International Journal of Mathematical Sciences
Vol. 17, Nos. 1-2, January-June 2018, pp. 49-59

OPTIMAL RECOUPMENT POLICY FOR A CUMULATIVE DAMAGE MODEL FOR A HUMAN SYSTEM IN TERMS OF ALDOSTERONE WITH TIME DETERIORATION

S. Lakshmi & R. Balakumar

Abstract: The replacement problem for shock models under additive damage has been studied by many authors. They have considered the replacement problem where a system is replaced at damage, that have been derived the optimal policy for a cumulative damage model. The authors have considered the replacement problem where a system suffers two kinds of damage and is replaced preventively at age $\boldsymbol{T}$ or at the N^{th} shock. According to this formula a human system has been considered two types of damages one with cumulative process and another with at that time of stress effect. Hence a part of the system is recouped by a new concept the expected cost per unit time and optimal values in terms of aldosterone have also been obtained.

AMS Classification: 60GXX.

Keywords: Aldosterone, Optimal replacement policy, NHPP.

1. INTRODUCTION

The replacement problem for shock models under additive damage has been studied by many authors **[3, 4, 7, 11, 12].** They have considered the replacement problem where a system is replaced at damage and ref **[8]** has derived the optimal policy for a cumulative damage model with restoration, using a control limit policy. In these papers for damage models, it is assumed that a system suffers damage only when shocks occur. However, most systems deteriorate continuously with time, e.g., deterioration resulting from wear, fatigue, crack, corrosion, and erosion. Recently, the authors **[9]** have considered the replacement problem where a system suffers two kinds of damage and is replaced preventively at age $\boldsymbol{T}$ or at the N^{th} shock.

It would be impossible to keep track of the age or the total operating time and the number of stress effects received up to a given time. In this case, it is better to adopt damage level as an indicator for replacement. That is, we check the damage immediately after a stress effect occurs and replace the system if the total damage exceeds a threshold level k.

The model formulation is as follows. Stress effects or shocks occur according to a nonhomogeneous Poisson process (NHPP). A system suffers two kinds of damage, which

are accumulated with time and by each stress effect, and fails when the total damage exceeds a failure level K. To prevent this, a part of the system is recouped by a new concept when the total damage exceeds a threshold level $k\,(0 \le k \le K)$, and this is done only at instants where stress effects occur. The expected cost per unit time in terms of aldosterone is obtained, using the theory of cumulative processes. Optimal k^* is the value of k which minimizes the total expected cost per unit time in terms of aldosterone, when stress effects occur in a Poisson process.

2. ASSUMPTION OF THE MODEL AND EXPECTED COST

Suppose that stress effects occur according to a nonhomogeneous Poisson process with intensity function $\lambda(t)$ and mean value function $R(t)$, i.e., $R(t) = \int_0^t \lambda(u)\,du$. Let $X_i (i = 1, 2, ...)$ be the i^{th} arrival time between successive stress effects and let $S_j = \sum_{i=1}^{j} X_i$ where $S_0 = 0$. The probability that at least j stress effects occur during $(0, t]$ is given by

$$H_j(t) = P\{S_j \le t\} = \sum_{i=j}^{\infty} \frac{[R(t)]^i}{i!} e^{-R(t)}, \qquad j = 0, 1, 2, ... \tag{1}$$

Each stress effect causes a random amount of damage to a system, and the damage is additive. Random variables Y_i, $(i = 1, 2, 3, ...)$ denote the amount of damage produced by the i^{th} stress effect, and are nonnegative, independent, and identically distributed. Each Y_i has a general distribution $G(x)$, i.e.,$G(x) = P\{Y_i \le x\}$ $(i = 1, 2, ...)$. The total amount of damage is $Z_j = \sum_{i=1}^{j} Y_i$ after j stress effects, with $Z_0 = 0$ and with the distribution function

$$P\{Z_j \le x\} = G^j(x), \qquad j = 0, 1, 2, ... \tag{2}$$

Where $G^{(j)}(x)$ is the j-fold Stieltjes convolution of $G(x)$ with itself, and $G^{(0)}(x) = 0$ for $x \ge 0$ and 0 for $x < 0$.

