**IJCTA,** 9(39), 2016, pp. 71-76


# Minimum Time Ascent Phase Trajectory Optimization using Steepest Descent Method

**Diksha Diva Singh* Vishnu G. Nair* and Dileep M. V.***

***Abstract :*** In this paper, ascent phase gravity turn trajectory of a launch vehicle is taken into account. Minimum-time problem of optimization is considered for specified structural and propulsive data. The objective is to achieve the target in minimum time and minimum error at terminal point by taking into account all the constraints. Trajectory generation is done for a single stage rocket in 2-D plane and computed for a given initial and final conditions. Hamiltonian is formulated by converting the given problem into secure final flight path angle problem and Pontryagin minimum principle is used to find the necessary conditions. The non-linear equations are solved using gradient method. Ultimately, the numerical results are evaluated and validity of result is exhibited.

***Keywords :*** Varying Inertia Weight; Trajectory Optimization; Minimum time.

## 1. INTRODUCTION

In this paper, ascent phase gravity turn trajectory of a launch vehicle is taken into account. The term trajectory is used to define the time history of the state variables of the system. The trajectory of a launch vehicle is generally optimized to meet the target by minimum fuel, minimum control or minimum time. The purpose of trajectory optimization in this paper is to ensure the terminal conditions are met accurately and the system takes minimum time to reach the injection point, while satisfying all the constraints. Reducing the total time taken for the entire mission can yield economic advantage and mission coherence.

The trajectory optimization of launch vehicles has been studied and analyzed using various approaches and constraints in the past. Reference [1] discusses the minimum control approach for the launch vehicle, in reference [2] ascent trajectory of multistage launch vehicle is considered. Many numerical procedure exist to solve the optimization problem. Gradient restoration algorithm [3] and shooting method [4] are the indirect methods whereas direct collocation [5] and differential inclusion [6] are the few examples of direct methods. Reference [7] describes the direct and indirect optimization and their relation. One of the standard procedure to solve the nonlinear equations is gradient or steepest-decent method. In reference [8] this method is applied to a launch vehicle carrying a hypersonic vehicle as payload. In reference [9] min-max technique is used for a satellite launch system to obtain an optimum pitch steering program while maximizing apogee velocity for a specified altitude and perigee of the satellite.

The launch vehicle trajectory dynamics basically consists of two segments [1], the first segment deals with the launcher clearance and the vertical raise while the second segment considers the gravity turn trajectory. This paper addresses the problem of gravity turn trajectory of a single stage launch vehicle with the specified initial and terminal characteristics. Well researched gradient method [10] is used to solve the optimization problem. The cost function in this paper consists of two terms. First, ensuring the accuracy of final conditions to be met and the later one for the time minimization part. The control variable and the weighting factor is carefully chosen by proper tuning [11]. By altering the weighting factor of the

* Manipal Institute of Technology, Manipal University Manipal Karnataka, India. *E-Mail: vishnu.nair@manipal.edu*

corresponding terminal condition we can choose how closely the terminal conditions can be met. In this paper the formulation of trajectory optimization problem is done on the basis of point mass equations of motion assuming the non-rotating spherical earth. The mass flow rate of fuel is assumed to be constant that is, the thrust is considered constant throughout the trajectory. Pontryagin minimum principle is used to find the necessary conditions of state, control and co-state variables.

The performance of the trajectory optimization is demonstrated by considering different initial and terminal conditions and the constraints throughout the trajectory.

## 2. PROBLEM FORMULATION

The mathematical modelling of the launch vehicle system is described in this section.

