp})$, then:

(a) $k = 0$ or

(b) $dim\, D \geq 2$ or

(c) H and H' are perpendiculars to $JD^{\perp}$ or

(d) M is totally geodesic with respect to $\bar{\nabla}$ and $\bar{\nabla}'$.

Theorem 4.22 [10]: Let M be a proper CR-statistical submanifold in a holomorphic statistical space form $\bar{M}(k)$. If M is totally umbilical with respect to $\bar{\nabla}$ and $\bar{\nabla}'$ such that $\nabla_X^{\perp} FY = F\nabla'_X Y$ for any $Y \in \Gamma(D^{\perp})$ and $X \in \Gamma(D)$, then we have: (a) $k = 0$ or (b) $dim\, D^{\perp} \geq 0$.

The above result is also true if M is totally geodesic with respect to $\bar{\nabla}$ and $\bar{\nabla}'$. Furthermore, Furuhata et al [7] show that a Lagrangian submanifold is of constant sectional curvature if the statistical shape operator and its dual operator commute.

Theorem 4.23 [7]: Let $(\bar{M},\bar{\nabla},\bar{g},J)$ be a holomorphic statistical manifold and (M,∇,g) a Lagrangian submanifold in $\bar{M}$. If $A_{JX}A'_{JY} = A'_{JY}A_{JX}$ for each $X, Y \in \Gamma(TM)$, then $(\bar{S}(X,Y)Z)^{\perp} = S(X,Y)Z$ for each $X, Y, Z \in \Gamma(TM)$. In particular, if $(\bar{M},\bar{\nabla},\bar{g},J)$ is of constant holomorphic sectional curvature k additionally, then M is of constant sectional curvature $k/4$.

Definition 4.24 [11]: Let M be a submanifold of a holomorphic statistical manifold $\bar{M}$.

(a) M is said to have parallel second fundamental form h with respect to the connection $\bar{\nabla}$, if $\bar{\nabla}h = 0$. Here $(\bar{\nabla}_X h)(Y,Z) = \nabla^{\perp}_X h(Y,Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z)$.

(b) M is called a semi-parallel submanifold for the connection $\bar{\nabla}'$, if $\bar{R}(X,Y)h = 0$ where $\bar{R}(X,Y)h(Z,W) = \left(\bar{\nabla}_X(\bar{\nabla}_Y h)\right)(Z,W) - \left(\bar{\nabla}_Y(\bar{\nabla}_X h)\right)(Z,W) - (\bar{\nabla}_{[X,Y]}h)\,(Z,W)$.

In [11] Milijevic studied semi-parallel submanifolds with respect to the connection $\bar{\nabla}$ and gave the following result:

Theorem 4.26 [11]: Let $(\bar{M}, \bar{\nabla}, \bar{g}, J)$ be a holomorphic statistical manifold and M a totally real submanifold of $\bar{M}$. Suppose

(a) $\nabla^{\perp}_X(fV) = f\nabla^{\perp}_X V$.

(b) (∇, g) is of constant curvature $k \neq 0$. If M is semi-parallel for $\bar{\nabla}$, then M is totally geodesic for $\bar{\nabla}$.

Corollary 4.27 [11]: Let $(\bar{M}, \bar{\nabla}, \bar{g}, J)$ be a holomorphic statistical manifold and M a Lagrangian submanifold of $\bar{M}$. If (∇, g) is of constant curvature $k \neq 0$ and M a semi-parallel for $\bar{\nabla}$, then M is totally geodesic for $\bar{\nabla}$.

It is remarked that the dual version of the above theorem holds namely if M is semi-parallel for $\bar{\nabla}'$, then M is totally geodesic for $\bar{\nabla}'$.

5. STATISTICAL REAL HYPERSURFACES

Real hypersurfaces in holomorphic statistical manifolds form an important class of CR-statistical manifolds. In this section, we give results given by [3] and [7] on statistical real hypersurfaces. Let $(\bar{M}, \bar{\nabla}, \bar{g}, J)$ be a $2m$-dimensional holomorphic statistical manifold and M be a $(2m-1)$-dimensional submanifold of $\bar{M}$, that is, M is a hypersurface of $\bar{M}$. Let V be a unit normal vector field of M. Then Gauss and Weingarton formulas are

$$\bar{\nabla}_X Y = \nabla_X Y + h(X,Y), \bar{\nabla}'_X Y = \nabla'_X Y + h'(X,Y),$$

$$\bar{\nabla}_X V = -AX + \mu(X)V, \bar{\nabla}'_X V = -A'X + \mu'(X)V,$$

where, $\mu(X) = \bar{g}(\nabla^{\perp}_X V, V), \mu'(X) = \bar{g}\left(\nabla'^{\perp}_X V, V\right), \mu(X) = -\mu'(X)$. Also $g(A'X,Y) = \bar{g}(h(X,Y),V), g(AX,Y) = \bar{g}(h'(X,Y),V)$

