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Some Results in Group Structure on the Set of Graph

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Abstract: Graph theory is a young but rapidly maturing subject. It is one of the important areas in mathematics and computer science. The structure of objects or technologies has to new inventions and modifications in the environment for enhancement those fields. In this paper we show that how the group theoretical properties can be applied on simple graphs and also discussed the properties of cut set and circuit spaces related to simple graph.

Keywords: Simple graph, group, symmetric difference, set of edges and vertices.

1. INTRODUCTION

Algebraic graph theory is a new interesting subject concerned with the relationship between algebra and graph theory. Algebraic properties can be used to give surprising and elegant proofs of graph theoretic facts, and there are so many algebraic objects associated with graphs. Algebraic graph theory uses linear algebra, group theory and other parts of algebra for investigation of graphs. Graph theory ideas are highly utilized by engineering applications. Especially in research areas such as image segmentation, image capturing, networking etc. A data structure can be designed in the form of tree which utilized vertices and edges. Similarly modeling of the network topologies can be completed using graph concepts. Similarly the concept of graph coloring is utilized in resource allocation, scheduling. Also walks, path and circuits are used in many applications say traveling salesman problem, database design concepts and resource networking. The wide scope of these and other applications has been well-documented in [1], [2]. In this paper, we present a few selected applications of graph theory to various fields in general. As well known the first paper on graph theory was written by L. Euler. The motivation for Euler's paper was the celebrated Konigsberg bridge problem which rose in Konigsberg.

L. Euler posed the following problem: can we find a walk through the city that use each bridge once and only once time, and start and ends at the same point? He published a scientific article where he showed that this was not possible.

After L. Euler, Cayley studied a class of graphs, says trees. A tree is a graph which has only one path between two vertices. Cayley solved a problem from differential calculus. His solution exert an influenced to development of theoretical chemistry. The technique he used focused on listing all graphs which had certain properties. Now a day is called Enumerative graph theory. Publications presented by Polya and De Bruijn. Cayley linked his results on trees for studies of chemical composition.

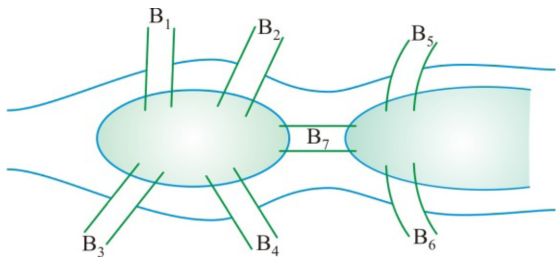

Figure 1: Konigsberg Seven Bridge Problem

Last 30 years was a period of intensive period of activity in graph theory. A great research has been done and is being done in graph theory. Algebraic graph theory is the most demanding field these days. Network graph theory is being used for systems biology [5], [3] and graphs. Godsil and Royle [4] and Chung [5] are good references of explore mappings from graph to graph, or Isomorphism and homomorphism.

There are so many papers on assigning a graph to a group, ring and of algebraic properties of group or ring using the associated graph, for instance, see [6], [2] [10], [9], [8]. Generally, there is an relationship between group theory and graph theory, and in many areas the properties of graphs give rise to some properties of groups and vice versa and concept of solvable graph for a finite group G, presented by Abe and Iiyori in [7].

The aim of this paper is to motivate and introduce the basic constructions and results, which have been developed in the group theory to graphs. Symmetric difference is used as binary relation on the set of vertices or edges of the graph.

2. GROUP REPRESENTATION FOR A GRAPH

To every graph we can always associate edge set and vertices set. By defining proper binary operation for these sets a group can be associated with every simple graph. It has been proved by the result below.

Theorem 1: If $G(V, E)$ is simple graph with edge set E and power set $P(E)$ then (P, Δ) denotes group for graph $G(V, E)$.

