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SPECTRAL MULTIPLIERS FOR THE DUNKL LAPLACIAN

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Abstract. In this paper we prove L^p-boundedness properties of spectral multipliers associated with Dunkl laplacian in the $\mathbb{Z}_2^d$ Dunkl setting.

1. Introduction

A fundamental object in harmonic analysis is the multiplier operator. Multipliers related to numerous classic kinds of orthogonal expansions were widely investigated. In particular, Stempak and Trebels (cf. [14]) studied multipliers of non-Laplace type in a one-dimensional Laguerre setting. Some earlier results concerning multiplier operators of Laplace type for discrete and continuous orthogonal expansions can be found in [3, 6, 10, 12] among others. A general treatment of Laplace type multipliers in a context of symmetric diffusion semigroups can be found in Stein's monograph (cf. [13]). In this paper, we focus on the Dunkl multiplier operator P_m which is defined, for a suitable function f, by

$$P_m f(x) = \mathcal{F}_\kappa(m\mathcal{F}_\kappa f)(-x), \quad x \in \mathbb{R}^d,$$

where m is a bounded measurable function and $\mathcal{F}_\kappa$ denotes the Dunkl transform (see the next section).

Consider the reflection group G generated by σ_j, $j = 1, ..., d$, the reflection with respect to the hyperplane perpendicular to e_j, the j-th coordinate vector, that is to say for every $x = (x_1, ..., x_d) \in \mathbb{R}^d$

$$\sigma_j(x) = x - 2\frac{< x, e_j >}{|e_j|^2} e_j,$$

where $e_1, ..., e_d$ is the standard basis of $\mathbb{R}^d$, $< ., . >$ is the usual inner product on $\mathbb{R}^d \times \mathbb{R}^d$ and $|.|$ is the associated norm.

The finite reflection group G is isomorphic to $\mathbb{Z}_2^d$ with the associated measure $h_\kappa^2(x)\, dx$ given by

$$h_\kappa(x) = \prod_{j=1}^{d} |x_j|^{\kappa_j} = \prod_{j=1}^{d} h_{\kappa_j}(x_j), \tag{1.1}$$

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where $\kappa_1, ..., \kappa_d$ are nonnegative real numbers (let us note that h_κ is homogeneous of degree $\gamma_\kappa = \sum_{j=1}^{d} \kappa_j$).

Associated with these objects, the Dunkl differential-difference operators T_j^κ, $j = 1, ..., d$, are given by (cf.[7])

$$T_j^\kappa f(x) = \partial_j f(x) + \kappa_j \frac{f(x) - f(\sigma_j(x))}{x_j}.$$

The Dunkl Laplacian is defined by $\Delta_\kappa = \sum_{j=1}^{d} \big(T_j^\kappa\big)^2$, or more explicitly by

$$\Delta_\kappa f(x) = \sum_{j=1}^{d} \Big(\frac{\partial^2 f(x)}{\partial^2 x_j} + \frac{2\kappa_j}{x} \frac{\partial f(x)}{\partial x_j} - \kappa_j \frac{f(x) - f(\sigma_j(x))}{x_j^2}\Big).$$

Following Stein (cf. [13]), we say that m is a multiplier of Laplace transform type when

$$m(y) = |y|^2 \int_0^\infty e^{-t|y|^2} \phi(t) dt, \quad y \in \mathbb{R}^d, \tag{1.2}$$

where ϕ is a bounded measurable function on $\mathbb{R}_+$.

The aim of this paper is to prove the following theorem, where we denote by $L^p(\mathbb{R}^d, h_\kappa^2)$, $1 \le p \le +\infty$, the space of measurable functions on $\mathbb{R}^d$ such that

$$\|f\|_{p,\kappa} = \Big(\int_{\mathbb{R}^d} |f(y)|^p h_\kappa^2(y) dy\Big)^{\frac{1}{p}} < +\infty \quad \text{if } 1 \le p < +\infty,$$

$$\|f\|_{\infty,\kappa} = ess \sup_{y \in \mathbb{R}^d} |f(y)| < +\infty \quad \text{otherwise.}$$

Theorem 1.1. *Assume that m is of Laplace transform type. Then, the Dunkl multiplier P_m is bounded from $L^p(\mathbb{R}^d, h_\kappa^2)$ into itself, for every $1 < p < +\infty$, and from $L^1(\mathbb{R}^d, h_\kappa^2)$ into $L^{1,\infty}(\mathbb{R}^d, h_\kappa^2)$.*

In order to prove Theorem 1.1, we investigate how to define the multiplier operator P_m in terms of its kernel, as a limit of truncated integrals more precisely, we represent P_m as a principal value integral operator when it acts on the space $\mathcal{C}_c^\infty(\mathbb{R}^d)$ of the $\mathcal{C}^\infty$ functions with compact support in $\mathbb{R}^d$ (see Proposition 4.2). Then, after proving L^p- boundedness properties for the maximal operator associated with the principal value integral operator (see Proposition 4.3), we extend the Dunkl multiplier P_m to $L^p(\mathbb{R}^d, h_\kappa^2)$, $1 \le p < +\infty$, as a principal value integral operator that is bounded from $L^p(\mathbb{R}^d, h_\kappa^2)$ into itself, when $1 < p < \infty$, and from $L^1(\mathbb{R}^d, h_\kappa^2)$ into $L^{1,\infty}(\mathbb{R}^d, h_\kappa^2)$ (see Proposition 4.4).

Given an initial distribution $f \in \mathcal{C}_b(\mathbb{R}^d)$, there is a function

$$u \in \mathcal{C}^2(\mathbb{R}^d \times]0, +\infty[) \cap \mathcal{C}_b(\mathbb{R}^d \times [0, +\infty[)$$

satisfying

$$\begin{cases} \Delta_\kappa u(x,t) = \partial_t u(x,t), & (x,t) \in \mathbb{R}^d \times]0, +\infty[; \\ u(.,0) = f. & \end{cases}$$

For smooth and rapidly decreasing initial data f an explicit solution is easy to obtain, it involves the generalized heat kernel

$$\Gamma_t^\kappa(x,y) = \frac{c_\kappa^{-1}}{(2t)^{\gamma_\kappa+\frac{d}{2}}} e^{-\frac{(|x|^2+|y|^2)}{4t}} E_\kappa\left(\frac{x}{\sqrt{2t}}, \frac{y}{\sqrt{2t}}\right),$$

where E_κ denotes the Dunkl kernel (see the next section) and c_κ is the Mehta type constant

$$c_\kappa^{-1} = \int_{\mathbb{R}^d} e^{-\frac{|x|^2}{2}} h_\kappa^2(x)dx = \prod_{j=1}^{d} c_{\kappa_j}^{-1}.$$

The Dunkl type heat kernel satisfies the following properties

$$\Gamma_t^\kappa(x,y) = c_\kappa^{-2} \int_{\mathbb{R}^d} e^{-t|\xi|^2} E_\kappa(ix,\xi)E_\kappa(-iy,\xi)h_\kappa^2(\xi)d\xi \tag{1.3}$$

$$\int_{\mathbb{R}^d} \Gamma_t^\kappa(x,y)h_\kappa^2(y)dy = 1 \tag{1.4}$$

$$\Gamma_t^\kappa(x,y) \leq \frac{c_\kappa^{-1}}{(2t)^{\gamma_\kappa+d/2}} e^{-\frac{(|x|-|y|)^2}{4t}}. \tag{1.5}$$

The Dunkl type heat kernel allows us to define a generalized heat operator (or Dunkl type heat operator). More precisely for every $f \in L^p(\mathbb{R}^d, h_\kappa^2)$, with $1 \leq p \leq \infty$ and for every $t \geq 0$, we set

$$H_t^\kappa f(x) = \begin{cases} \int_{\mathbb{R}^d} f(y)\Gamma_t^\kappa(x,y)h_\kappa^2(y)dy, & t > 0; \\ f, & t = 0. \end{cases}$$

For every p satisfying $1 \leq p \leq \infty$, the family $\{H_t^\kappa f\}_{t\geq 0}$ is a symmetric diffusion semigroup (cf. [5]) in the sense of Stein on $L^p(\mathbb{R}^d, h_\kappa^2)$. Moreover, the Dunkl multiplier P_m is actually a spectral multiplier associated with $(-\Delta_\kappa)$. Then, by [13] P_m is bounded from $L^p(\mathbb{R}^d, h_\kappa^2)$ into itself, for every $1 < p < \infty$. In Theorem 1.1, we prove as a new result that P_m defines a bounded operator from $L^1(\mathbb{R}^d, h_\kappa^2)$ into $L^{1,\infty}(\mathbb{R}^d, h_\kappa^2)$. Moreover, in Proposition 4.4 we establish a representation of the operator P_m as a principal value integral operator in $L^p(\mathbb{R}^d, h_\kappa^2), p \geq 1$.