The structure vector U is defined by $U = -JV \in \Gamma(TM)$. For $X \in \Gamma(TM)$, JX can be decomposed as

$$JX = PX = g(X,U)V.$$

Proposition 5.1 [10]: Let M be a real hypersurface of a holomorphic statistical manifold $\bar{M}$. For $X, Y \in \Gamma(TM)$, the following relations hold:

1. $\nabla_X U = PA'X + \mu'(X)U = PA'X - \mu(X)U$,
2. $(\nabla_X P)Y = \mu(Y)AX - g(AX,Y)U - P\nabla_X Y + P\nabla'_X Y$.

Lemma 5.2 [7]: Let (M, ∇, g) be a statistical hypersurface in $\bar{M}$. The following formulas hold for $X, Y \in \Gamma(TM)$:

1. $g(U,U) = 1, PU = 0, uoP = 0$, where $u(X) = g(U,X)$,
2. $P^2 X = -X + u(X)U, g(PX,PY) = g(X,Y) - u(X)u(Y)$,
3. $\mu(X) = u(\nabla'_X U)$,
4. $\nabla'_X Y = -P\nabla_X U + u(A'X)U$,
5. $A'X = -P\nabla_X U + u(A'X)U$.

The Gauss and Codazzi equations can be derived as follows:

$$\begin{aligned}\bar{R}(X,Y)Z = &\ R(X,Y)Z - \{g(A'Y,Z)AX - g(A'X,Z)AY\} + g\big((\nabla'_X A')\big)Y,Z)V \\ &-g((\nabla'_Y A')X,Z)V + g(A'Y,Z)\mu(X)V - g(A'X,Z)\mu(Y)V,\end{aligned}$$

$$\begin{aligned}\bar{R}'(X,Y)Z = &\ R'(X,Y)Z - \{g(AY,Z)A'X - g(AX,Z)A'Y\} + g\big((\nabla_X A)Y,Z\big)V \\ &-g\big((\nabla_Y A)X,Z\big)V + g(AY,Z)\mu'(X)V - g(AX,Z)\mu'(Y)V.\end{aligned}$$

Therefore

$$\begin{aligned}\bar{S}(X,Y)Z = &\ S(X,Y)Z + \tfrac{1}{2}\{g(A'X,Z)AY - g(A'Y,Z)AX + g(AX,Z)A'Y \\ &-g(AY,Z)A'X + g\big((\nabla'_X A' + \nabla_X A)Y,Z\big)V \\ &-g\big((\nabla'_X A' + \nabla_Y A)X,Z\big)\xi + g(A'Y,Z)\mu(X)V - g(A'X,Z)\mu(Y)V \\ &+g(AY,Z)\mu'(X)V - g(A'X,Z)\mu(Y)V + g(AY,Z)\mu'(X)V \\ &-g(AX,Z)\mu'(Y)V\},\end{aligned}$$

$$\begin{aligned}\bar{g}(\bar{S}(X,Y)Z,W) = &\ g(S(X,Y)Z,W) + \tfrac{1}{2}\{g(A'X,Z)g(AY,W) \\ &-g(A'Y,Z)g(AX,W) + g(AX,Z)g(A'Y,W) \\ &-g(AY,Z)g(A'X,W)\},\end{aligned}$$

(the Gauss equation)

$$\begin{aligned}\bar{g}(\bar{S}(X,Y)Z,V) = &\ \tfrac{1}{2}\{g((\nabla'_X A' + \nabla_X A)Y,Z) - g(\nabla'_Y A' + \nabla_Y A)X,Z) \\ &+g((A' - A)Y,Z)\mu(X) - g((A' - A)X,Z)\mu(Y)\},\end{aligned}$$

(the Codazzi equation)

where, $X, Y, Z, W \in \Gamma(TM)$ and $V \in \Gamma(TM^{\perp})$.