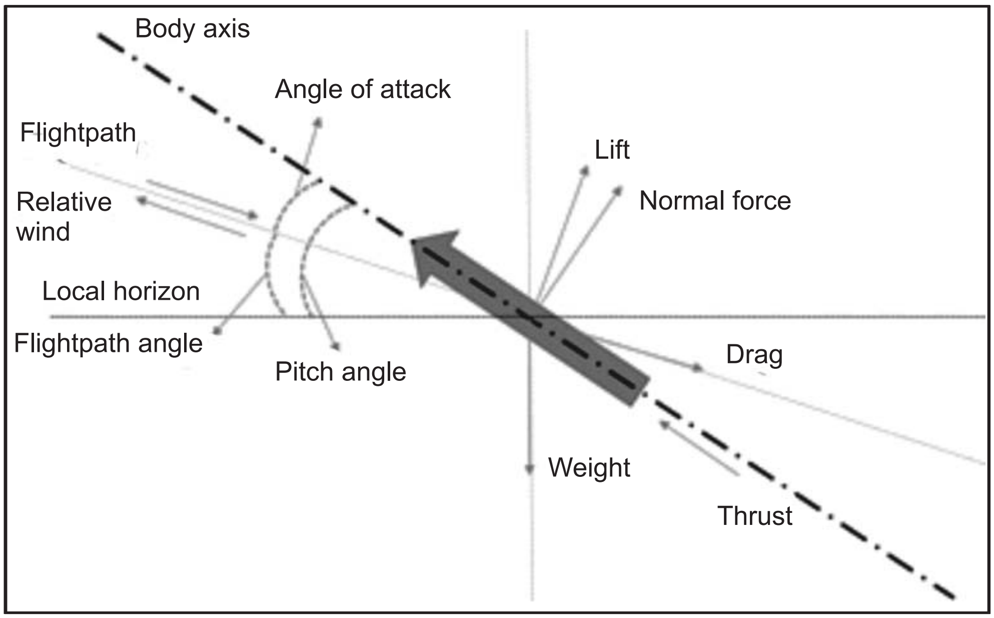


**Figure 1: Force components for an air/space vehicle**

### 2.1. Dynamics of the System

Point mass equation of motion with spherical earth is given by,

$$\dot{r} = V \sin \gamma \tag{1}$$

$$\dot{V} = \frac{1}{mV}(T \cos \alpha - D - mg \sin \gamma) \tag{2}$$

$$\dot{\gamma} = \frac{1}{mV}(T \cos \alpha + L) + \left(\frac{V}{r} - \frac{g}{V}\right)\cos \gamma \tag{3}$$

Where $m$ is mass of launch vehicle, T is thrust generated by the launch vehicle which is considered constant in this paper, D is the drag, L is lift and g is the acceleration due to gravity.

In this case ($r$, $v$, $\gamma$) distance from centre of earth, velocity and flight path angle respectively are the state variables and ($\alpha$) angle of attack is the control parameter. To include the variation of gravity, it is modelled as function of height.

$$g = g_0\left(\frac{r_e}{r}\right)^2 \tag{4}$$

The above equations are dependent on time. But for applying those in minimum time optimization problem we have first convert them into flight path angle dependent problem thereby converting the flight path angle an independent variable rather than time.

$$\dot{r} = F(\gamma) \; ; \dot{v} = G(\gamma) \; ; \gamma = I(\gamma) \tag{5}$$

### 2.2. Physical Constraints

Next the physical constraints are defined. Constraints are the function that describe the relationship among the variable and that define allowable value of variables.

**In this paper the constraints taken into account are :**

- The state variables at injection point should have minimum error *i.e* the final altitude, flight path angle and the velocity at injection point should be met as accurately as possible.
- The vehicle should follow the trajectory, which takes minimum time to reach the injection point.

### 2.3. Performance Measure

In order to evaluate performance of the system, performance measure is selected and is denoted as J throughout the paper also called cost function. Here the problem is a combination of minimum-time problem and the terminal control problem, the performance measure is given by[12]:

$$J = \sum S_x \| x_f - x_{t_f} \|_2^2 + \int_{t_0}^{t_f} dt \tag{6}$$

Where, $s_r$, $s_v$ and $s_\gamma$ are the weighing factors. The relative weighing factor for different constraints are adjusted by proper selection of corresponding weighing factors. To reformulate the performance measure, $\gamma$ is chosen as an independent variable. Therefore, the cost function can be substituted as,

$$J = \sum S_x \| x_f - x_{t_f} \|_2^2 + \int_{\gamma_0}^{\gamma_f} \frac{dt}{d\gamma} d\gamma \tag{7}$$

In dealing with optimal control problem [10], formulation of Hamiltonian, state equations, co-state equations, boundary conditions and optimal control equations are entailed. Thus obtained nonlinear equations and it is solved by gradient method.

## 3. ALGORITHM

**The steps involved in the process are summarized below :**

- Start with an initial guess control $\alpha^0(\gamma)$, where $\gamma_0 \le \gamma \le \gamma_f$
- Propagate the states from $\gamma_0$ to $\gamma_f$ using $a^k(\gamma)$ with initial conditions $X_0$
- Obtain $\lambda_f(\gamma_f)$ by using the terminal boundary conditions.
- Propagate the co-state vector from $\gamma_f$ to $\gamma_0$
- Calculate the gradient $\frac{\partial H}{\partial \alpha}$ from $\gamma_0$ to $\gamma_f$
- Calculate the control update, $a^{k+1}(\gamma) = a^k(\gamma) + \frac{\partial H}{\partial \alpha}\tau$ Where $\tau \in (0, 1))$ is the learning rate.
- Repeat from step (2) until optimality conditions are met with in a specified tolerance.

