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Exact Solutions of the Variable Coefficient K-dV Burger Equation and Modified K-dV Equation

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ABSTRACT

In this paper we look for the exact solutions of certain types of nonlinear and variable coefficient K-dV Burger and modified K-dV equations by using modified tanh method. In particular, the solitary wave solutions are found. Such equations arise in a variety of contexts in physical and biological problems.

Keywords: Nonlinear diffusion, Reaction equation, Auxiliary equation method, Solitary wave solutions.

1. INTRODUCTION

Mathematical modeling of blood flow through a stenosed artery becomes important since the main cause of more than half of all mortality in industrialized countries is hardening of artery or atherosclerosis. During the blood flow, deposits of fatty substances, cholesterol, cellular waste products, calcium and other substances build up in the inner lining of an artery. This build up is called plaque. Plaques can grow large enough (stenosis) to significantly reduce the flow of blood through an artery. It is for this reason that the study of the effects of a stenosis on blood flow in arteries becomes extremely important. By treating the arteries as circularly cylindrical, long, thin, homogeneous and isotropic elastic tubes with a bump. In essence, the arteries are inhomogeneous and have variable radius along the axis of the tube, the blood as an incompressible Newtonian .fluid whose viscosity changes with the radial coordinate, and by employing the reductive perturbation method in the longwave approximation to the nonlinear equations of that model, Hilim Demiray has studied weakly nonlinear waves in such a medium and obtained the variable coefficient Korteweg-deVries-Burgers (KdV-B) and modified Korteweg-deVries equation [15-16].

$$u_\tau + \mu_1 u u_\zeta - \mu_2 u_{\zeta\zeta} + \mu_3 u_{\zeta\zeta\zeta} - \mu_4 h(\tau) u_\zeta = 0\,, \tag{1}$$

$$u_\tau + \mu_4 u^2 u_\zeta + \mu_2 u_{\zeta\zeta\zeta} + \mu_3 h_2(\tau) u_\zeta = 0\,. \tag{2}$$

2. EXACT SOLUTIONS OF EQ. (1)

Based on the method of auxiliary equation, we first transform the partial differential to a total differential equation by defining a variable [2]

$$x = p(\tau)\zeta + q(\tau) \tag{3}$$

By using this expression Eq. (1) can be expressed as

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$$\left[\frac{d}{d\tau}p(\tau)\,x+\frac{d}{d\tau}q(\tau)\right]\left[a_1(\tau)\frac{dz(x)}{dx}+2a_2(\tau)\,z(x)\frac{dz(x)}{dx}\right]+\left[\frac{d}{d\tau}a_1(\tau)\right]z(x)$$
$$+\left[\frac{d}{d\tau}a_2(\tau)\right]z^2(x)+\mu_1 u(x)\,p(\tau)\frac{du(x)}{dx}-\mu_2 p^2(\tau)\frac{d^2u(x)}{dx^2}$$
$$+\mu_3 p^3(\tau)\frac{d^3u(x)}{dx^3}-\mu_4 h(\tau)\,p(\tau)\frac{du(x)}{dx}+\frac{d}{d\tau}a_0(\tau)=0, \tag{4}$$

For the solution of Eq. (4) we make ansatz [13]

$$u(x)=\sum_{i=0}^{l}a_i(\tau)\,z^i(x), \tag{5}$$

where a_i are all functions of τ to be determined, l is a positive integer which can be determined by balancing the highest order derivative term with the highest order nonlinear term in Eq. (1), and $z(x)$ satisfies the following auxiliary ordinary differential equation [13]

$$\frac{dz}{dx}=b+z(x)^2, \tag{6}$$

Where b will be determined later. Eq. (6) has the following general solution s:

(i) If $b<0$, then

$$z(x)=-\sqrt{-b}\tanh(\sqrt{-b}x),\qquad \text{or}\qquad z(x)=-\sqrt{-b}\coth(\sqrt{-b}x).$$

(ii) If $b>o$, then

$$z(x)=\sqrt{b}\tan(\sqrt{b}x),\qquad \text{or}\qquad z(x)=-\sqrt{b}\cot(\sqrt{b}x).$$