For a manifold $\bar{M}$ of constant holomorphic sectional curvature k, the Gauss and Codazzi equation, respectively reduces to

$$S(X,Y)Z = \tfrac{1}{2}\{g(A'Y,Z)AX - g(A'X,Z)AY + g(AY,Z)A'X - g(AX,Z)A'Y\}$$

$$+\frac{k}{4}\left\{\begin{matrix} g(Y,Z)X - g(X,Z)Y + g(PY,Z)PX \\ -g(PX,Z)PY + 2g(X,PY)PZ \end{matrix}\right\}$$

$$S(X,Y)Z = \frac{k}{2}\{g(X,U)PY - g(Y,U)PX + 2g(PX,PY)U\} = (\nabla'_X A' + \nabla_X A)Y$$
$$-(\nabla'_Y A' + \nabla_Y A)X + \mu(X)(A' - A)Y - \mu(Y)(A' - A)X$$

Theorem 5.3 [10]: Let M be a real-hypersurface of a holomorphic statistical manifold $\bar{M}$ of constant holomorphic sectional curvature k. If for the shape operators A, A' of M and functions α and β; $AX = \alpha X$ and $A'X = \beta X$ then $k = 0$.

Proposition 5.4 [10]: Let M be a real-hypersurface of a holomorphic statistical manifold $\bar{M}$ of constant holomorphic sectional curvature k. If for the shape operators A, A' of M and functions α and β; $AX = \alpha X$ and $A'X = \beta X$ then $\alpha + \beta = constant$ if and only if V is parallel with respect to the normal connections $\nabla^{\perp}(\nabla'^{\perp})$.

Theorem 5.5 [10]: Let M be an n-dimensional CR-submanifold of maximal CR-dimension in an $(n+p)$-dimensional holomorphic statistical manifold $\bar{M}$ of constant holomorphic sectional curvature k and let $p < n$. If for the shape operators A, A' of the distinguished normal vector field V and functions α and β; $AX = \alpha X$ and $A'X = \beta X$ then $k = 0$.

Theorem 5.6 [7]: Let $\bar{M}(k)$ denote a holomorphic statistical manifold $(\bar{M}, \bar{\nabla}, \bar{g}, J)$ of constant holomorphic sectional curvature k and (M, ∇, g) be a statistical hypersurface in $\bar{M}(k)$. If M is totally umbilical with respect to $\bar{\nabla}$ and $\bar{\nabla}'$, then $k = 0$ and $X(\lambda + \lambda') = (\lambda - \lambda')\mu(X), S(X,Y)Z = \lambda\lambda'\{g(Y,Z)X - g(X,Z)Y\}$, for any $X, Y, Z \in \Gamma(TM)$ where λ and λ' denote the eigenvalues of A and A', respectively. Moreover, if $\lambda = \lambda'$, then λ is constant and M is of constant sectional curvature λ^2.

6. SUBMANIFOLDS OF SASAKIAN STATISTICAL MANIFOLDS

In this section, we give the notion of statistical structure on Sasakian manifolds. A typical example of Sasakian manifolds is an odd-dimensional sphere. Furuhata et al [8] discuss standard Sasakian statistical structure on odd-dimensional sphere which is also an example of compact statistical manifold.

Definition 6.1 [8]: A triple $(\bar{g}, \phi, \xi)$ is called an almost contact metric structure on $\bar{M}$, if the following equations hold for any $X, Y \in \Gamma(T\bar{M})$:

$$\phi\xi = 0, \bar{g}(\xi, \xi) = 1,$$

$$\phi^2 X = -X + \bar{g}(X, \xi)\xi,$$

$$\bar{g}(\phi X, Y) + \bar{g}(X, \phi Y) = 0,$$

where, $\phi \in \Gamma(T\bar{M}^{(1,1)})$ and $\xi \in \Gamma(T\bar{M})$.

Definition 6.2 [8]: An almost contact structure on $\bar{M}$ is called a Sasakian structure, if

$$\left(\bar{\nabla}^g_X \phi\right)Y = \bar{g}(Y, \xi)X - \bar{g}(Y, X)\xi,$$

holds for any $X, Y \in \Gamma(T\bar{M})$. We call a manifold equipped with a Sasakian structure a Sasakian manifold.