As an application of Theorem 1.1, we can show L^p-boundedness properties for the imaginary powers of $(-\Delta_\kappa)$. We define, for every $\beta \in \mathbb{R}$, the function

$$\phi_\beta(t) = \frac{t^{-i\beta}}{\Gamma(1-i\beta)}, \quad t \in \mathbb{R}_+.$$

A formal computation based on the formula

$$\lambda^{-i\gamma} = \frac{1}{\Gamma(i\gamma)} \int_0^\infty e^{-t\gamma} t^{i\gamma-1} dt, \quad \lambda > 0,$$

gives

$$m_\beta(y) = |y|^2 \int_0^\infty e^{-t|y|^2} \phi_\beta(t)dt = |y|^{2i\beta}, \; y \in \mathbb{R}^d, \; \beta \in \mathbb{R}.$$

For every $\beta \in \mathbb{R}$, the $i\beta$- power $\Delta_\kappa^{i\beta}$ of Δ_κ is defined by $(-\Delta_\kappa)^{i\beta} = P_{m_\beta}$. From Theorem 1.1, we deduce the following result.

Corollary 1.2. *Let $\beta \in \mathbb{R}$. Then, the operator $(-\Delta_\kappa^{i\beta})$ is bounded from $L^p(\mathbb{R}^d, h_\kappa^2)$ into itself, for every $1 < p < +\infty$ and from $L^1(\mathbb{R}^d, h_\kappa^2)$ into $L^{1,\infty}(\mathbb{R}^d, h_\kappa^2)$.*

This paper is organized as follows. In section 2, we collect some definitions and results related to Dunkl's analysis. In section 3, we establish some estimates for the Dunkl heat kernel and its derivatives. Finally, we prove the main result of this paper.

Throughout this paper we will always denote by C a suitable positive constant that can change from line to the other one. Also, we will use repeatedly without saying it that, for every $k \in \mathbb{N}$, $\sup_{z>0} z^k e^{-z} < \infty$.

2. Preliminaries

This section is devoted to the preliminaries and background. We only focus on the aspects of the Dunkl theory which will be relevant for the sequel. For a large survey about this theory, the reader may especially consult [7, 8, 4, 11].

First of all we recall that for every $f \in \mathcal{S}(\mathbb{R}^d)$, $g \in \mathcal{C}_b^1(\mathbb{R}^d)$, one then has the following property of integration by parts (cf. [9])

$$\int_{\mathbb{R}^d} T_j^\kappa f(x) g(x) h_\kappa^2(x) dx = -\int_{\mathbb{R}^d} f(x) T_j^\kappa g(x) h_\kappa^2(x) dx. \tag{2.1}$$

The operators ∂_j and T_j^κ are intertwined by a linear isomorphism V_κ of $\bigoplus_{n\geq 0} \mathcal{P}_n$ determined uniquely by

$$V_\kappa(\mathcal{P}_n) = \mathcal{P}_n, \quad V_\kappa(1) = 1, \quad T_j^\kappa V_\kappa = V_\kappa \partial_j, \quad j = 1, ..., d.$$

with $\mathcal{P}_n$ the subspace of homogeneous polynomials of degree n in d variables.

An explicit formula of V_κ is not known in general. However, in our setting, the operator V_κ is given according to [16] by the following integral representation

$$V_\kappa f(x) = \int_{[-1,1]^d} f(x_1 t_1, ..., x_d t_d) \prod_{j=1}^d M_{\kappa_j} (1 + t_j)(1 - t_j^2)^{\kappa_j - 1} dt,$$

with $M_{\kappa_j} = \frac{\Gamma(\kappa_j + \frac{1}{2})}{\sqrt{\pi}\Gamma(\kappa_j)}$.

For every $y \in \mathbb{C}^d$, the simultaneous eigenfunction problem

$$\begin{cases} T_j^\kappa u(x,y) = y_j u(x,y), & 1 \leq j \leq d; \\ u(0,y) = 1. \end{cases}$$

has a unique solution $x \to E_\kappa(x,y)$, which is given by

$$E_\kappa(\lambda, x) = V_\kappa(e^{<\lambda,.>})(x), \quad x \in \mathbb{R}^d.$$

Furthermore $x \mapsto E_\kappa(x,y)$ extends to a holomorphic function on $\mathbb{C}^d$ and it satisfies the following basic properties: $E_\kappa(x,y) = E_\kappa(y,x)$ for $x, y \in \mathbb{C}^d$, $|E_\kappa(ix,y)| \leq 1$ for $x, y \in \mathbb{R}^d$ and $E_\kappa(\lambda x, y) = E_\kappa(x, \lambda y)$ for $x, y \in \mathbb{C}^d$ and $\lambda \in \mathbb{C}$.

Considering the definition of E_κ together with the explicit formula for V_κ gives us

$$E_\kappa(x,y) = \prod_{j=1}^d E_{\kappa_j}(x_j, y_j). \tag{2.2}$$

The Dunkl kernel E_κ is of particular interest as it gives rise to an integral transform which is taken with respect to the measure $h_\kappa^2(x)dx$. More precisely, for $f \in L^1(\mathbb{R}^d, h_\kappa^2)$, the Dunkl transform of f, denoted by $\mathcal{F}_\kappa f$, is defined by

$$\mathcal{F}_\kappa f(x) = c_\kappa^{-1} \int_{\mathbb{R}^d} f(y) E_\kappa(x, -iy) h_\kappa^2(y) dy, \quad x \in \mathbb{R}^d.$$

Let us point out that the Dunkl transform coincides with the Euclidean Fourier transform when $\kappa_1 = ... = \kappa_d = 0$ and that it is more or less a Hankel transform when $d = 1$.

We list some known properties of the Dunkl transform:

(i) The Dunkl transform is a topological automorphism of the Schwartz space $\mathcal{S}(\mathbb{R}^d)$.

(ii) (Plancherel Theorem) The Dunkl transform extends to an isometric automorphism of $L^2(\mathbb{R}^d, h_\kappa^2)$.

(iii) (Inversion formula) For every $f \in \mathcal{S}(\mathbb{R}^d)$, and more generally for every $f \in L^1(\mathbb{R}^d, h_\kappa^2)$ such that $\mathcal{F}_\kappa f \in L^1(\mathbb{R}^d, h_\kappa^2)$, we have

$$f(x) = \mathcal{F}_\kappa^2 f(-x), \qquad x \in \mathbb{R}^d.$$

(iv) A formula connecting Dunkl transform and Dunkl Laplacian is the following

$$\mathcal{F}_\kappa(\Delta_\kappa f)(x) = -|x|^2 \mathcal{F}_\kappa f(x),\ x \in \mathbb{R}^d,\ f \in \mathcal{C}_c^\infty(\mathbb{R}^d). \tag{2.3}$$

3. Some Estimates Involving Dunkl Kernel

We will establish in this section three technical lemmas. In order to do that, we first recall some facts related to Dunkl's kernel and the confluent hypergeometric function.

In the one dimensional case, E_κ can be expressed in terms of Bessel functions (cf. [15]). Specifically

$$E_\kappa(x, y) = j_{\kappa-\frac{1}{2}}(ixy) + \frac{xy}{2\kappa+1} j_{\kappa+\frac{1}{2}}(ixy),$$

where

$$j_\alpha(z) = 2^\alpha \Gamma(\alpha+1) \frac{J_\alpha(z)}{z^\alpha} = \Gamma(\alpha+1) \sum_{n\geq 0} (-1)^n \frac{(z/2)^{2n}}{n!\Gamma(n+\alpha+1)},$$

are normalized Bessel functions.