The damage would be increasing with time at a constant rate 'a' independent of stress effects. Hence, the total damage at the j^{th} stress effect is given by $aS_j + Z_j$. A system fails if the total damage exceeds a failure level K.

It is difficult to monitor the state of a system continuously. In our model, we check the total damage after a stress effect occurs. A preventive recoupment is done before failure if the total damage exceeds a threshold level $k\,(0 \le k \le K)$ at checking. Similarly, a failure recoupment is done only when the total damage at the end of stress effect is greater than K.

The probability that a system is recouped at failure, i.e., the total damage has exceeded a failure level K, after a stress effect occurs, is

$$\sum P\{Z_j + aS_j \le k, Z_{j+1} + aS_{j+1} > K\}$$

$$= \sum_{j=0}^{\infty} \int_0^{k/a} \int_0^{k-at} P\{Y_{j+1} + aX_{j+1} > K - at - z \backslash S_j = t\}\, dG^{(j)}(z)\, dH_j(t)$$

$$= \sum_{j=0}^{\infty} \int_0^{k/a} \int_0^{k-at} \left\{1 - \int_0^{(k-z-at)/a} G[K - z - a(t+y)]\, dH_{j+1}(y,t)\right\} dG^{(j)}(z)\, dH_j(t) \quad (3)$$

where $H_{j+1}(y, t) = P\{Z_{j+1} \le y/S_j = t\}$ and we put that $x/a = \infty$ for any $x > 0$ when $a = 0$. Further, the probability that a part of the system is recouped before failure, i.e., the total damage has exceed a threshold level k and is less than K, after a stress effect occurs, is

$$\sum_{j=0}^{\infty} P\{Z_j + aS_j \le k, \quad k < Z_{j+1} + aS_{j+1} \le K\}$$

$$= \int_0^{k/a} \int_0^{k-at} P\{k - z - at < Y_{j+1} + aX_{j+1} \le K - z - a(t+y) \backslash S_j = t\}\, dG^{(j)}(z)\, dH_{j+1}(y,t)$$

$$= \sum_{j=0}^{\infty} \int_0^{k/a} \int_0^{k-at} \left\{ \begin{array}{l} \displaystyle\int_0^{(k-z-at)/a} G[K - z - a(t+y)]\, dH_{j+1}(y,t) \\ \displaystyle -\int_0^{(k-z-at)/a} G[K - z - a(t+y)]\, dH_{j+1}(y,t) \end{array} \right\} dG^{(j)}(z)\, dH_j(t) \quad (4)$$

Thus, the mean time to recoupment is

$$\sum_{j=0}^{\infty} \int_0^{k/a} \int_0^{k-at} \int_0^{\infty} (t+y)\, dH_{j+1}(y,t)\, dG^{(j)}(z)\, dH_j(t)$$

$$-\sum_{j=0}^{\infty} \int_0^{k/a} \int_0^{k-at} \int_0^{(k-z-at)/a} (t+y)\, G(k - z - a(t+y))\, dH_{j+1}(y,t)\, dG^{(j)}(z)\, dH_j(t)$$

$$= \sum_{j=0}^{\infty} \int_0^{k/a} \int_0^{\infty} y\, dH_{j+1}(y,t)\, G^{(j)}(k - at)\, dH_j(t) \quad (5)$$

Let c_1 be the preventive recoupment cost of a system and c_2 ($> c_1$) be the cost for a failure recoupment. Then, the expected cost rate in terms of aldosterone is, from (3)-(5),

$$C(k) = \frac{1}{D(k)} \left\{ c_1 + (c_2 - c_1) \sum_{j=0}^{\infty} \left[\begin{array}{l} \displaystyle\int_0^{k/a} G^{(j)}(k - at)\, dH_j(t) \\ \displaystyle -\int_0^{k/a} \int_0^{(K-z-at)/a} G[K - z - a(t+y)]\, dH_{j+1}(y,t)\, dG^{(j)}(z)\, dH_j(t) \end{array} \right] \right\} \quad (6)$$

where

$$D(k) = \sum_{j=0}^{\infty} \int_{0}^{k/a} \int_{0}^{\infty} y\, dH_{j+1}(y,t)\, G^{(j)}(k-at)\, dH_j(t)$$