Let, $\eta \in \Gamma(T\bar{M}^*)$ and $\omega \in \Gamma(TM^{(0,2)})$ as

$$\eta(X) = \bar{g}(X,\xi), \omega(X,Y) = \bar{g}(X,\phi Y),$$

For any $X, Y \in \Gamma(T\bar{M})$, we have the following formulas easily:

$$\begin{aligned} \eta o \phi &= 0, \\ \bar{g}(\phi X, \phi Y) &= \bar{g}(X,Y) - \eta(X)\eta(Y), \\ (\bar{\nabla}^g_X \omega)(Y,Z) &= \bar{g}\big(Y, (\bar{\nabla}^g_X \phi)Z\big) = \bar{g}(Y,X)\bar{g}(Z,\xi) - \bar{g}(Z,X)\bar{g}(Y,\xi) \\ &= \eta\{\bar{g}(X,Y)Z - \bar{g}(X,Z)Y\}. \end{aligned}$$

Lemma 6.3 [8]: Let $(\bar{\nabla}, \bar{g})$ be a statistical structure and $(\bar{g}, \phi, \xi)$ be an almost contact metric structure on $\bar{M}$. Then

$$(\bar{\nabla}_X \omega)(Y,Z) = \bar{g}(Y, \bar{\nabla}'_X(\phi Z) - \phi(\bar{\nabla}_X Z)),$$

$$\big(\bar{\bar{\nabla}}_X\, \omega\big)(Y,Z) - (\bar{\nabla}'_X \omega)(Y,Z) = -2\bar{g}\big(Y, K(X,\phi Z) + \phi K(X,Z)\big),$$

$$\bar{\nabla}_X(\phi Y) - \phi(\bar{\nabla}'_X Y) = \big(\bar{\nabla}^g_X \phi\big)Y + K(X,\phi Y) + \phi K(X,Y).$$

Definition 6.4 [6]: A quadruple $(\bar{\nabla}, \bar{g}, \phi, \xi)$ is called a Sasakian statistical structure on $\bar{M}$, if :

1. $(\bar{g}, \phi, \xi)$ is a Sasakian structure and
2. $(\bar{\nabla}, \bar{g})$ is a statistical structure and
3. $K \in \Gamma(T\bar{M}^{(1,2)})$ for $(\bar{\nabla}, \bar{g})$ satisfies $K(X,\phi Y) + \phi K(X,Y) = 0$ for $X, Y \in \Gamma(T\bar{M})$.

Proposition 6.5 [8]: Let $(\bar{\nabla}, \bar{g}, \phi, \xi)$ be a Sasakian statistical structure on $\bar{M}$ and $\lambda \in C^\infty(\bar{M})$. Set $\bar{\nabla}^\lambda = \bar{\nabla}^g + \lambda K$. Then $(\bar{\nabla}^\lambda, \bar{g}, \phi, \xi)$ is a Sasakian statistical structure. In particular, so is $(\bar{\nabla}', \bar{g}, \phi, \xi)$.

Proposition 6.6 [6]: For a Sasakian statistical manifold $(\bar{M}, \bar{\nabla}, \bar{g}, \phi, \xi)$, we have $S(X,Y)\xi = \bar{g}(Y,\xi)X - \bar{g}(X,\xi)Y$ holds for $X, Y \in \Gamma(T\bar{M})$.

Definition 6.7 [8]: Let $(\bar{M}, \bar{\nabla}, \bar{g}, \phi, \xi)$ be a Sasakian statistical manifold and $k \in \mathcal{R}$. The Sasakian statistical structure is said to be of constant ϕ-sectional curvature k if

$$\begin{aligned} S(X,Y)Z = \tfrac{k+3}{4}\{\bar{g}(Y,Z)X - \bar{g}(X,Z)Y\} + \tfrac{k-1}{4}\{&\bar{g}(\phi Y,Z)\phi X - \bar{g}(\phi X,Y)\phi Y \\ &-2\bar{g}(\phi X,Y)\phi Z - \bar{g}(Y,\xi)\bar{g}(Z,\xi)X + \bar{g}(X,\xi)\bar{g}(Z,\xi)Y \\ &+\bar{g}(Y,\xi)\bar{g}(Z,X)\xi - \bar{g}(X,\xi)\bar{g}(Z,Y)\xi\}, \end{aligned}$$

holds for $X, Y, Z \in \Gamma(T\bar{M})$.

A Sasakian statistical manifold is of constant ϕ-sectional curvature if and only if the sectional curvature has the same value for each ϕ-section and for each point. At a point $x \in \bar{M}$, by definition, a ϕ-section means a plane spanned by X and ϕX in $T_x\bar{M}$, where X is a unit tangent vector orthogonal to ξ_x.