The integral representation of the Dunkl kernel E_κ is given by

$$E_\kappa(z, \omega) = \frac{\Gamma(\kappa+1/2)}{\Gamma(1/2)\Gamma(\kappa)} \int_{-1}^1 e^{tz\omega}(1-t)^{\kappa-1}(1+t)^\kappa dt = e^{z\omega}. {}_1F_1(\kappa, 2\kappa+1, -2z\omega). \tag{3.1}$$

where ${}_1F_1$ is the confluent hypergeometric function defined by (cf. [1])

$${}_1F_1(a, b, z) = \sum_{n=0}^\infty \frac{(a)_n}{(b)_n} \frac{z^n}{n!}, \quad b \notin \{0, -1, -2, ...\}.$$

The function ${}_1F_1$ has well-known asymptotic expansions. They are of the form

$${}_1F_1(a, b, z) = \frac{\Gamma(b)}{\Gamma(a)} e^z z^{a-b} \big(1 + \frac{(b-a)(1-a)}{z} + \mathcal{O}(\frac{1}{|z|^2})\big), \quad \Re z > 0, \tag{3.2}$$

$$ {}_1F_1(a,b,z)=\frac{\Gamma(b)}{\Gamma(b-a)}(-z)^{-a}\big(1+\frac{a(1+a-b)}{z}+\mathcal{O}(\frac{1}{|z|^2})\big),\quad \Re z<0. \tag{3.3}$$

Specializing, we thus obtain

$$E_\kappa(z,\omega)=\frac{\Gamma(2\kappa+1)}{2^\kappa\Gamma(\kappa+1)}e^{z\omega}(z\omega)^{-\kappa}\big(1-\frac{\kappa^2}{2z\omega}+\mathcal{O}(\frac{1}{|z\omega|^2})\big),\quad z\omega\to+\infty, \tag{3.4}$$

$$E_\kappa(z,\omega)=\frac{\Gamma(2\kappa+1)}{2^{\kappa+1}\Gamma(\kappa)}e^{-z\omega}(-z\omega)^{-\kappa-1}\big(1-\frac{1-\kappa^2}{2z\omega}+\mathcal{O}(\frac{1}{|z\omega|^2})\big),\quad z\omega\to-\infty \tag{3.5}$$

Also, the next properties of the Dunkl kernel and the confluent hypergeometric function are very useful in the sequel

$$|\partial_x^\alpha E_\kappa(x,y)|\le|y|^\alpha E_\kappa(x,y),\quad x\in\mathbb{R},\ y\in\mathbb{C},\ \alpha\ \in\mathbb{Z}_+. \tag{3.6}$$

$$\frac{d^m}{dz^m}\ {}_1F_1(a,b,z)=\frac{(a)_m}{(b)_m}\ {}_1F_1(a+m,b+m,z),\quad m>0. \tag{3.7}$$

Now, we establish some estimates involving Dunkl kernel that we will needed in the following section. In the sequel we assume that $\mathbb{K}=\mathbb{K}_1\cup\mathbb{K}_2$, where $\mathbb{K}_1$ (resp. $\mathbb{K}_2$) is a compact subset of $]0,+\infty[$ (resp. $]-\infty,0[$) and we denote by $\mathbb{W}_t$ the classical heat kernel given by

$$\mathbb{W}_t(x,y)=\frac{e^{-|x-y|^2/4t}}{(4\pi t)^{d/2}},\qquad x,\ y\in\mathbb{R}^d,\ t>0.$$

Lemma 3.1. *Let $\kappa>0$. Then*

(i)

$$\Gamma_t^\kappa(u,v)\le C\begin{cases} t^{-\kappa-\frac12}e^{-(u^2+v^2)/4t}, & |uv|\le t;\\ |uv|^{-\kappa}\frac{e^{-(|u|-|v|)^2/4t}}{\sqrt{t}}, & |uv|>t.\end{cases}$$

(ii) $\Gamma_t^\kappa(u,v)\le Ct^{-\kappa-\frac12}e^{-v^2/20t}$, *$t>0$ and $2|u|<|v|<\infty$.*

(iii) $\Big|\Gamma_t^\kappa(u,v)-(uv)^{-\kappa}\mathbb{W}_t(u,v)\Big|\le C\sqrt{t}(uv)^{-\kappa-1}e^{-(u-v)^2/4t}$, *$uv>t>0$.*

Proof. Due to (3.4), (3.5) and $|E_\kappa(\frac{u}{\sqrt{2t}},\frac{v}{\sqrt{2t}})|\le C$ if $\frac{|uv|}{2t}<1$, we can deduce (i). Suppose now that $2|u|<|v|<\infty$ and $t>0$. If $|uv|\le t$, (i) implies (ii). Also if $|uv|>t$, by using (i) we can write

$$\Gamma_t^\kappa(u,v)\le C|uv|^{-\kappa-1/2}\sqrt{\frac{|uv|}{t}}e^{-v^2/16t}\le Ct^{-\kappa-\frac12}e^{-v^2/20t}$$

and (ii) is shown. If $uv>t$, then using (3.4) and the duplication formula we obtain

$$\Gamma_t^\kappa(u,v)=(uv)^{-\kappa}\mathbb{W}_t(u,v)\Big(1+\mathcal{O}(\frac{t}{uv})\Big).$$

Thus, (iii) is proved. □

Lemma 3.2. *Let $\kappa>0$ and $u,v\in\mathbb{K}$. Then*

(i)

$$\Big|\frac{\partial}{\partial u}\Gamma_t^\kappa(u,v)\Big|\le C\begin{cases} t^{-\kappa-\frac32}, & t\ge1;\\ \frac1t e^{-\frac{(|u|-|v|)^2}{8t}}, & 0<t<1.\end{cases}$$

(ii) $\Big|\frac{\partial^2}{\partial u^2}\Gamma_t^\kappa(u,v)\Big|\le Ct^{-\kappa-3/2}$, *$t\ge1$.*

(iii) $\left|\frac{\partial^2}{\partial u^2}\Gamma_t^\kappa(u,v)-(uv)^{-\kappa}\frac{\partial^2}{\partial u^2}\mathbb{W}_t(u,v)\right|\le C\frac{e^{-(u-v)^2/8t}}{t}$, $uv>0$ *and* $0<t<1$.

(iv) $\left|\frac{\partial^2}{\partial u^2}\Gamma_t^\kappa(u,v)\right|\le C\frac{e^{-(u+v)^2/8t}}{t}$, $uv<0$ *and* $0<t<1$.

Proof. Let $u,v\in\mathbb{R}$ and $t>0$, due to (3.6) we can write

$$\Big|\frac{\partial}{\partial u}\Gamma_t^\kappa(u,v)\Big|\le\frac{c_\kappa^{-1}}{(2t)^{\kappa+\frac12}}\Big(\frac{|u|}{2t}+\frac{|v|}{2t}\Big)E_\kappa(\frac{u}{\sqrt{2t}},\frac{v}{\sqrt{2t}})e^{-\frac{(u^2+v^2)}{4t}}.$$

According to (3.4), (3.5) and $|E_\kappa(\frac{u}{\sqrt{2t}},\frac{v}{\sqrt{2t}})|\le C$ if $\frac{|uv|}{2t}<1$ we deduce (i).

Due to (3.1) and (3.7), we are lead after simplifications to $\frac{\partial^2}{\partial u^2}\Gamma_t^\kappa(u,v)$

$$=\frac{\sqrt{2\pi}}{c_\kappa(2t)^\kappa}e^{-\frac{uv}{2t}}\Big\{-\frac{v^2}{4t^2}e^{\frac{uv}{2t}}.{}_1F_1(\kappa+2,2\kappa+3,-\frac{uv}{t})\mathbb{W}_t(u,v)$$
$$-\frac{2\kappa v}{t(\kappa+1)}e^{\frac{uv}{2t}}.{}_1F_1(\kappa+1,2\kappa+2,-\frac{uv}{t})\frac{\partial}{\partial u}\mathbb{W}_t(u,v)+E_\kappa(\frac{u}{\sqrt{2t}},\frac{v}{\sqrt{2t}})\frac{\partial^2}{\partial u^2}\mathbb{W}_t(u,v)\Big\},$$

to show (ii), we use that $E_\kappa(0,v)=\ {}_1F_1(a,b,0)=1$. In order to establish (iii) we estimate three different parts on the last equation using (3.3) and (3.4).