This equation agrees with equation (5) when $a = 0$. In particular, if a part of the system is always recouped at first stress effect, i.e., $k = 0$, the expected cost rate is

$$C(0) = \frac{c_1 + (c_2 - c_1)\left\{1 - \int_{0}^{K/a} G(K-ay)\, dH_1(y,0)\right\}}{\int_{0}^{\infty} y\, dH_1(y,0)} \tag{7}$$

When a system is recouped only at failure, i.e., the expected cost rate is

$$C(K) = \frac{c_2}{\sum_{j=0}^{\infty} \int_{0}^{k/a} \int_{0}^{\infty} y\, dH_{j+1}(y,t)\, G^{(j)}(K-at)\, dH_j(t)} \tag{8}$$

3. OPTIMAL POLICY

Suppose that $G(x)$ has a density $g(x)$ and stress effects occur in a Poisson process with rate λ

i.e., $\dfrac{dG(x)}{dx} = g(x)$ and $\lambda(t) = \lambda$. Then, we have

$$H_j(t) = \sum_{i=j}^{\infty} \frac{(\lambda t)^i}{i!} e^{-\lambda t}, \qquad j = 0, 1, 2, ..., \tag{9}$$

$$H_{j+1}(y,t) = H_1(y) = 1 - e^{-\lambda y}, \qquad j = 0, 1, 2, ..., \tag{10}$$

We seek an optimal level k^* which minimizes $C(k)$ in (6). Differentiating $C(k)$ with respect to k and setting it equal to zero, we have

$$\sum_{j=0}^{\infty} \int_{0}^{k/a} \int_{0}^{k-at} \int_{0}^{(kK-z-at)/a} G[K - z - a(t+y)]\, dH_1(y)\, dG^{(j)}(z)\, dH_j(t)$$
$$- \int_{0}^{(K-k)/a} G(K-k-ay)\, dH_1(y) \sum_{j=0}^{\infty} \int_{0}^{k/a} G^{(j)}(k-at)\, dH_j(t) = \frac{c_1}{c_2 - c_1} \tag{11}$$

Denote the left hand side of (11) by $L(k/a)$. It is evident that $L(0/a) = 0$,

$$L(K/a) = \sum_{j=1}^{\infty} \int_0^{K/a} G^{(j)}(K - at)\, dH_j(t), \tag{12}$$

$$\frac{\partial L(k/a)}{\partial k} = \int_0^{(K-k)/a} g(K - k - ay)\, dH_1(y) \sum_{j=0}^{\infty} \int_0^{k/a} G^{(j)}(k - at)\, dH_j(t) > 0. \tag{13}$$

Thus, $L(k/a)$ is strictly increasing from 0 to $L(k/a)$. Therefore, we have the following optimal recoupment policy.

(a) If, $L(k/a) > \frac{c_1}{c_2 - c_1}$ then there exists a finite and unique k^* which satisfies (11), and the resulting cost rate in terms of aldosterone is

$$C(k^*) = \lambda(c_2 - c_1)\left[1 - \int_0^{(K-k^*)/a} G(K - k^* - ay)\, dH_1(y)\right] \tag{14}$$

(b) If, $L(k/a) \le \frac{c_1}{c_2 - c_1}$ then $k^* = K$, i.e., no scheduled recoupment should be made, and the expected cost rate in terms of aldosterone is (8)

In the particular case $a = 0$

$$L(k/0) = \int_{K-k}^{K} [1 + M(K - x)]\, dG(x) \tag{15}$$

Which is equal to equation (10), where $M(x) = \sum_{i=1}^{\infty} G^{(i)}(x)$.