Proposition 6.8 [8]: Let $(\bar{\nabla}, \bar{g})$ be a statistical structure and $(\bar{g}, \phi, \xi)$ a Sasakian structure on $\bar{M}$. Then $(\bar{\nabla}, \bar{g}, \phi, \xi)$ is a Sasakian statistical structure if and only if two of $\bar{\nabla}\omega, \bar{\nabla}'\omega, \bar{\nabla}^g\omega$ coincide with each other.

Theorem 6.9 [8]: Let $(\bar{\nabla}, \bar{g})$ be a statistical structure and $(\bar{g}, \phi, \xi)$ an almost contact metric structure on $\bar{M}$. Then $(\bar{\nabla}, \bar{g}, \phi, \xi)$ is a Sasakian statistical structure if and only if the following formulas hold:

$$\bar{\nabla}_X(\phi Y) - \phi(\bar{\nabla}'_X Y) = \bar{g}(Y,\xi)X - \bar{g}(Y,X)\xi,$$

$$\bar{\nabla}_X\xi = \phi X + \bar{g}(\bar{\nabla}_X\xi,\xi)\xi.$$

The following relations hold for a Sasakian statistical manifold:

$$K(X,\xi) = \lambda\bar{g}(X,\xi)\xi, \bar{g}(K(X,Y),\xi) = \lambda\bar{g}(X,\xi)\bar{g}(Y,\xi),$$

where, $\lambda = \bar{g}(K(\xi,\xi),\xi)$.

Proposition 6.10 [8]: $(\bar{M}, \bar{g}, \phi, \xi)$ be a Sasakian manifold. Set $\bar{\nabla} = \bar{\nabla}^g + fK$ for $f \in C^\infty(\bar{M})$. Then $(\bar{\nabla}, \bar{g}, \phi, \xi)$ is a Sasakian statistical structure on $\bar{M}$. Conversely, we define $\bar{\nabla}_X Y = \bar{\nabla}^g_X Y + L(X,Y)V$ for some unit vector field V and $L \in \Gamma(T\bar{M}^{(0,2)})$. If $(\bar{\nabla}, \bar{g}, \phi, \xi)$ is a Sasakian statistical structure then $L \otimes V$ is written as $L(X,Y)V = f\bar{g}(X,\xi)\bar{g}(Y,\xi)$, for some $f \in C^\infty(\bar{M})$.

Lemma 6.11 [8]: Let $(\bar{M}, \bar{\nabla}, \bar{g}, J)$ be a holomorphic statistical manifold and M be a hypersurface with a unit normal vector field V. Let (g, ϕ, ξ) be the induced almost contact metric structure on M defined as:

$$\phi = -JV, JX = \phi X + \eta(X)V,$$

for, $X, \xi \in \Gamma(TM), \eta \in \Gamma(T^*)$.

We have the following formulas and their duals hold for $X, Y \in \Gamma(TM)$:

$$-AX = \phi\nabla'_X\xi - g(AX,\xi)\xi,$$

$$\mu(X) = g(\nabla'_X\xi,\xi),$$

$$\nabla_X(\phi Y) - \phi(\nabla'_X Y) = g(Y,\xi)AX - g(Y,AX)\xi,$$

$$\nabla_X\xi = \phi A'X - \mu(X)\xi = \phi A'X + g(\nabla_X\xi,\xi)X.$$

Lemma 6.12 [8]: Suppose that a hypersurface M in a holomorphic statistical manifold $(\bar{M}, \bar{\nabla}, \bar{g}, J)$ satisfies $AX = X + g(AX - X,\xi)\xi, A'X = X + g(A'X - X,\xi)\xi$ for any $X \in \Gamma(TM)$. Then $(M, \nabla, g, \phi, \xi)$ is a Sasakian statistical manifold and $\mu(X) = \mu(\xi)\eta(X)$ holds for any $X \in \Gamma(TM)$.

Theorem 6.13 [8]: Let M with a contact metric structure (g, ϕ, ξ) be a hypersurface of a holomorphic statistical manifold $(\bar{M}, \bar{\nabla}, \bar{g}, J)$. Then the quadruple (∇, g, ϕ, ξ) is a Sasakian statistical structure on M if and only if the shape operators satisfy $AX = X + g(AX - X, \xi)\xi, A'X = X + g(A'X - X, \xi)\xi$.

Moreover, if the ambient space is of constant holomorphic sectional curvature k, then the Sasakian statistical structure (∇, g, ϕ, ξ) is of constant ϕ-sectional curvature $k + 1$.