Thanks to (3.2), (3.5) and (3.7), we obtain, for $0<t\le1$, $-uv>t$ and $u,v\in\mathbb{K}$,

$$\Big|\frac{\partial^2}{\partial u^2}\Gamma_t^\kappa(u,v)\Big|\le C\Big(t\frac{\partial^2}{\partial u^2}\mathbb{W}_t(u,-v)+\frac{\partial}{\partial u}\mathbb{W}_t(u,-v)\Big)\le C\frac{e^{-(u+v)^2/8t}}{t}.$$

Then (iv) is proved. □

Lemma 3.3. *Let* $\kappa>0$. *Then*

(i)

$$\Big|\frac{\partial}{\partial t}\Gamma_t^\kappa(u,v)\Big|\le C\begin{cases}\frac{e^{-(u^2+v^2)/8t}}{t^{\kappa+\frac32}}, & |uv|\le t;\\ (uv)^{-\kappa}\frac{e^{-(u-v)^2/8t}}{t^{3/2}}, & uv>t;\\ (-uv)^{-\kappa-1}\frac{e^{-\frac{(u+v)^2}{8t}}}{\sqrt t}, & -uv>t.\end{cases}$$

(ii) $\Big|\frac{\partial}{\partial t}\Gamma_t^\kappa(u,v)\Big|\le C\frac{e^{-v^2/40t}}{t^{\kappa+\frac32}}$, $t>0$ *and* $0<2|u|<|v|<\infty$.

(iii) $\Big|\frac{\partial}{\partial t}\Gamma_t^\kappa(u,v)-(uv)^{-\kappa}\frac{\partial}{\partial t}\mathbb{W}_t(u,v)\Big|\le C(uv)^{-\kappa-1}\frac{e^{-(u-v)^2/8t}}{\sqrt t}$, $\quad uv>t$.

Proof. Due to (3.7) and using that

$$a.{}_1F_1(a+1,b,z)=a.{}_1F_1(a,b,z)+\frac{az}{b}.{}_1F_1(a+1,b+1,z),$$

we can write

$$\frac{\partial}{\partial t}\Gamma_t^\kappa(u,v)=c_\kappa^{-1}e^{-\frac{(u-v)^2}{4t}}\Big\{\big(\frac{(u-v)^2}{(2t)^{\kappa+5/2}}-\frac{1}{(2t)^{\kappa+3/2}}\big){}_1F_1(\kappa,2\kappa+1,-\frac{uv}{t})$$
$$-\frac{2\kappa}{(2t)^{\kappa+3/2}}.{}_1F_1(\kappa+1,2\kappa+1,-\frac{uv}{t})\Big\}.$$

Using that ${}_1F_1(a,b,0)=1$, then (i) is deduced for $|uv|\le t$. We now use (3.3) in order to obtain

$$\Big|\frac{\partial}{\partial t}\Gamma^\kappa_t(u,v)\Big|\le C\frac{e^{-\frac{(u-v)^2}{4t}}}{t^{3/2}}(uv)^{-\kappa}(1+\frac{(u-v)^2}{t})\le C\frac{e^{-\frac{(u-v)^2}{8t}}}{t^{3/2}}(uv)^{-\kappa},\ uv>t.$$

By similar arguments we can prove the third inequality of (i).

When $|uv|\le t$, (ii) can be inferred immediately from (i). Also, if $uv>t$, from (i) we deduce that for $0<2|u|<|v|<\infty$

$$\Big|\frac{\partial}{\partial t}\Gamma^\kappa_t(u,v)\Big|\le C\frac{e^{-v^2/32t}}{t}\Big(\frac{(uv)}{t}\Big)^{1/2}(uv)^{-\kappa-1/2}\le C\frac{e^{-v^2/32t}}{t^{\kappa+3/2}}\frac{|v|}{\sqrt{t}}\le C\frac{e^{-v^2/40t}}{t^{\kappa+3/2}}.$$

From (3.1) and (3.7), we can write $\frac{\partial}{\partial t}\Gamma^\kappa_t(u,v)$

$$=\sqrt{2\pi}c_\kappa^{-1}\frac{e^{-\frac{uv}{2t}}}{(2t)^\kappa}\Big\{\frac{\partial}{\partial t}\mathbb{W}_t(u,v)E_\kappa(\frac{u}{\sqrt{2t}},\frac{v}{\sqrt{2t}})$$
$$+\mathbb{W}_t(u,v)\Big(-\frac{2\kappa}{2t}E_\kappa(\frac{u}{\sqrt{2t}},\frac{v}{\sqrt{2t}})+\frac{4\kappa uve^{\frac{uv}{2t}}}{(2\kappa+1)(2t)^2}.{}_1F_1(\kappa+1,2\kappa+2,-\frac{uv}{t})\Big)\Big\}.$$

Using (3.3) and (3.4) we get that,

$$\frac{\partial}{\partial t}\Gamma^\kappa_t(u,v)=(uv)^{-\kappa}\frac{\partial}{\partial t}\mathbb{W}_t(u,v)+O\Big(t(uv)^{-\kappa-1}\frac{\partial}{\partial t}\mathbb{W}_t(u,v)\Big),\ uv>t.$$

It is not hard to see that

$$\Big|\frac{\partial}{\partial t}\mathbb{W}_t(u,v)\Big|\le\frac{C}{t^{3/2}}e^{-\frac{(u-v)^2}{8t}},$$

and (iii) easily follows. □

4. Proof of Theorem 1.1

In order to prove Theorem 1.1, we need to establish a pointwise integral representation for the Dunkl multiplier operator P_m as a principal value integral operator. In the sequel we assume that m satisfies (1.2).

Firstly we prove the following result.

Lemma 4.1. *Let $\kappa=(\kappa_1,...,\kappa_d)\in\mathbb{R}^d_+$ and $f\in\mathcal{C}^\infty_c(\mathbb{R}^d)$. Then*

$$P_mf(x)=\int_0^\infty\phi(t)\mathcal{F}_\kappa^{-1}\Big(|y|^2e^{-t|y|^2}\mathcal{F}_\kappa f(y)\Big)(x)dt,\quad a.e.\ x\in\mathbb{R}^d.$$

Proof. Assume that $d=1$, when $d\ge2$ we can proceed in a similar way. Let $g\in\mathcal{C}^\infty_c(\mathbb{R})$, by using Plancherel theorem we claim that

$$\int_{\mathbb{R}}P_m(f)(x)g(x)h_\kappa^2(x)dx=\int_{\mathbb{R}}|y|^2\int_0^\infty e^{-t|y|^2}\phi(t)dt\mathcal{F}_\kappa(f)(y)\mathcal{F}_\kappa^{-1}(g)(y)h^2_\kappa(y)dy$$
$$=\int_0^\infty\phi(t)\int_{\mathbb{R}}|y|^2e^{-t|y|^2}\mathcal{F}_\kappa(f)(y)\mathcal{F}_\kappa^{-1}(g)(y)h^2_\kappa(y)dy.$$

The interchange of the order of integration is justified by using Hölder inequality and that $\mathcal{F}_\kappa$ is an isometry in $L^2(\mathbb{R},h^2_\kappa)$. Plancherel theorem leads to

$$\int_{\mathbb{R}}P_mf(x)g(x)h^2_\kappa(x)dx=\int_0^\infty\phi(t)\int_{\mathbb{R}}\mathcal{F}_\kappa^{-1}\Big(|y|^2e^{-t|y|^2}\mathcal{F}_\kappa(f)(y)\Big)(x)g(x)h^2_\kappa(x)dx.$$

Since $|E_\kappa(ix,y)| \le C,\ |xy| \le 1$ and $|E_\kappa(ix,y)| \le C|xy|^{-\kappa},\ |xy| \in [1,+\infty[$, we can deduce that

$$\int_0^{1/|x|}\int_{1/|x|}^{\infty} \Big|E_\kappa(ix,y)\Big||y|^2 e^{-t|y|^2}\Big|\mathcal{F}_\kappa(f)(y)\Big|h_\kappa^2(y)dy$$
$$\le C\int_{\mathbb{R}} |y|^2 e^{-t|y|^2}\Big|\mathcal{F}_\kappa(f)(y)\Big||xy|^{-\kappa}h_\kappa^2(y)dy.$$