A function $L(K/a)$ is an important factor which decides whether to recoup a system before failure or at failure. We evidently have $\frac{\partial L(K/a)}{\partial a} < 0$, hence, $L(K/a)$ is strictly decreasing from $M(K)$ to 0. Thus, there exists a finite and unique $\overline{a}$ which satisfies $L(K/\overline{a}) > \frac{c_1}{c_2 - c_1}$ for K such that $M(K) > \frac{c_1}{c_2 - c_1}$. In other words, if a coefficient 'a' is greater than $\overline{a}$, then no preventive recoupment should be made.

4. APPLICATION

We consider the organ system subject to stress effects (shock threats), each stress effect weaken the system and made it more expensive to run and considered periodic recoupment of the system have conditions for the existence of an optimal finite period of stress effect was a nonhomogeneous Poisson process and cost structure did not depend on time but its particularly possible only if the cost structure is time dependent in the case when aldosterone is taken as stress effect. **[6]**

Theorem 1[7]: The optimal value of the periodic recoupment time always exists and is equal to the unique solution of the integral equations $\int_0^t [M(T) - m(t)]\, dt = \frac{c_0}{c}$.

To make the model more realistic the recoupment time are included here and is fitted for real life situations. Mean sample values of aldosterone secretions in successive times form an exponential distribution.

Therefore, in this study, we test whether ouabain stimulates aldosterone production with a time course consistent with early membrane depolarization as suggested by the previously reported early increase in cytosolic calcium. To study the time course of aldosterone production, we developed a perfusion technique that allows an examination of the initial effects of ouabain on aldosterone production. The results show that ouabain rapidly stimulates aldosterone production. Continuous perfusion with 0.25 or 1 mmol/*L* ouabain induced a brisk, robust increase in aldosterone production, followed by a decrease to near baseline over 60 minutes. Ouabain-stimulated aldosterone production was dependent on the presence of extracellular calcium and calcium influx through voltage-gated calcium channels. Our results support the hypothesis that the inhibition of *Na*, *K*-ATPase in rat adrenal glomerulosa cells immediately depolarizes the membrane potential and opens voltage-gated calcium channels. **[1, 2, 10]**

Initial studies were performed to determine the concentration response to ouabain in statically incubated cells to verify that the rat glomerulosa cells in our studies produced similar responses to those previously reported. Ouabain stimulated the production of aldosterone in a concentration-dependent manner up to 100 μ mol/*L* (Fig. 1).

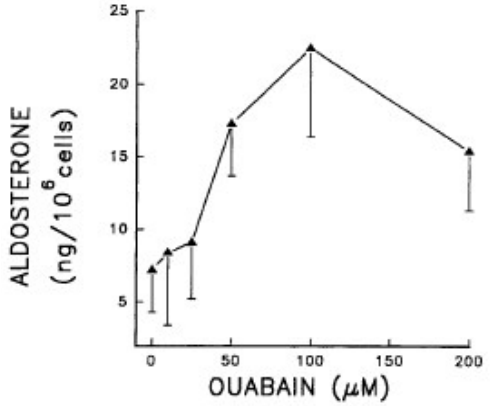

Figure 1: Effect of Ouabain on Aldosterone Production by Statically Incubated Rat Adrenal Glomorulosa Cells, Aldosterone Production was Quantified over a 60-minute Incubation with Quabain (*n* = 3 Experiments). Results are the Mean Error bars are the SE

Ouabain increased aldosterone production by perfused glomerulosa cells. The rapidity of the response was dependent on the perfusion concentration of ouabain. Continuous perfusion with ouabain concentrations of 250 μ mol/*L* or greater induced an initial rapid increase in aldosterone production followed by a decline to near baseline. In contrast, perfusion with 100 μ mol/*L* ouabain over 60 minutes led to a gradual increase in aldosterone production to a plateau during the period of observation (Fig. 2).