Proof:

(I) ***P*(*E*), is closure with symmetric difference** Δ**:** Let $E_1^*, E_2^* \in P(E)$ then $E_1^* \Delta E_2^* \in P(E)$

As $E_1^* - E_2^* \subset E$ & $E_2^* - E_1^* \subset E$ hence $(E_1^* - E_2^*) \cup (E_2^* - E_1^*) \subset E$

Thus $E_1^* \Delta E_2^* = (E_1^* - E_2^*) \cup (E_2^* - E_1^*) \in P(E)$

Hence $P(E)$, is satisfied closure under symmetric difference operation Δ.

(II) ***P*(*E*) is associative with symmetric difference** Δ**:** Let $E_1^*, E_2^*, E_3^* \in P(E)$ then $E_1^* \Delta(E_2^* \Delta E_3^*) = (E_1^* \Delta E_2^*)\Delta E_3^*$

To show this property we will take by the help of Venn's diagram.

$$E_1^* \Delta(E_2^* \Delta E_3^*) = \{E_1^* - (E_2^* - E_3^*) \cup (E_3^* - E_2^*)\} \cup \{(E_2^* - E_3^*) \cup (E_3^* - E_2^*) - E_1^*\}$$

$$(E_1^* \Delta E_2^*)\Delta E_3^* = \{(E_1^* - E_2^*) \cup (E_2^* - E_1^*) - E_3^*\} \cup \{E_3^* - (E_1^* - E_2^*) \cup (E_2^* - E_1^*)\}$$

Venn diagram for $(E_2^* - E_3^*) \cup (E_3^* - E_2^*)$ is shown in Figure 2.

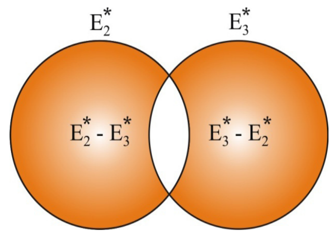

Figure 2: $(E_2^* - E_3^*) \cup (E_3^* - E_2^*)$

$(E_1^* - (E_2^* - E_3^*) \cup (E_3^* - E_2^*))$ is shown by orange color in Figure 3.

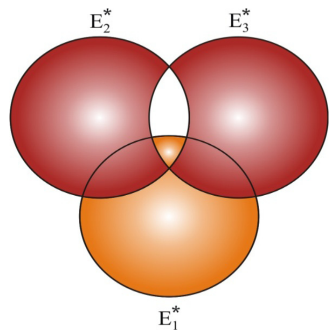

Figure 3: $\{E_1^* - (E_2^* - E_3^*) \cup (E_3^* - E_2^*)\}$

The shaded region in Figure 4 is representing $((E_2^* - E_3^*) \cup (E_3^* - E_2^*) - E_1^*)$

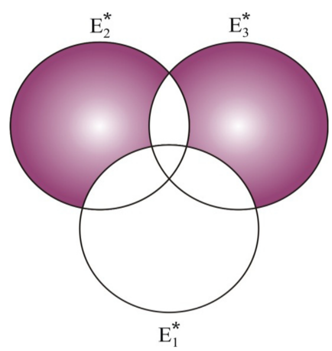

Figure 4: $((E_2^* - E_3^*) \cup (E_3^* - E_2^*) - E_1^*)$

Then shade region by colors orange and brown is representing $E_1^* \Delta(E_2^* \Delta E_3^*) = \{E_1^* - (E_2^* - E_3^*) \cup (E_3^* - E_2^*)\} \cup \{(E_2^* - E_3^*) \cup (E_3^* - E_2^*) - E_1^*\}$ in Figure 5.

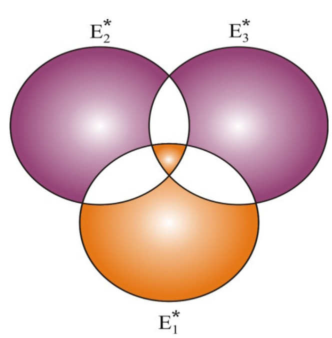

Figure 5: $E_1^* \Delta(E_2^* \Delta E_3^*)$

Now for the second case $(E_1^* - E_2^*) \cup (E_2^* - E_1^*)$ is shown in Figure 6.