(iii) If $b=0$,

$$z(x)=-1/x.$$

Using the balancing procedure we get $l = 2$ for Eq. (4). This suggest the choice of $u(x)$ in Eq. (5) as

$$u(x)=a_0(\tau)+a_1(\tau)\,z(x)+a_2(\tau)\,z^2(x). \tag{7}$$

Substituting (7) along with (6) into Eq. (4) and then setting the coefficients of $z(x)^j=(j=0,1,...,3)$ to zero in the resultant expression, one obtains a Set of equations involving $a_0(\tau)$, $a_1(\tau)$, $a_2(\tau)$, $p(\tau)$ and $q(\tau)$ as

$$2\mu_1 p(\tau)a_2^2(\tau)+24\mu_3 p^3(\tau)a_2(\tau)=0,$$

$$-6\mu p^2(\tau)a_2(\tau)+3\mu_1 p(\tau)a_1(\tau)a_2(\tau)+6\mu_3 p^3(\tau)a_1(\tau)=0,$$

$$3\mu_1 p(\tau)\,a_1(\tau)\,a_2(\tau)\,b+\frac{dp(\tau)}{d\tau}\zeta a_1(\tau)-8\mu_2 p^2(\tau)\,a_2(\tau)\,b+\mu_1 p(\tau)\,a_0(\tau)\,a_1(\tau)$$
$$-\mu_4 h(\tau)\,p(\tau)\,a_1(\tau)+8\mu_3 p^3(\tau)\,a_1(\tau)\,b+\frac{dq(\tau)}{d\tau}a_1(\tau)+\frac{da_2(\tau)}{d\tau}=0,$$

$$2\frac{dq(\tau)}{d\tau}a_2(\tau)b+\mu_1 p(\tau)a_1^2(\tau)b+2\mu_1 p(\tau)a_0(\tau)a_2(\tau)b+16\mu_3 p^3(\tau)a_2(\tau)b^2$$
$$+\frac{da_1(\tau)}{d\tau}-2\mu_4 h(\tau)p(\tau)a_2(\tau)b-2\mu_2 p^2(\tau)a_1(\tau)b+2\frac{dp(\tau)}{d\tau}\zeta a_2(\tau)b=0,$$

$$2\mu_1 p(\tau)a_2^2(\tau)b-2\mu_4 h(\tau)p(\tau)a_2(\tau)+\mu_1 p(\tau)a_1^2(\tau)+2\mu_1 p(\tau)a_0(\tau)a_2(\tau)$$
$$+2\frac{dq(\tau)}{d\tau}a_2(\tau)+2\frac{dp(\tau)}{d\tau}\zeta a_2(\tau)-2\mu_2 p^2(\tau)a_1(\tau)+40\mu_3 p^3(\tau)a_2(\tau)b=0,$$

$$-2\mu_2 p^2(\tau)a_2^2(\tau)b^2+\frac{dq(\tau)}{d\tau}a_1(\tau)b+\mu_1 p(\tau)a_0(\tau)a_1(\tau)b+2\mu_3 p^3(\tau)a_1(\tau)b^2$$
$$+\frac{da_0}{d\tau}-\mu_4 h(\tau)p(\tau)a_1(\tau)b+\frac{dp(\tau)}{d\tau}\zeta a_1(\tau)b=0. \tag{8}$$

This is a set of coupled nonlinear ordinary differential equations. To solve the previous system we will use the symbolic computation technique of Maple to get the following solutions

$$a_0(\tau)=a_0,\qquad a_1(\tau)=\frac{12\mu_2^2}{50\mu_3\mu_1\sqrt{-b}},$$
$$a_2(\tau)=\frac{3\mu_2^2}{25b\mu_3\mu_1},\qquad p(\tau)=\frac{\mu_2}{10\mu_3\sqrt{-b}}, \tag{9}$$
$$q(\tau)=\int\frac{(25h(\tau)\mu_2\mu_3\mu_4+3\mu_2^3-25\mu_2\mu_1\mu_3 a_0)}{250\sqrt{-b}\,\mu_3^2}\,d\tau.$$