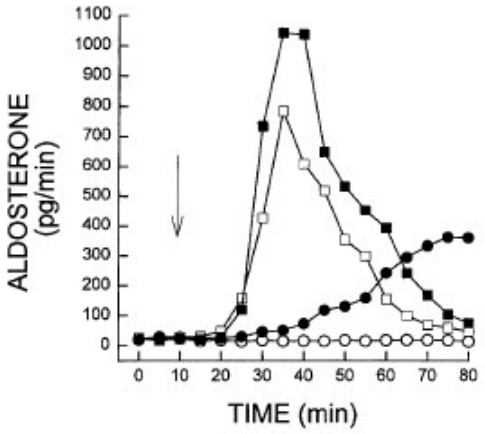

Figure 2: Effect of Ouabain on Aldosterone Production by Perfused Rat Glomerulosa Cells. A Representative Study is Shown from 3 Independent Experiments in which Ouabain was Continuously Perfused beginning at 10 minutes (Arrow). Ouabain Concentrations were 0 (○), 100 μ mol/*L* (●), 250 μ mol/*L* (■) and 1 m mol/*L* (□).

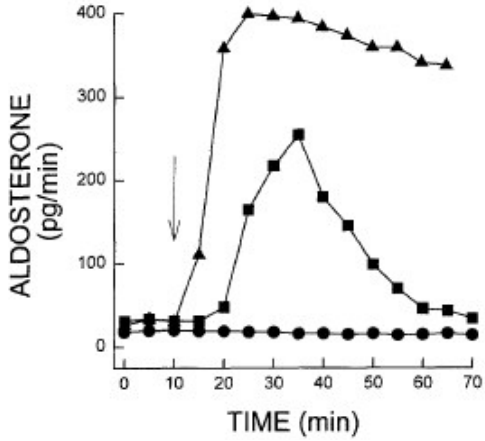

Figure 3: Comparison of Aldosterone Production in Perfused Glomerulosa Cells in Response to Ouabain and Potassium. A Representative Study is Shown from 2 Independent Experiments in which Cells were Continuously Perfused with 1 m mol/*L*. Ouabain (■), 10 m mol/*L* KCI (▲), or Buffer (●) beginning at the Arrow.

We compared the aldosterone response to ouabain with that of potassium, because potassium is known to depolarize the cell membrane. We used a high concentration of ouabain in these studies to inhibit *Na*, *K*-ATPase as rapidly as possible. The aldosterone response to 1 mmol/*L* ouabain by perfused cells was brisk, albeit slightly delayed in comparison to the response to l0 mmol/*L* potassium, probably due to the time required for ouabain to bind to *Na*, *K*-ATPase. Importantly, the response to ouabain declined over time, which may reflect the collapse of ion gradients (Fig. 3).

If ouabain-stimulated aldosterone production is due to membrane depolarization, the production should be dependent on the presence of extracellular calcium and be inhibitable by a calcmm-channel blocker. These are requisite characteristics of other stimuli of aldosterone release that depolarize the membrane, namely potassium and angiotensin II. To determine if the response to ouabain is calcium-dependent, ceils were incubated with 100 μ mol/*L* ouabain and with calcium concentrations of 0 to 4 μ mol/*L*. With no added calcium in the buffer, ouabain failed to stimulate aldosterone production. Increasing concentrations of calcium led to greater aldosterone production in response to ouabain (Fig. 4). In addition, calcium influx through voltage-gated calcium channels was required for ouabain-stimulated aldosterone production.

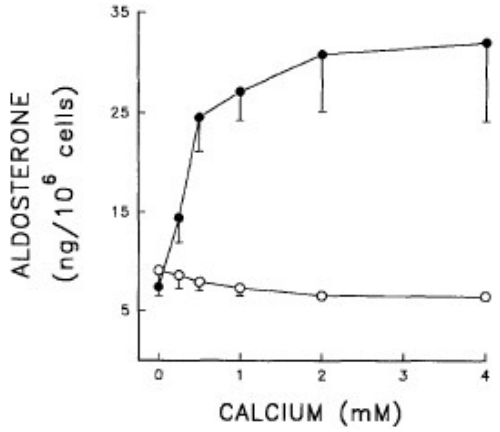

Figure 4: Effect of Calcium on Ouabain Stimulated Aldosterone Production by Statically Incubated Glomerulosa Cells. Cells were Incubated with (●) and without (○) 100 μ mol/*L* Ouabain at Calcium Concentrations of 0 to 4 m mol/*L* (*n* = 3 Experiments). Results are the Mean; Error Bars are the SE and may be located within the Symbols