Then, since $|y|^l\mathcal{F}_\kappa f$ is bounded on $\mathbb{R}$, for every $l \in \mathbb{N}$, and $g \in \mathcal{C}_c^\infty(\mathbb{R})$ we get, after simplifications,

$$\int_0^\infty |\phi(t)|\int_{\mathbb{R}}\int_0^{1/|x|}\int_{1/|x|}^{\infty}\Big|E_\kappa(ix,y)\Big||y|^2e^{-t|y|^2}\Big|\mathcal{F}_\kappa(f)(y)\Big|h_\kappa^2(y)dy\Big|g(x)\Big|h_\kappa^2(x)dxdt$$
$$\le C\Big(\int_{\mathbb{R}}\Big|g(x)\Big||x|^{-\kappa}h_\kappa^2(x)dx\Big)\Big(\int_{\mathbb{R}}\Big|\mathcal{F}_\kappa(f)(y)\Big||y|^{-\kappa}h_\kappa^2(y)dy\Big) < \infty.$$

We conclude that $\int_{\mathbb{R}} P_m f(x)g(x)h_\kappa^2(x)dx$

$$= \int_{\mathbb{R}}\Big\{\int_0^\infty \phi(t)\mathcal{F}_\kappa^{-1}\Big(|y|^2e^{-t|y|^2}\mathcal{F}_\kappa f(y)\Big)(x)dt\Big\}g(x)h_\kappa^2(x)dx.$$

Thus, the proof of this lemma finishes. □

Proposition 4.2. *Let* $f \in \mathcal{C}_c^\infty(\mathbb{R}^d)$. *Then,*

$$P_m f(x) = -\lim_{\epsilon\to 0^+}\Big(\alpha(\epsilon)f(x) + \int_{|y-x|>\epsilon} f(y)K_\kappa^\phi(x,y)h_\kappa^2(y)dy\Big),\ a.e.\ x \in \mathbb{R}^d,$$

where

$$K_\kappa^\phi(x,y) = \int_0^\infty \phi(t)\frac{\partial}{\partial t}\Gamma_t^\kappa(x,y)dt,\ x,y \in \mathbb{R}^d,\ x \ne y,$$

and α *is a bounded function on* $\mathbb{R}_+$. *Moreover, if the limit* $\phi(0^+) = \lim_{t\to 0^+}\phi(t)$ *exists, then*

$$P_m f(x) = C\phi(0^+)f(x) - \lim_{\epsilon\to 0^+}\Big(\int_{|y-x|>\epsilon} f(y)K_\kappa^\phi(x,y)h_\kappa^2(y)dy\Big),\ a.e.\ x \in \mathbb{R}^d,$$

where C *is a positive constant.*

Proof. Assume that $d = 1$. When $d \ge 2$ we can proceed in a similar way. By Lemma 4.1 and (2.3), the Dunkl multiplier can be written as

$$P_m f(x) = \int_0^\infty \phi(t)\mathcal{F}_\kappa^{-1}\Big(e^{-t|y|^2}\mathcal{F}_\kappa(-\Delta_\kappa f)(y)\Big)(x)dt,\quad a.e.\ x \in \mathbb{R}. \tag{4.1}$$

By interchanging the order of integration and using (1.3), we can write

$$\mathcal{F}_\kappa^{-1}\Big(e^{-t|y|^2}\mathcal{F}_\kappa\big(-\Delta_\kappa f\big)(y)\Big)(x)$$
$$= c_\kappa^{-2}\int_{\mathbb{R}}\big(-\Delta_\kappa f\big)(z)\int_{\mathbb{R}} e^{-ty^2}E_\kappa(ix,y)E_\kappa(-iz,y)h_\kappa^2(y)dyh_\kappa^2(z)dz$$
$$= \int_{\mathbb{R}}\big(-\Delta_\kappa f\big)(z)\Gamma_t^\kappa(x,z)h_\kappa^2(z)dz,\quad t > 0. \tag{4.2}$$

Let $a>1$ such that $suppf \subset \mathbb{K}$, where $\mathbb{K}=[-a,-\frac{1}{a}]\bigcup[\frac{1}{a},a]$. Due to mean value Theorem and (3.6), we can assert that

$$\Big|\Gamma_t^\kappa(x,y)-\frac{t^{-(\kappa+\frac{1}{2})}}{c_\kappa 2^{\kappa+\frac{1}{2}}}\Big|\leq C\frac{1}{t^{\kappa+\frac{3}{2}}},\quad t\geq 1,\ y\in\mathbb{K}. \tag{4.3}$$

On the other hand, (2.3) yields

$$\int_{\mathbb{R}}(-\Delta_\kappa f)(z)h_\kappa^2(z)dz=\lim_{y\to 0}c_\kappa\mathcal{F}_\kappa(-\Delta_\kappa f)(y)=\lim_{y\to 0}c_\kappa|y|^2\mathcal{F}_\kappa f(y)=0. \tag{4.4}$$

According to (4.1), (4.2) and (4.4)(suggested by (4.3)), we are lead to

$$P_m f(x)=\int_0^\infty\phi(t)\int_{\mathbb{R}}(-\Delta_\kappa f)(z)\Big(\Gamma_t^\kappa(x,z)-\frac{\chi_{[1,\infty[}(t)t^{-(\kappa+\frac{1}{2})}}{c_\kappa 2^{\kappa+\frac{1}{2}}}\Big)h_\kappa^2(z)dzdt.$$

Using (1.4) and (4.3), we deduce that the last integral is absolutely convergent.

By (4) we can rewrite P_m as

$$P_m f(x)$$
$$=-\lim_{\epsilon\to 0}\int_0^\infty\phi(t)\int_{|z-x|>\epsilon}(\Delta_\kappa f)(z)\Big(\Gamma_t^\kappa(x,z)-\frac{\chi_{[1,\infty[}(t)t^{-(\kappa+\frac{1}{2})}}{c_\kappa 2^{\kappa+\frac{1}{2}}}\Big)h_\kappa^2(z)dzdt.$$

Assume that ϵ is small enough, for instance, $0<\epsilon<\frac{|x|}{2}$. We now analyze the integral

$$I^\epsilon(x,t)=\int_{\{z\in\mathbb{K}/|z-x|>\epsilon\}}(\Delta_\kappa f)(z)\Big(\Gamma_t^\kappa(x,z)-\frac{\chi_{[1,\infty[}(t)t^{-(\kappa+\frac{1}{2})}}{c_\kappa 2^{\kappa+\frac{1}{2}}}\Big)h_\kappa^2(z)dz.$$

We can write for $t>0$

$$I^\epsilon(x,t)=\Big(\int_{-\infty}^{x-\epsilon}+\int_{x+\epsilon}^{+\infty}\Big)\Delta_\kappa f(z)\Big(\Gamma_t^\kappa(x,z)-\frac{\chi_{[1,\infty[}(t)t^{-(\kappa+\frac{1}{2})}}{c_\kappa 2^{\kappa+\frac{1}{2}}}\Big)h_\kappa^2(z)dz. \tag{4.5}$$

By integration by parts, we obtain, for $t>0$,

$$\Big(\int_{-\infty}^{x-\epsilon}+\int_{x+\epsilon}^{+\infty}\Big)(\Delta_\kappa f)(z)\Big(\Gamma_t^\kappa(x,z)-\frac{\chi_{[1,\infty[}(t)t^{-(\kappa+\frac{1}{2})}}{c_\kappa 2^{\kappa+\frac{1}{2}}}\Big)h_\kappa^2(z)dz$$
$$=\Big(\int_{-\infty}^{x-\epsilon}+\int_{x+\epsilon}^{+\infty}\Big)f(z)(\Delta_\kappa\Gamma_t^\kappa)(x,z)h_\kappa^2(z)dz$$
$$+H_1(x,x-\epsilon,t)-H_1(x,x+\epsilon,t)-H_2(x,x-\epsilon,t)+H_2(x,x+\epsilon,t)$$
$$+\kappa\int_{|z-x|>\epsilon}\frac{f(-z)}{z^2}\Big(\Gamma_t^\kappa(x,z)-\frac{\chi_{[1,\infty[}(t)t^{-(\kappa+\frac{1}{2})}}{c_\kappa 2^{\kappa+\frac{1}{2}}}\Big)h_\kappa^2(z)dz$$
$$-\kappa\int_{|z-x|>\epsilon}\frac{f(z)}{z^2}\Big(\Gamma_t^\kappa(x,-z)-\frac{\chi_{[1,\infty[}(t)t^{-(\kappa+\frac{1}{2})}}{c_\kappa 2^{\kappa+\frac{1}{2}}}\Big)h_\kappa^2(z)dz,$$

where

$$H_1(x,z,t)=h_\kappa^2(z)\frac{\partial}{\partial z}f(z)\Big(\Gamma_t^\kappa(x,z)-\frac{\chi_{[1,\infty[}(t)t^{-(\kappa+\frac{1}{2})}}{c_\kappa 2^{\kappa+\frac{1}{2}}}\Big),$$

and

$$H_2(x,z,t) = h_\kappa^2(z)\frac{\partial}{\partial z}\Big(\Gamma_t^\kappa(x,z)\Big)f(z).$$