## 4. RESULT ANALYSIS

Hamiltonian is formed by converting minimum-time problem to fixed final flight path angle problem. The independent variable considered is flight path angle instead of time, which is the convention followed in literature. The data for numerical analysis is taken from [12,13]. Where the final time calculated to achieve the desired terminal conditions following the given constraints throughout the trajectory is 150 seconds.

The objective of this work is to reduce the final time by 2-3% using proposed optimization technique. Since the gradient method being an indirect method, an initial guess of control variable is required. For finding the accurate steering profile taking an assumption that is acceleration is a linearly increasing quantity. So rate of change of velocity is approximated as a linear function. Using the above, state variable profiles are determined followed by the calculation of co-state variables. The control variable is updated using gradient method.

**Table 1**
**State variables and numerical results**

| *Parameters* | *Initial conditions* | *Terminal conditions* | *Current results* | *Deviation* |
|---|---|---|---|---|
| Time (s) | 0 | 150 | 145.8 | 4.2 |
| Altitude (m) | 6377.353 | 6438.553 | 6438.5586 | 0.0056 |
| Velocity (m/s) | 65 | 2630 | 2631.04 | 1.04 |
| Flightpath angle (deg) | 89.5 | 0 | 2.85 | –2.85 |

The simulation is done in Matlab environment and corresponding results are given below. Figure. 2 represents the radial distance from centre of earth with respect to time. The radial distance variation is smooth and not having sudden changes. Figure. 3 and 4 represents the velocity and flight path angle variations with respect to time. In the flightpath angle variation graph the error occurred in the terminal conditions are clearly visible.

Table. 1 represents the error encountered while achieving the terminal conditions and minimum time with boundary conditions. From the results it is clear that the injection criteria is met satisfactorily.