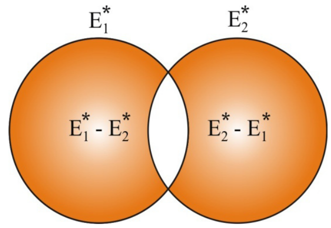

Figure 6: $(E_1^* - E_2^*) \cup (E_2^* - E_1^*)$

$(E_3^* - (E_1^* - E_2^*) \cup (E_2^* - E_1^*)$ is shown by orange color in Figure 7.

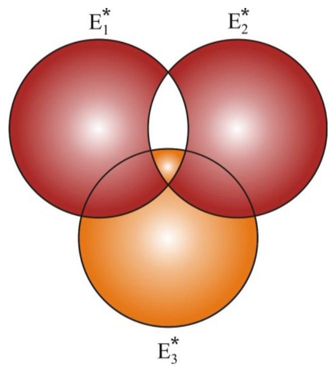

Figure 7: $(E_3^* - (E_1^* - E_2^*) \cup (E_2^* - E_1^*)$

$((E_1^* - E_2^*) \cup (E_2^* - E_1^*) - E_3^*)$ is shown by shaded region in Figure 8.

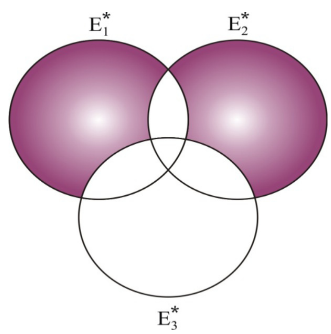

Figure 8: $((E_1^* - E_2^*) \cup (E_2^* - E_1^*) - E_3^*)$

Thus, $(E_1^* \Delta E_2^*)\Delta E_3^* = \{(E_1^* - E_2^*) \cup (E_2^* - E_1^*) - E_3^*\} \cup \{E_3^* - (E_1^* - E_2^*) \cup (E_2^* - E_1^*)\}$ is the shaded region by orange and brown colors in Figure 9.

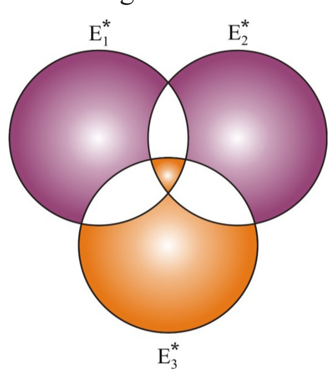

Figure 9: $(E_1^* \Delta E_2^*)\Delta E_3^*$

Hence by Figure 5 and Figure 9 it clear that shaded region of both Figures are same thus $E_1^* \Delta(E_2^* \Delta E_3^*) = (E_1^* \Delta E_2^*)\Delta E_3^*$.

(III) ***P(E)* has an identity:** We have $\phi \in P(E)$. If $E^* \in P(E)$ then we have $\phi\Delta E^* = E^*\Delta\phi = E^*$

As
$$\phi\Delta E^* = (\phi - E^*) \cup (E^* - \phi)$$
$$= \phi \cup E^* = E^*$$

Similarly $E^*\Delta\phi = E^*$

Thus ϕ is an identity element.

(IV) **Existence of inverse:** Let $E_1^*, E_2^* \in P(E)$ if $E_1^* \Delta E_2^* = \phi = E_2^* \Delta E_1^*$ then E_1^* is the inverse of E_2^* and E_2^* is the inverse of E_1^*. In P (E) it is possible only if $E_1^* = E_2^*$.

i.e.,
$$E_1^* \Delta E_2^* = (E_1^* - E_1^*) \cup (E_1^* - E) = \phi \cup \phi = \phi$$

Hence every element of P (E) is inverse of itself.

Thus by the above properties (I) to (IV) we can say that *P(E)* is a group with respect to symmetric difference.

Theorem 2: Consider a simple graph $G(V, E)$ and let W_G be the power set of edge set E. Let $\otimes$ be a multiplication operation between the elements of $GF(2)$ and those of W_G defined as, $1 \otimes E_i = E_i$ and $0 \otimes E_i = \phi$ for $E_i \in W_G$. Then W_G will be a vector space over $GF(2)$.