We note that

$$\lim_{\epsilon\to 0}\int_0^{+\infty}\phi(t)\int_{|z-x|>\epsilon}\frac{f(-z)}{z^2}\Big(\Gamma_t^\kappa(x,z)-\frac{\chi_{[1,\infty[}(t)t^{-(\kappa+\frac{1}{2})}}{c_\kappa 2^{\kappa+\frac{1}{2}}}\Big)h_\kappa^2(z)dz$$
$$=\lim_{\epsilon\to 0}\int_0^{+\infty}\phi(t)\int_{|z-x|>\epsilon}\frac{f(z)}{z^2}\Big(\Gamma_t^\kappa(x,-z)-\frac{\chi_{[1,\infty[}(t)t^{-(\kappa+\frac{1}{2})}}{c_\kappa 2^{\kappa+\frac{1}{2}}}\Big)h_\kappa^2(z)dz$$
$$=\int_0^{+\infty}\phi(t)\int_{\mathbb{R}}\frac{f(z)}{z^2}\Big(\Gamma_t^\kappa(x,-z)-\frac{\chi_{[1,\infty[}(t)t^{-(\kappa+\frac{1}{2})}}{c_\kappa 2^{\kappa+\frac{1}{2}}}\Big)h_\kappa^2(z)dz.$$

Using (1.4) and (4.3), we assert that the last integral is absolutely convergent.

Since $f\in\mathcal{C}_c^\infty(\mathbb{R})$ then by mean value theorem, Lemma 3.2 (i) and (4.3), we get

$$\int_1^\infty|\phi(t)|\Big|H_1(x,x-\epsilon,t)-H_1(x,x+\epsilon,t)\Big|dt\le C\epsilon\int_1^\infty\frac{1}{t^{\kappa+\frac{3}{2}}}dt\to 0,\ \epsilon\to 0^+.$$

Also, due to Lemmas 3.1 (i) and 3.2 (i), we deduce that for $0<t<1$,

$$\Big|H_1(x,x-\epsilon,t)-H_1(x,x+\epsilon,t)\Big|\le C\Big\{\frac{\epsilon}{\sqrt{t}}+\int_{x-\epsilon}^{x+\epsilon}\frac{1}{t}e^{-\frac{(|x|-|z|)^2}{8t}}dz\Big\}. \tag{4.6}$$

Using (4.6) it follows

$$\int_0^1|\phi(t)|\Big|H_1(x,x-\epsilon,t)-H_1(x,x+\epsilon,t)\Big|dt\to 0,\ \epsilon\to 0^+.$$

Due to Lemma 3.2 (i) and (ii) and mean value theorem we can assert

$$\int_1^\infty|\phi(t)|\Big|H_2(x,x-\epsilon,t)-H_2(x,x+\epsilon,t)\Big|dt\le C\epsilon\int_1^\infty\frac{1}{t^{\kappa+3/2}}dt\to 0,\ \epsilon\to 0^+.$$

By mean value theorem and Lemma 3.2 (i) it has

$$\int_0^1|\phi(t)|\Big|(x-\epsilon)^{2\kappa}f(x-\epsilon)-(x+\epsilon)^{2\kappa}f(x+\epsilon)\Big|\Big|\frac{\partial}{\partial z}\Gamma_t^\kappa(x,x+\epsilon)\Big|dt$$
$$\le C\epsilon\int_0^1\frac{e^{-\frac{\epsilon^2}{8t}}}{t}dt\le C\epsilon,$$

which goes to 0 when $\epsilon\to 0^+$. Also, we write for each $0<t<1$,

$$(x-\epsilon)^{2\kappa}f(x-\epsilon)\Big(\frac{\partial}{\partial z}\Gamma_t^\kappa(x,x-\epsilon)-\frac{\partial}{\partial z}\Gamma_t^\kappa(x,x+\epsilon)\Big)$$
$$=\Big((x-\epsilon)^{2\kappa}f(x-\epsilon)-x^{2\kappa}f(x)\Big)\Big(\frac{\partial}{\partial z}\Gamma_t^\kappa(x,x-\epsilon)-\frac{\partial}{\partial z}\Gamma_t^\kappa(x,x+\epsilon)\Big)$$
$$+x^{2\kappa}f(x)\Big(\frac{\partial}{\partial z}\Gamma_t^\kappa(x,x-\epsilon)-\frac{\partial}{\partial z}\Gamma_t^\kappa(x,x+\epsilon)\Big).$$

By proceeding as above and using Lemma 3.2 (i) we obtain

$$\int_0^1\phi(t)\Big|(x-\epsilon)^{2\kappa}f(x-\epsilon)-x^{2\kappa}f(x)\Big|\Big|\frac{\partial}{\partial z}\Gamma_t^\kappa(x,x-\epsilon)-\frac{\partial}{\partial z}\Gamma_t^\kappa(x,x+\epsilon)\Big|dt\to 0,\ \epsilon\to 0^+.$$

From the above estimates and using that $\Delta_\kappa \Gamma_t^\kappa(x,z) = \frac{\partial}{\partial t}\Gamma_t^\kappa(x,z)$, we conclude that

$$\int_0^\infty \phi(t)\int_{\mathbb{R}}(\Delta_\kappa f)(z)\Big(\Gamma_t^\kappa(x,z) - \frac{\chi_{[1,\infty[}(t)t^{-(\kappa+\frac{1}{2})}}{c_\kappa 2^{\kappa+\frac{1}{2}}}\Big)h_\kappa^2(z)dzdt$$

$$= \lim_{\epsilon\to 0^+}\Big[\int_0^\infty \phi(t)\int_{|z-x|>\epsilon} f(z)\frac{\partial}{\partial t}\Gamma_t^\kappa(x,z)h_\kappa^2(z)dz - x^{2\kappa}f(x)\int_0^1\phi(t)\Big(\int_{|x-z|<\epsilon, xz>0}\frac{\partial^2}{\partial z^2}\Gamma_t^\kappa(x,z)dt + \int_{|x-z|<\epsilon, xz<0}\frac{\partial^2}{\partial z^2}\Gamma_t^\kappa(x,z)dzdt\Big)\Big].$$

Using Lemma(3.2) (iv) we obtain

$$\lim_{\epsilon\to 0^+}\int_0^1|\phi(t)|\int_{|x-z|<\epsilon, xz<0}\Big|\frac{\partial^2}{\partial z^2}\Gamma_t^\kappa(x,z)\Big|dt = 0.$$

For $xz > 0$, using Lemma (iii) and proceeding as in [2] we conclude that

$$P_m(f)(x) = -\lim_{\epsilon\to 0^+}\Big[\int_0^\infty \phi(t)\int_{|x-z|>\epsilon} f(z)\frac{\partial}{\partial t}\Gamma_t^\kappa(x,z)h_\kappa^2(z)dzdt - f(x)\int_0^1\phi(t)\int_{|y|<\epsilon}\frac{\partial^2}{\partial y^2}\frac{e^{-|y|^2/4t}}{2\sqrt{\pi t}}dydt\Big]. \tag{4.7}$$

Suppose that $\phi(0^+) = \lim_{t\to 0}\phi(t)$, then by making changes of variables and using the dominated convergence theorem we have

$$\lim_{\epsilon\to 0^+}\int_0^1\phi(t)\int_{|y|<\epsilon}\frac{\partial^2}{\partial y^2}\frac{e^{-|y|^2/4t}}{2\sqrt{\pi t}}dydt = -M\phi(0^+). \tag{4.8}$$

where $M = \frac{1}{2\sqrt{\pi}}\int_0^{+\infty}\frac{e^{-1/4s}}{s^{3/2}}\,ds$.