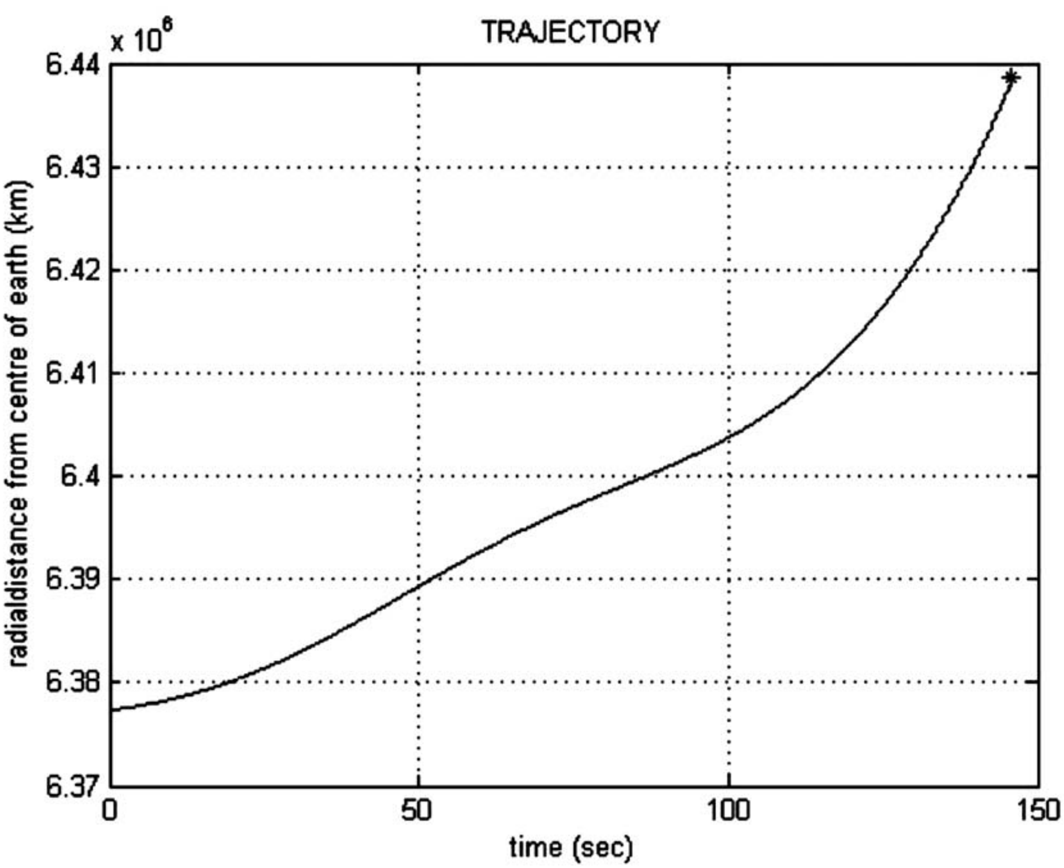


**Figure 2: Radial distance from center of earth with respect to time**

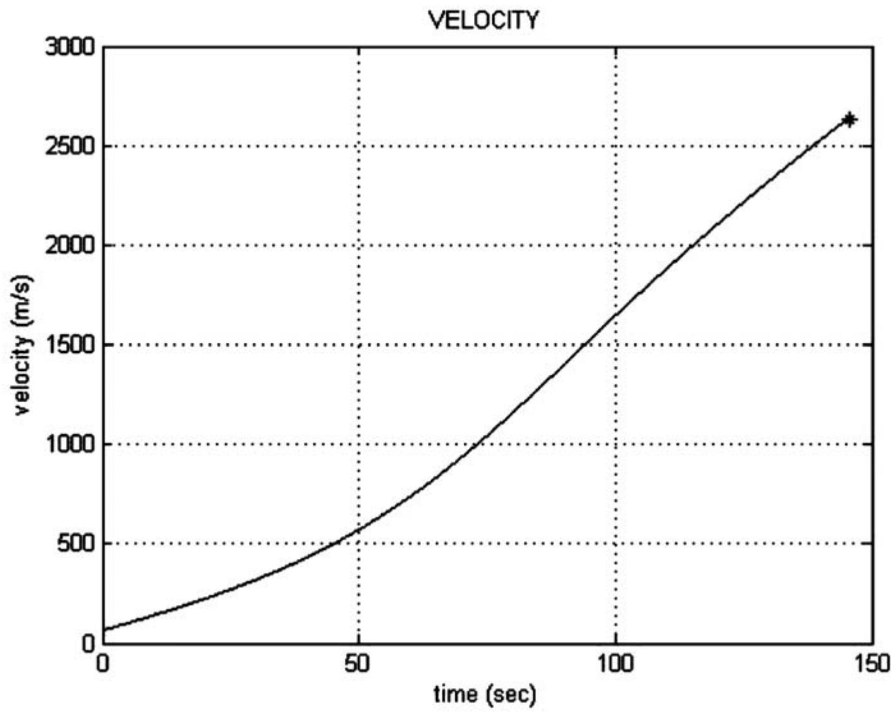


**Figure 3: Launch vehicle velocity with respect to time**

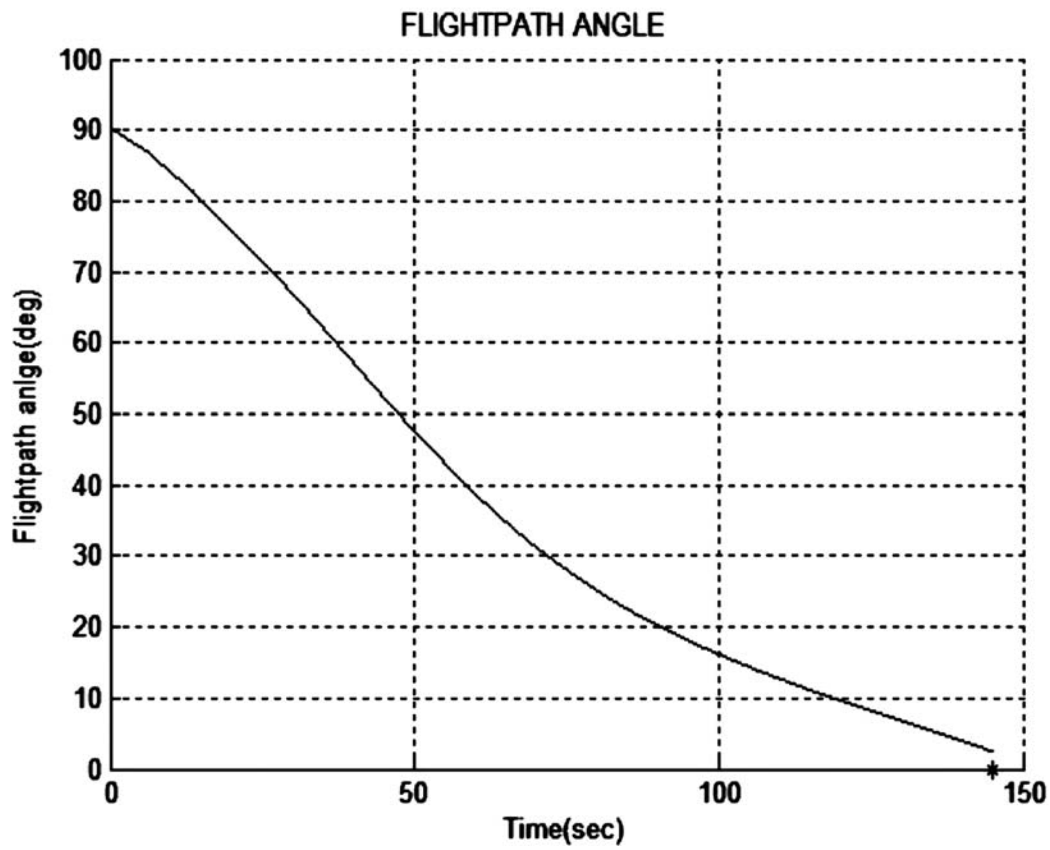


**Figure 4: Flight path angle with respect to time**

## 5. CONCLUSION

The trajectory of a single stage launch vehicle is studied in this paper. The gravity turn trajectory is formulated for given initial and final conditions. The target is achieved by minimum time and minimum error at injection point. Hamiltonian is formed by converting minimum-time problem to fixed final flight path angle problem. Pontryagin minimum principle is applied for obtaining necessary conditions. Appropriate constraints are followed throughout the trajectory and necessary boundary conditions are applied. The non-linear equations are solved using steepest descent method/gradient method. Auspicious results are obtained.

## 6. ACKNOWLEDGEMENT

We would like to thank Manipal Institute of Technology, Manipal University,Manipal for providing all kind of support for fulfilling this research.

## 7. REFERENCES

1. Lukkana Varaprasad, Radhakant Padhi "Ascent Phase Trajectory Optimization of A Generic Launch Vehicle" Xxxii National Systems Conference, IIT-Roorkee. NSC 2008
2. Mauro Pontani and Giampaolo Cecchetti "Ascent trajectories of multistage launch vehicles: numerical optimization with second order condition verification" Operations Research Volume 2013 (2013) Article ID 498765, 9 pages
3. A. Miele and T. Wang "Multiple-subarc gradient-restoration algorithm, part 1: algorithm structure," Journal of Optimization Theory and Applications, vol. 116, no. 1, pp. 1–17, 2003.
4. K. R. Brown, E. F. Harrold, and G. W. Johnson "Rapid optimization of multiple-burn rocket flights," NASA Contractor Reports, 1969.