Proof:

(1) (W_G, Δ) is an abelian group, which has already been proved in the theorem 1.

(2) If $\alpha, \beta \in GF(2)$ and $u, w \in W_G$, then vector addition 'Δ' and scalar multiplication $\otimes$ satisfy the following:

(i) $\alpha \otimes (u\Delta w) = (\alpha \otimes u)\Delta(\alpha \otimes w)$: Since $\alpha \in GF(2)$ then either $\alpha = 0$ or $\alpha = 1$. So when $\alpha = 0$ then $\alpha \otimes (u\Delta w) = 0 \otimes (u\Delta w) = \phi = (0 \otimes u)\Delta(0 \otimes w) = (\alpha \otimes u)\Delta(\alpha \otimes w)$ and if $\alpha = 1$ then $\alpha \otimes (u\Delta w) = 1 \otimes (u\Delta w) = (u\Delta w) = (1 \otimes u)\Delta(1 \otimes w) = (\alpha \otimes u)\Delta(\alpha \otimes w)$.

Hence, $\alpha \otimes (u\Delta w) = (\alpha \otimes u)\Delta(\alpha \otimes w)$

(ii) $(\alpha +_2 \beta) \otimes u = (\alpha \otimes u)\Delta(\beta \otimes u)$ Because $\alpha, \beta \in GF(2)$ then $(\alpha +_2 \beta) = 0$ or 1 by the addition table for $GF(2)$

Case I: When $(\alpha +_2 \beta) = 0$, then we have two possibilities: (a) $\alpha = 0, \beta = 0$ & (b) $\alpha = 1, \beta = 1$ in both case we have L.H.S. $= 0 \otimes u = \phi$. Now if $\alpha = 0, \beta = 0$ then R.H.S. $= (\alpha \otimes u)\Delta(\beta \otimes u) = (0 \otimes u)\Delta(0 \otimes u) = \phi\Delta\phi = \phi$ and if $\alpha = 1, \beta = 1$ then R.H.S. $= (\alpha \otimes u)\Delta(\beta \otimes u) = (1 \otimes u)\Delta(1 \otimes u) = u\Delta u = \phi$. Thus $(\alpha +_2 \beta) \otimes u = (\alpha \otimes u)\Delta(\beta \otimes u)$.

Case II: When $(\alpha +_2 \beta) = 1$ we have following two cases: (a) $\alpha = 0, \beta = 1$ & (b) $\alpha = 1, \beta = 0$ and in both case we have L.H.S. $= 1 \otimes u = u$. We have R.H.S. $= (\alpha \otimes u)\Delta(\beta \otimes u) = (0 \otimes u)\Delta(1 \otimes u) = \phi\Delta u = u$ for $\alpha = 0, \beta = 1$. And if $\alpha = 1, \beta = 0$ then R.H.S. $= (1 \otimes u)\Delta(0 \otimes u) = u\Delta\phi = u$. So for $(\alpha +_2 \beta) = 1$ we have $(\alpha +_2 \beta) \otimes u = (\alpha \otimes u)\Delta(\beta \otimes u)$.

(iii) $(\alpha +_2 \beta) \otimes u = \alpha \otimes (\beta \otimes u)$:

Case I: When $(\alpha \times_2 \beta) = 0$ then L.H.S. $= (\alpha \times_2 \beta) \otimes u = 0 \otimes u = \phi$. Further we have three possibilities for $\alpha, \beta \in GF(2)$ namely (a) $\alpha = 0, \beta = 0$, (b) $\alpha = 0, \beta = 1$ (c) $\alpha = 1, \beta = 1$. So in this case for all three possibilities when we will have R.H.S. = $\alpha \otimes (\beta \otimes u) = \phi$ proof is same as in (ii). Hence we have $(\alpha \times_2 \beta) \otimes u = \alpha \otimes (\beta \otimes u)$.