Since $f \in \mathcal{C}_c^\infty(\mathbb{R})$ and thanks to Lemma 3.3 (i) it follows that, for every $\epsilon > 0$

$$\int_0^\infty|\phi(t)|\int_{|x-z|>\epsilon}|f(z)|\Big|\frac{\partial}{\partial t}\Gamma_t^\kappa(x,z)\Big|h_\kappa^2(z)dzdt < \infty.$$

Then, we can interchange the order of integration on the integrals in (4.7) and due to (4.8) we conclude that

$$P_m f(x) = -\lim_{\epsilon\to 0^+}\int_{|x-z|>\epsilon} f(z)\int_0^\infty\phi(t)\frac{\partial}{\partial t}\Gamma_t^\kappa(x,z)dt h_\kappa^2(z)dz + C\phi(0^+)f(x),$$

for a certain $C > 0$. □

In the rest of this section we analyze the L^p-boundedness properties for the maximal operator of the heat semigroup in the Dunkl setting.

Proposition 4.3. *Suppose that m is of Laplace transform type associated with $\phi \in L^\infty(\mathbb{R}_+)$. Then the maximal operator P_m^* defined by*

$$P_m^* f(x) = \sup_{\epsilon>0}\Big|\int_{|x-y|>\epsilon} f(y)K_\kappa^\phi(x,y)h_\kappa^2(y)dy\Big|, \quad x\in\mathbb{R}^d,$$

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is bounded from $L^p(\mathbb{R}^d, h_\kappa^2)$ into itself, for every $1 < p < \infty$, and from $L^1(\mathbb{R}^d, h_\kappa^2)$ into $L^{1,\infty}(\mathbb{R}^d, h_\kappa^2)$, K_κ^ϕ being as in Proposition 4.2.

Proof. Assume that $d = 1$, when $d \geq 2$ we can proceed analogously. Consider the operator

$$P^*_{m,loc} f(x) = \sup_{\epsilon>0} \Big| \int_{L(x),|x-y|>\epsilon} f(y)\chi_{\{xy>0\}}(xy)^{-\kappa} H^\phi(x,y) h_\kappa^2(y)dy \Big|, \quad x \in \mathbb{R},$$

where, for every $x \in \mathbb{R}$,

$$L(x) = \{y \in \mathbb{R}, \quad |x|/2 < |y| < 2|x|\}; \quad \text{and} \quad H^\phi(x,y) = \int_0^\infty \phi(t) \frac{\partial}{\partial t} \mathbb{W}_t(x,y)dt.$$

We can write

$$P^*_m f(x) \leq \mathcal{G}_\kappa(|f|)(x) + \mathcal{L}_\kappa(|f|)(x) + P^*_{m,loc} f(x), \quad x \in \mathbb{R},$$

where

$$\mathcal{G}_\kappa(|f|)(x) = \int_{\mathbb{R}\setminus L(x)} |f(y)| \Big| K_\kappa^\phi(x,y) \Big| h_\kappa^2(y)dy,$$

and

$$\mathcal{L}_\kappa(|f|)(x) = \int_{L(x)} |f(y)| \big| K_\kappa^\phi(x,y) - \chi_{\{xy>0\}}(xy)^{-\kappa} H^\phi(x,y) \big| h_\kappa^2(y)dy.$$

We are going to show the L^p-boundedness properties for the operators $\mathcal{G}_\kappa$, $\mathcal{L}_\kappa$ and $P^*_{m,loc}$.

4.1. The operator $P^*_{m,loc}$. For every $j \in \mathbb{Z}$, the dyadic interval $\mathcal{Q}_j$ is defined by

$$\mathcal{Q}_j = \{y \in \mathbb{R} : \quad 2^j \leq |y| < 2^{j+1}\},$$

and the interval $\widetilde{\mathcal{Q}_j}$ is given by

$$\widetilde{\mathcal{Q}_j} = \{y \in \mathbb{R} : \quad 2^{j-1} \leq |y| < 2^{j+2}\}.$$

It is clear that if $j \in \mathbb{Z}$, $x \in \mathcal{Q}_j$ and $y \in L(x)$, then $y \in \widetilde{\mathcal{Q}_j}$. We can write for $x \in \mathcal{Q}_j$, $j \in \mathbb{Z}$ and $\epsilon > 0$

$$\int_{L(x),|x-y|>\epsilon} f(y)\chi_{\{xy>0\}}(xy)^{-\kappa} H^\phi(x,y) h_\kappa^2(y)dy$$
$$= \Big(\int_{\widetilde{\mathcal{Q}_j},|x-y|>\epsilon} - \int_{\widetilde{\mathcal{Q}_j}\setminus L(x),|x-y|>\epsilon} \Big) f(y)\chi_{\{xy>0\}} (\frac{y}{x})^\kappa H^\phi(x,y)dy.$$

Let $j \in \mathbb{Z}$. It has $\widetilde{\mathcal{Q}_j} \setminus L(x) = \widetilde{\mathcal{Q}_j^+} \cup \widetilde{\mathcal{Q}_j^-}$, where

$$\widetilde{\mathcal{Q}_j^+} = \{y \in \mathbb{R} : \ 2|x| < |y| < 2^{j+2}\}$$

and

$$\widetilde{\mathcal{Q}_j^-} = \{y \in \mathbb{R} : \ 2^{j-1} < |y| < |x|/2\}.$$

For every $\epsilon > 0$, we get

$$\Big|\int_{\widetilde{\mathcal{Q}_j}\setminus L(x),|x-y|>\epsilon} f(y)\chi_{\{xy>0\}}(\frac{y}{x})^\kappa H^\phi(x,y)dy\Big|$$
$$\leq \int_{\widetilde{\mathcal{Q}_j}\setminus L(x)}\Big|f(y)\Big||\frac{y}{x}|^\kappa\Big|H^\phi(x,y)\Big|dy$$
$$\leq C\int_{\widetilde{\mathcal{Q}_j^+}\cup\widetilde{\mathcal{Q}_j^-}}\frac{|f(y)|}{|x-y|}dy,\ \ x\in\mathcal{Q}_j.$$

where we have used that

$$\big|H^\phi(x,y)\big|\leq C\frac{1}{|x-y|},\quad x,y\in\mathbb{R}.$$

Then, for each $x\in\mathcal{Q}_j$

$$\sup_{\epsilon>0}\Big|\int_{\substack{\widetilde{\mathcal{Q}_j}\setminus L(x)\\ |x-y|>\epsilon}} f(y)\chi_{\{xy>0\}}(\frac{y}{x})^\kappa H^\phi(x,y)dy\Big|\leq C\int_{\widetilde{\mathcal{Q}_j^+}\cup\widetilde{\mathcal{Q}_j^-}}\frac{|f(y)|}{|x-y|}dy. \tag{4.9}$$

The operator of the right hand side of (4.9) is bounded from $L^p(\mathbb{R},h_\kappa^2)$ into itself, $1<p<\infty$ and from $L^1(\mathbb{R},h_\kappa^2)$ into $L^{1,\infty}(\mathbb{R},h_\kappa^2)$. On the other hand we introduce the following maximal operator $T^{m,*}$ defined by

$$T^{m,*}f(x)=\sup_{\epsilon>0}\Big|\int_{|x-y|>\epsilon}f(y)H^\phi(x,y)dy\Big|,\quad x\in\mathbb{R}.$$

By proceeding in a similar way as in [2] and using the fact that $T^{m,*}$ is bounded from $L^p(\mathbb{R},dx)$ into itself, $1<p<\infty$, and from $L^1(\mathbb{R},dx)$ into $L^{1,\infty}(\mathbb{R},dx)$, we can see that $P^*_{m,loc}$ is bounded from $L^p(\mathbb{R},h_\kappa^2)$ into itself, $1<p<\infty$ and from $L^1(\mathbb{R},h_\kappa^2)$ into $L^{1,\infty}(\mathbb{R},h_\kappa^2)$.