5. C. R. Hargraves and S. W. Paris "Direct trajectory optimization using nonlinear programming and collocation," Journal of Guidance, Control, and Dynamics, vol. 10, no. 4, pp. 338–342, 1987 .
6. H. Seywald. "Trajectory optimization based on differential inclusion," Journal of Guidance, Control, and Dynamics, vol. 17, no. 3, pp. 480–487, 1994
7. John T. Betts ―Survey of Numerical Methods for Trajectory Optimization‖ Journal Of Guidance, Control, And Dynamics Vol. 21, No. 2, AIAA 1998..
8. C. V. Sirisha and M. Manickavasagam "A Specific Energy Based Guidance Algorithm for the Hypersonic Launch Vehicle", International Conference on Avionics Systems–2008Conference, RCI, Hyderabad
9. S. Vathsal, K.A.P. Menon and R. Swaminathan. "Minmax Approach to Trajectory Optimization of Multistage Launch Vehicles", IEEE Transactions on Aerospace & Electronics Systems, Volume AES-13, Issue 2, March 1977, pp 179-187
10. D. E. Krik."Optimal Control Theory: An Introduction", Prentice Hal, 1970
11. Vishnu G Nair, Dileep M V and Prahalad K R. Article: New Programming Approach for Steepest Descent Optimization of Rocket Trajectories. International Journal of Computer Applications 48(19):21-27, June 2012
12. Joshi, Ashok, and B. Sudhir Kumar. "Effect of Initial Flight Path Angle Error and Control Constraint on the Optimized Ascent Trajectory of a Typical Launch Vehicle", 19th AIAA International Space Planes and Hypersonic Systems and Technologies Conference, 2014.
13. Martin S. K. Leung and Anthony J. Calise. "Hybrid approach to near-optimal launch vehicle guidance", Journal of Guidance, Control, and Dynamics, Vol. 17, No. 5 (1994), pp. 881-888. doi: 10.2514/3.21285
14. Ping Lu. "Nonlinear trajectory tracking guidance with application to a launch vehicle", Journal of Guidance, Control, and Dynamics, Vol. 19, No. 1 (1996), pp. 99-106. doi: 10.2514/3.21585.