Where a_0 is a constant. Finally, the solution $u(x)$ of Eq. (4) turns out to be

If $b<0$, then

$$u(x)=a_0-\left(\frac{12\mu_2^2}{50\mu_3\mu_1}\right)[\tanh(\sqrt{-b}x)]-\left(\frac{3\mu_2^2}{25\mu_3\mu_1}\right)[\tanh(\sqrt{-b}x)]^2,\qquad \text{or}$$
$$u(x)=a_0-\left(\frac{12\mu_2^2}{50\mu_3\mu_1}\right)[\coth(\sqrt{-b}x)]-\left(\frac{3\mu_2^2}{25\mu_3\mu_1}\right)[\coth(\sqrt{-b}x)]^2$$

If $b>0$, then

$$u(x)=a_0+\left(\frac{12\mu_2^2}{50\mu_3\mu_1\sqrt{-b}}\right)[\sqrt{b}\tan(\sqrt{b}x)]+\left(\frac{3\mu_2^2}{25\mu_3\mu_1}\right)[\tan(\sqrt{b}x)]^2,\qquad \text{or}$$
$$u(x)=a_0-\left(\frac{12\mu_2^2}{50\mu_3\mu_1\sqrt{-b}}\right)[\sqrt{b}\cot(\sqrt{b}x)]+\left(\frac{3\mu_2^2}{25\mu_3\mu_1}\right)[\cot(\sqrt{b}x)]^2$$

If $b=0$, then

$$u(x)=a_0-\left(\frac{12\mu_2^2}{50\mu_3\mu_1\sqrt{-b}x}\right)+\left(\frac{3\mu_2^2}{25b\mu_3\mu_1 x^2}\right). \tag{10}$$

Where x as in Eq. (3)

$$x = \frac{\mu_2}{10\mu_3\sqrt{-b}}\zeta + \int \frac{(25h(\tau)\mu_2\mu_3\mu_4 + 3\mu_2^3 - 25\mu_2\mu_1\mu_3 a_0)}{250\sqrt{-b}\mu_3{}^2}\,d\tau.$$

As an example to illustrate the properties of the solution we plot the first solution in Fig. (1), if we take the constant integration = 0 and $b = -1, \mu_4 = -1, a_0 = 1, \mu_2 = \mu_3 = \mu_1 = 1$ and $h(\tau) = \tau$

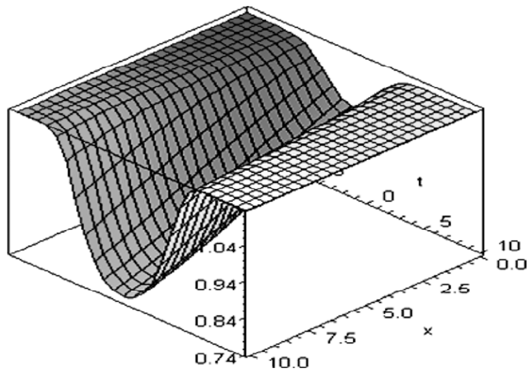

Figure 1: Which is a Solitary Wave Solution of Eq. (1)

3. EXACT SOLUTIONS OF EQ. (2)

By using the transformation $\zeta = p(\tau)x + q(\tau)$, we write Eq. (2) as

$$\left[\frac{d}{d\tau}p(\tau)\zeta + \frac{d}{d\tau}q(\tau)\right]a_1(\tau)\frac{dz(x)}{dx} + \frac{d}{d\tau}a_0(\tau) + \frac{d}{d\tau}a_1(\tau)z(x) + \mu_4 u^2(x)p(\tau)\frac{du(x)}{dx}$$
$$+\mu_2 p(\tau)^3\frac{d^3u(x)}{dx^3} + \mu_3 h_2(\tau)p(\tau)\frac{du(x)}{dx} = 0. \qquad (11)$$