Definition 6.14: A submanifold M of a Sasakian manifold $(\bar{M}, \bar{g}, \bar{\phi}, \bar{\xi})$ is said to be an invariant submanifold if $\bar{\xi}_x \in T_xM, \bar{\phi}X \in T_xM$ for any $X \in T_xM, x \in M$. Let $g \in \Gamma(TM^{(0,2)}), \phi \in \Gamma(TM^{(1,2)})$ and $\xi \in \Gamma(TM)$ be the restrictions of $\bar{g}, \bar{\phi}$ and $\bar{\xi}$, respectively. Then (g, ϕ, ξ) is a Sasakian structure on M.

In [6], Furuhata et al also discussed the notion of invariant submanifolds of a Sasaian statistical manifolds and gave the following results:

Theorem 6.15 [6]: Let $(\bar{M}, \bar{\nabla}, \bar{g}, \bar{\phi}, \bar{\xi})$ be a Sasakian statistical manifold and M be an invariant submanifold of $\bar{M}$ with g, ϕ, ξ as defined earlier. Then the following hold:

1. A quadruple $(M, \nabla, g, \phi, \xi)$ is a Sasakian statistical manifold.
2. $h(X, \xi) = h'(X, \xi)$ for any $X \in \Gamma(TM)$.
3. $h(X, \phi Y) = h(\phi X, Y) = \bar{\phi}h'(X, Y)$ for any $X, Y \in \Gamma(TM)$. In particular, $tr_g h = tr_g h' = 0$.
4. If h is parallel with respect to the Van der Weaden-Bertolotti connection $\bar{\nabla}'$ for $\bar{\nabla}$, then h and h' vanish, Namely, if $(\bar{\nabla}'_X h)(Y, Z) = \nabla^{\perp}_X h(Y, Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z) = 0$ for $Z \in \Gamma(TM)$ then $h'(X, Y) = 0$.
5. $\bar{g}(\bar{S}(X, \bar{\phi}X)\bar{\phi}X - S(X, \phi X)\phi X, X) = 2\bar{g}(h'(X, X), h(X, X))$ for $X \in \Gamma(TM)$.

Also the induced Sasakian statistical structure on M has constant ϕ-sectional curvature k if and only if $\bar{g}(h'(X, X), h(X, X)) = 0$, for any $X \in \Gamma(TM)$ orthogonal to ξ since $g(S(X, \phi X)\phi X, X) = k(g(X, X))^2$ for any $X \in \Gamma(TM)$ such that $g(X, \xi) = 0$.

Definition 6.16 [14]: If $\pi: (M, \nabla, g) \to (B, \bar{\nabla}, g_B)$ is a semi-Riemannian statistical submersion such that $(M, \nabla, g, \phi, \xi, \eta)$ is an almost contact metric manifold of certain kind then π is said to be an almost contact metric submersion of certain kind. We say that a statistical submersion $\pi: (M, \nabla, g) \to (B, \bar{\nabla}, g_B)$ is a Sasaki-like statistical submersion if $(M, \nabla, g, \phi, \xi, \eta)$ is a Sasaki-like statistical manifold, each fibre is a ϕ-invariant semi-Riemannian submanifold of M and tangent to the vector ξ.

Takano [14] discusses the notion of Sasaki-like statistical submersions and gave the following results:

Theorem 6.17 [14]: Let $\pi: \bar{M} \to B$ be an almost contact metric submersion of certain kind. Then the base space is an almost Hermite-like manifold and each fibre is an almost contact metric manifold of certain kind.

Theorem 6.18 [14]: If $\pi: (M, \nabla, g) \to (B, \overline{\nabla}, g_B)$ is a Sasaki-like statistical submersion, then the base space $(B, \overline{\nabla}, g_B, \bar{\phi})$ is a Kahler-like statistical manifold and each fibre $(\overline{M}, \overline{\nabla}, \bar{g}, \bar{\phi}, \xi, \eta)$ is a Sasaki-like statistical manifold.

Theorem 6.19 [14]: Let $\pi: (M, \nabla, g) \to (B, \overline{\nabla}, g_B)$ is a Sasaki-like statistical submersion. If $rank(\bar{\phi} + \bar{\phi}') = dim\overline{M} - 1$, then we have $A_X Y = -g(X, \phi Y)\xi$ for $X, Y \in \Gamma(TM)$. If $\bar{\phi} = \bar{\phi}'$, then we have $A_X Y = -g(X, \phi Y)\xi$, for $X, Y \in \Gamma(TM)$.

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