Suppose that the amount of damage due to each stress effect has an exponential distribution, i.e. $G(x) = 1 - e^{-\mu x}$, Then, equations (11) and (14) are written, respectively, in the following. For

$$\frac{\lambda}{\lambda-\mu a}e^{-\mu(K-k)}\sum_{j=0}^{\infty}\int_{0}^{k/a}G^{(j+1)}(k-at)\,dH_j(t)$$
$$-\frac{\mu a}{\lambda-\mu a}e^{-(\lambda/a)(K-k)}\sum_{j=0}^{\infty}\int_{0}^{k}H_{j+1}\left(\frac{k-x}{a}\right)dG^{(j)}(x)=\frac{c_1}{c_2-c_1} \tag{16}$$

And the expected cost rate in terms of aldostreone is

$$C(k^*)=\lambda(c_2-c_1)\left[\frac{\lambda}{\lambda-\mu a}e^{-\mu(K-k^*)}-\frac{\mu a}{\lambda-\mu a}e^{-(\lambda/a)(K-k^*)}\right] \tag{17}$$

For $\lambda=\mu a$,

$$e^{-\mu(K-k)}\sum_{j=0}^{\infty}\left\{\mu(K-k)\int_{0}^{k/a}G^{(j+1)}(k-at)dH_j(t)+\int_{0}^{k/a}G^{(j+2)}(k-at)dH_j(t)\right\}=\frac{c_1}{c_2-c_1} \tag{18}$$

and

$$C(k^*)=\mu a(c_2-c_1)[1+\mu(K-k^*)]\,e^{-\mu(K-k^*)} \tag{19}$$

Where $H_j(t)$ is given in (9) and

$$G^{(j)}(x)=\sum_{i=j}^{\infty}\frac{(\mu x)^i}{i!}e^{-\mu x},\qquad j=0,1,2,3,...,$$

In the particular case of $a = 0$, equations (16) and (17) are, respectively,

$$\mu k e^{-\mu(K-k)}=\frac{c_1}{c_2-c_1} \tag{20}$$

$$C(k^*)=\lambda(c_2-c_1)e^{-\mu(K-k^*)}. \tag{21}$$

5. RESULTS

In Fig. (5) & (6), the optimal threshold level k^* gives the corresponding expected cost rate in terms of aldosterone, i.e., if the damage exceeds k^* of a failure level, a part of a unit should be recouped before failure.

In Fig. (5) *L*1, *L*2, *L*3, *L*4 reflects the expected cost rate in terms of aldosterone by the corresponding k^* values, which denotes the effect of ouabain on aldosterone production by perfused rat glomerulosa cells. Here *L*1, *L*2, *L*3 and *L*4 are obtained from the ouabain concentrations that were 0, 100 μ mol/*L*, 250 μ mol/*L* and 1 mmol/*L* respectively.

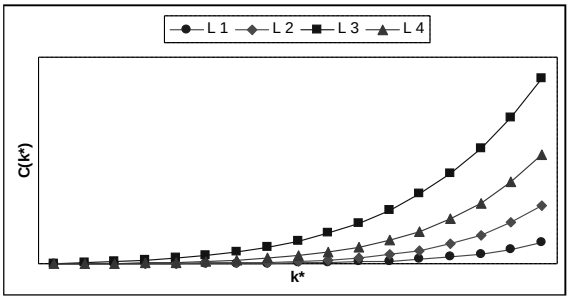

Figure 5

In Fig. (6) *L*5, *L*6, *L*7 reflects the expected cost rate in terms of aldosterone by the corresponding k^* values, which denote the mean effect of ouabain and potassium on aldosterone production by perfused rat glomerulosa cells. In which cells were continuously perfused with 1 m mol/*L* ouabain, 10 m mol/*L* potassium and beginning of the concentrations.