Again using the balancing procedure we get $l = 1, u(x) = a_0(\tau) + a_1(\tau)z(x)$, Substituting Eq. (5) and Eq. (6) in Eq. (11) we get a set of algebraic equations

$$2\mu_4 p(\tau)a_0(\tau)a_1(\tau)^2 = 0,$$

$$\frac{da_1(\tau)}{d\tau} + 2\mu_4 p(\tau)a_0(\tau)a_1^2(\tau)b = 0,$$

$$6\mu_2 p^3(\tau)a_1(\tau) + \mu_4 p(\tau)a_1^3(\tau) = 0,$$

$$\frac{dp(\tau)}{d\tau}\zeta a_1(\tau)b + a_1(\tau)\frac{dq(\tau)}{d\tau}b + \mu_3 h_2(\tau)p(\tau)a_1(\tau)b$$
$$+\frac{da_0(\tau)}{d\tau} + \mu_4 p(\tau)a_1(\tau)a_0^2(\tau)b + 2\mu_2 p^3(\tau)a_1(\tau)b^2 = 0,$$

$$\frac{dp(\tau)}{d\tau}\zeta a_1(\tau) + a_1(\tau)\frac{dq(\tau)}{d\tau} + \mu_4 p(\tau)a_1^3(\tau)b + \mu_4 p(\tau)a_1(\tau)a_0^2(\tau)$$
$$+\mu_3 p(\tau)a_1(\tau)h_2(\tau) + 8\mu_2 p^3(\tau)a_1(\tau)b = 0. \qquad (12)$$

After solving the set of coupled nonlinear differential equations, one obtains

$$a_0(\tau) = 0, \qquad a_1(\tau) = a_1, \qquad p(\tau) = \frac{\sqrt{-6\mu_2\mu_4}\,a_1}{6\mu_2},$$

$$q(\tau) = \left(\frac{\sqrt{-6\mu_2\mu_4}\,a_1}{6\mu_2}\right)\int\left[-\mu_3 h_2(\tau) + \frac{\mu_4 a_1^{\,2} b}{3}\right]d\tau. \tag{13}$$

Finally the solution $u(x)$ of Eq. (11) becomes

If $b < 0$, then

$$u(x) = -a_1\sqrt{-b}\,\tanh(\sqrt{-b}x), \qquad \text{or} \qquad u(x) = -a_1\sqrt{-b}\,\coth(\sqrt{-b}x),$$

If $b > 0$, then

$$u(x) = a_1\sqrt{b}\,\tan(\sqrt{b}x), \qquad \text{or} \qquad u(x) = -a_1\sqrt{b}\,\cot(\sqrt{b}x)],$$

If $b = 0$, then

$$u(x) = -a_1/x. \tag{14}$$

Where x as in Eq. (3)

$$x = \frac{\sqrt{-6\mu_2\mu_4}\,a_1}{6\mu_2}\zeta + \left(\frac{\sqrt{-6\mu_2\mu_4}\,a_1}{6\mu_2}\right)\int\left[-\mu_3 h_2(\tau) + \frac{\mu_4 a_1^2 b}{3}\right]d\tau.$$

As an example to illustrate the properties of the solution we plot Fig. (2) The first solution, If we take the constant integration = 0 and $b = -1$, $\mu_4 = -1$, $a_1 = 1$, $\mu_2 = \mu_3 = 1$ and $h_2(\tau) = \tau$.

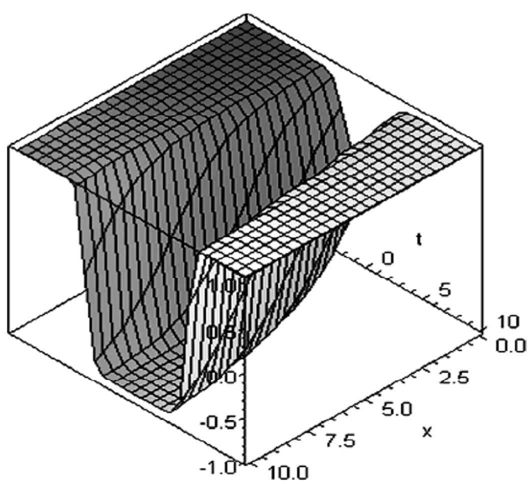

Figure 2: Which is a Solitary Wave Solution of Eq. (2)

4. CONCLSION

In this study, we have applied the modified tanh method to obtain the generalized solitary wave solutions of the K dV Burger and the modified K dV equations. Also, the obtained solitary wave solutions obtained, solutions are shown in Figs. 1 and 2.

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