4.2. The operator $\mathcal{L}_\kappa$. We have that

$$\begin{aligned}&\mathcal{L}_\kappa(|f|)(x)\\ &\leq C\int_{L(x)}|f(y)|\int_0^\infty\Big|\frac{\partial}{\partial t}\Gamma_t^\kappa(x,y)-\chi_{\{xy>0\}}(xy)^{-\kappa}\frac{\partial}{\partial t}\mathbb{W}_t(x,y)\Big|h_\kappa^2(y)dtdy\\ &\leq\mathcal{L}_\kappa^1(f)(x)+\mathcal{L}_\kappa^2(f)(x),\quad x\in\mathbb{R},\end{aligned}\tag{4.10}$$

where

$$\mathcal{L}_\kappa^1(f)(x)=\int_{L(x)}|f(y)|\int_0^\infty\chi_{\{xy>0\}}\Big|\frac{\partial}{\partial t}\Gamma_t^\kappa(x,y)-(xy)^{-\kappa}\frac{\partial}{\partial t}\mathbb{W}_t(x,y)\Big|dth_\kappa^2(y)dy,$$

and

$$\mathcal{L}_\kappa^2(f)(x)=\int_{L(x)}|f(y)|\int_0^\infty\chi_{\{xy<0\}}\Big|\frac{\partial}{\partial t}\Gamma_t^\kappa(x,y)\Big|dth_\kappa^2(y)dy.$$

Due to Lemma 3.3 (i) and (iii), we obtain the following

$$\begin{aligned}\mathcal{L}_\kappa^1(f)(x)\leq\int_{L(x)}|f(y)|\Big\{&\int_0^{xy}\frac{(xy)^{-\kappa-1}}{\sqrt{t}}e^{-\frac{(x-y)^2}{16t}}dt\\ &+\int_{xy}^\infty\Big(\frac{1}{t^{\kappa+3/2}}+(xy)^{-\kappa}\frac{1}{t^{3/2}}\Big)dt\Big\}h_\kappa^2(y)dy,\end{aligned}$$

and

$$\mathcal{L}_\kappa^2(f)(x) \le \int_{L(x)} |f(y)| \Big\{ \int_0^{|xy|} \frac{|xy|^{-\kappa-1}}{\sqrt{t}} e^{-\frac{(|x|-|y|)^2}{8t}} dt + \int_{|xy|}^{\infty} \frac{1}{t^{\kappa+3/2}} dt \Big\} h_\kappa^2(y) dy.$$

Then, the operators $\mathcal{L}_\kappa^1$ and $\mathcal{L}_\kappa^2$ are controlled by an operators of the following type

$$\Lambda_\kappa g(x) = \sup_{t>0} \Big| \int_{\frac{|x|}{2}}^{2|x|} |x|^{-2\kappa-1} |xy|^{-\kappa} \frac{e^{-\frac{(|x|-|y|)^2}{16t}}}{\sqrt{t}} g(y) h_\kappa^2(y) dy \Big|,$$

the operator Λ_κ is bounded from $L^p(\mathbb{R}, h_\kappa^2)$ into itself, $1 < p < \infty$, and from $L^1(\mathbb{R}, h_\kappa^2)$ into $L^{1,\infty}(\mathbb{R}, h_\kappa^2)$. Thus from (4.10), we conclude that $\mathcal{L}_\kappa$ is bounded from $L^p(\mathbb{R}, h_\kappa^2)$ into itself, $1 < p < \infty$, and from $L^1(\mathbb{R}, h_\kappa^2)$ into $L^{1,\infty}(\mathbb{R}, h_\kappa^2)$.

4.3. The operator $\mathcal{G}_\kappa$. The operator $\mathcal{G}_\kappa$ can be written as

$$\mathcal{G}_\kappa(|f|)(x) = \mathcal{G}_\kappa^1(f)(x) + \mathcal{G}_\kappa^2(f)(x), \tag{4.11}$$

where

$$\mathcal{G}_\kappa^1(f)(x) = \int_{2|x|}^{+\infty} |f(y)| \Big| K_\kappa^\phi(x,y) \Big| h_\kappa^2(y) dy$$

and

$$\mathcal{G}_\kappa^2(f)(x) = \int_0^{|x|/2} |f(y)| \Big| K_\kappa^\phi(x,y) \Big| h_\kappa^2(y) dy.$$

Thanks to Lemma 3.3 (ii), we infer

$$\begin{aligned} \Big| \mathcal{G}_\kappa^1(f)(x) \Big| &\le C \int_{2|x|}^{+\infty} |f(y)| \int_0^{+\infty} \frac{e^{-\frac{|y|^2}{40t}}}{t^{\kappa+3/2}} dt h_\kappa^2(y) dy \\ &\le C \int_{2|x|}^{+\infty} \frac{|f(y)|}{|y|^{2\kappa+1}} h_\kappa^2(y) dy \le C \mathcal{S}_k(f)(x), \quad x \in \mathbb{R}, \end{aligned}$$

where $\mathcal{S}_k(f)(x) = \displaystyle\int_{2|x|}^{+\infty} \frac{|f(y)|}{|y|} dy, \quad x \in \mathbb{R}.$

The operator $\mathcal{S}_k$ is a bounded operator from $L^p(\mathbb{R}, h_\kappa^2)$ into itself, $1 \le p < \infty$. Hence the operator $\mathcal{G}_\kappa^1$ satisfies the same boundedness properties.

By taking into account symmetries, Lemma 3.3 (ii), and using a change of variable we get

$$\begin{aligned} \Big| \mathcal{G}_\kappa^2(f)(x) \Big| &\le C \int_0^{|x|/2} |f(y)| \int_0^{+\infty} \frac{e^{-\frac{|x|^2}{40t}}}{t^{\kappa+3/2}} dt h_\kappa^2(y) dy \\ &\le \frac{C}{|x|^{2\kappa+1}} \int_0^{|x|/2} |f(y)| h_\kappa^2(y) dy. \end{aligned}$$

The operator $\mathbb{H}_\kappa$ given by

$$\mathbb{H}_\kappa(f)(x) = \frac{1}{|x|^{2\kappa+1}} \int_0^{|x|/2} |f(y)| h_\kappa^2(y) dy$$

is bounded from $L^p(\mathbb{R}, h_\kappa^2)$ into itself, when $1 < p < \infty$, and from $L^1(\mathbb{R}, h_\kappa^2)$ into $L^{1,\infty}(\mathbb{R}, h_\kappa^2)$. Then, from (4.11) the operator $\mathcal{G}_\kappa$ has the same L^p boundedness properties. Thus the proof of this Proposition is completed. □

From Proposition 4.3, we can deduce by using standard arguments the following result.

Proposition 4.4. *Let* $(\kappa_1, ..., \kappa_d) \in \mathbb{R}^d_+$. *Assume that* m *is of Laplace transform type associated with* $\phi \in L^\infty(\mathbb{R}_+)$. *For every* $f \in L^p(\mathbb{R}^d, h^2_\kappa)$, $1 \le p < \infty$, *the limit*

$$\lim_{\epsilon \to 0^+} \int_{|x-y|>\epsilon} f(y) K^\phi_\kappa(x, y) h^2_\kappa(y) dy,$$

exists, for almost all $x \in \mathbb{R}^d$. *Here* K^ϕ_κ *and* α *are defined as in Proposition 4.2.*

Moreover, the operator $\mathbb{P}_m$ *defined by*

$$\mathbb{P}_m f(x) = - \lim_{\epsilon \to 0^+} \int_{|x-y|>\epsilon} f(y) K^\phi_\kappa(x, y) h^2_\kappa(y) dy, \quad a.e.\ x \in \mathbb{R}^d,$$

is bounded from $L^p(\mathbb{R}^d, h^2_\kappa)$ *into itself, for every* $1 < p < \infty$, *and from* $L^1(\mathbb{R}^d, h^2_\kappa)$ *into* $L^{1,\infty}(\mathbb{R}^d, h^2_\kappa)$.

Since $\mathcal{C}^\infty_c(\mathbb{R}^d)$ is a dense subspace of $L^p(\mathbb{R}^d, h^2_\kappa)$, $1 \le p < \infty$, it follows that, for every $f \in L^2(\mathbb{R}^d, h^2_\kappa)$,

$$P_m f(x) = \lim_{\epsilon \to 0^+} \Big(\alpha(\epsilon) f(x) - \int_{|y-x|>\epsilon} f(y) K^\phi_\kappa(x, y) h^2_\kappa(y) dy \Big), \ a.e.\ x \in \mathbb{R}^d,$$

where α is a bounded function on $\mathbb{R}_+$, and P_m can be extended from $L^2(\mathbb{R}^d, h^2_\kappa) \cap L^p(\mathbb{R}^d, h^2_\kappa)$ to $L^p(\mathbb{R}^d, h^2_\kappa)$ as a bounded operator from $L^p(\mathbb{R}^d, h^2_\kappa)$ into itself, for every $1 < p < \infty$, and from $L^1(\mathbb{R}^d, h^2_\kappa)$ into $L^{1,\infty}(\mathbb{R}^d, h^2_\kappa)$. The proof of Theorem 1.1 is finished.

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