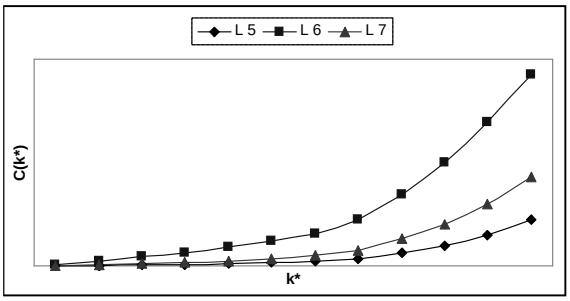

Figure 6

6. CONCLUSION

In this model we have assumed that stress effects occur according to a non homogeneous Poisson process. The human system suffers two kinds of damage in order to secretion of aldosterone which is accumulated with time and by each stress effect and fails when the total damage exceeds a failure level *k*. The expected cost per unit time in terms of aldosterone is obtained using the theory of cumulative process. Optimal k^* is obtained which minimizes the expected cost $C(k^*)$ in terms of aldosterone per unit time. The threshold optimal value k^* increases the corresponding $C(k^*)$ increases in terms of aldosterone, so the marginal value of aldosterone is obtained. In fig (5) & (6) show clearly the continuous perfusion with ouabain concentration of 250 μ mol/*L* or greater induced an initial rapid increase in aldosterone production followed by a decline to near baseline, perfusion with 100 μ mol/*L* ouabain over 60 minutes led to a gradual increase in aldosterone production during the period of observation.

REFERENCES

[1] Antonipillal I., Schick K., and Horton R., (1996), "Ouabain is a Potent Inhibitor of Aldosterone Secretion and Angiotensin Action in the Human Adrenal", *J. Ctin Endocrinol Metab*, **81**: 2335–2337.

[2] Beck M., Szalay K. S., and Nagy G. M., (1996), *et al.*, "Production of Ouabain by Rat Adrenocortical Cells", *Endocr Res.*, **22**: 845–849.

[3] Chikte S. D., and Deshmukh S. D., (1981), "Preventive Maintenance and Replacement Under Additive Damage", *Naval Res. Logist. Quart.,* **28**: 33-46.

[4] Feldman R. M., (1976), "Optimal Replacement with Semi-Markov Shock Models", *J. Applied Probability*, **13**: 108–117.

[5] Lakshmi S., (2004), "Stochastic Model for Stress Effects of Homeostatic Systems", *J. Indian Acad. Math.*, **26**(2): 411–417.

[6] Lakshmi S., and Geetha T., (2007), "An Application to Stochastic Clearing System", *J. Indian Acad. Math.*, **29**(1): 79–86.

[7] Nakagawa T., (1976), "On a Replacement Problem of a Cumulative Damage Model", *Opl. Res. Quart.,* **27**: 895–900.

[8] Posner M. J. M., and Zuckerman D., (1984), "A Replacement Model for an Additive Damage Model with Restoration", *Opns. Res. Letters*, **3**: 141–148.

[9] Stow T., and Nakagawa T., (1997), "Replacement Policies for a Shock Model with Two Kinds of Damage, Stochastic Modelling in Innovative Manufacturing", *Springer Lecture Notes in Economics and Mathematical Systems*, **445**: 185–195.

[10] Tamura M., Piston D. W., and Tani M., (1996), *et al.*, "Ouabain Increases Aldosterone Release from Bovine Adrenal Glomerulosa Cells: Role of Renin-Angiotensin System", *Am J. Physiol*, **270**: E27–E35.

[11] Taylor H. M., (1975), "Optimal Replacement Under Additive Damage and Other Failure Models", *Naval Res. Logist. Quart*., **22**: l–18.

[12] Zuckerman D., (1977), "Replacement Models Under Additive Damage", *Naval Res. Logist. Quart.,* **24**: 549–558.

S. Lakshmi
Associate Prof. of Mathematics,
K.N. Govt. Arts College for Women,
Thanjavur, Tamilnadu, India
E-mail: lakshmi291082@yahoo.co.in

R. Balakumar
Lecturer in Mathematics,
Ponnaiyah Ramajayam Institute of Science &
Technology (PRIST) University, Vallam,
Thanjavur, Tamilnadu, India.