Case II: When $(\alpha \times_2 \beta) = 1$, it is only possible when is $\alpha = 1$, $\beta = 1$ and hence L.H.S. = $(\alpha \times_2 \beta) \otimes u = 1 \otimes u = u$. And R.H.S. = $\alpha \otimes (\beta \otimes u) = 1 \otimes (1 \otimes u) = 1 \otimes u = u$. So for $(\alpha \times_2 \beta) = 1$ we have $(\alpha \times_2 \beta) \otimes u = \alpha \otimes (\beta \otimes u)$.

(iv) $1 \otimes u = u$: By the definition of $\otimes$ it is trivial since 1 is the identity element of $GF(2)$.

Thus all the axioms for vector space are satisfied over $GF(2)$. So $(W_G, \Delta, \otimes)$ is vector space over $GF(2)$.

Theorem 4.3.2: For a graph $G(V, E)$ with n-edges, W_G is an n-dimensional vector space over $GF(2)$.

Proof: Let $G(V, E)$ is a graph with edge set $E = \{e_1, e_2, ..., e_n\}$. $W_G = P(E)$ (Power set of E) i.e. $W_G = P(E) = \{\phi, \{e_1\}, \{e_2\}, \{e_3\}, ..., \{e_n\}, \{e_1, e_2\}, \{e_1, e_3\}, ..., E\}$.

Then the subsets $\{e_1\}, \{e_2\}, ..., \{e_n\}$ will constitute a basis 'β' for W_G. Since any subset of E can be express as a linear combination of elements of 'β' also β is linearly independent. Thus, β is basis of W_G. Because β contains n-elements. Then $\dim(W_G) = n$.

Hence W_G is an n-dimensional vector space over GF(2) for a graph G with n-edges.

Theorem 3: W_e (set of all edge induced subgraph of G including null graph) is subspace of W_G.

Proof: Let W_G on vector addition '⊕' is defined by symmetric difference operation and external multiplication '⊗' is defined as $1 \otimes G_i = G_i$, $0 \otimes G_i = \phi$, where, G_i is any edge-induced subgraph of G.

Now to show that W_e is subspace of W_G it is sufficient to prove that W_e is closed with respect to scalar multiplication and vector addition.

W_e is closed with respect to scalar multiplication: If $\alpha \in GF(2)$ and $u \in W_e$ then $\alpha = 1 \Rightarrow \alpha \otimes u = \phi \in W_e$ and similarly $\alpha = 1 \Rightarrow \alpha \otimes u = u \in W_e$ i.e. in both cases $(\alpha \otimes u) \in W_e$. So W_e is closed with respect to scalar multiplication.

W_e is closed with respect to vector addition: Let $u, v \in W_e$ then $u \oplus v \in W_e$ where, $u \oplus v = (u - v) \cup (v - u)$ as $\otimes$ is defined by symmetric difference.

Since W_e is the set of all edge induced subgraphs. Then u and v are edge induced subgraphs. So $(u - v)$ and $(v - u)$ are also edge induced subgraphs.

$\Rightarrow$ $(u - v) \cup (v - u)$ is also edge induced subgraph.

$\Rightarrow$ $(u \oplus v)$ is edge induced subgraph.

So $(u \oplus v) \oplus W_e$. So W_e is closed with respect to $\oplus$

Hence W_e is subspace of W_G.

Theorem 4: W_c (set of all circuits and union of edge disjoint circuits of G including null graph) is a subspace of W_G in G.

Proof: We know that a graph can be expressed as the union of edge disjoint circuits if and only if every vertex in the graph is of even degree. So W_c is the set of all edge induced subgraphs of G, in which all vertices are of even degree.

Let $C_1, C_2 \in W_c$ i.e. C_1 and C_2 are edge induced subgraphs with even degree of all their vertices.

Now let $C_3 = C_1 \oplus C_2$

Consider any vertex v in C_3. Then $v \in C_1$ or $v \in C_2$.

Suppose λ_i, $(i = 1, 2, 3)$ be the set of edges incident on v in C_i and $|\lambda_i|$ denotes the number of edges in λ_i i.e. $|\lambda_i|$ is the degree of vertex v in C_i.

Now from above equation, we have

$$\lambda_3 = \lambda_1 \oplus \lambda_2$$

and

$$|\lambda_3| = |\lambda_1| + |\lambda_2| - 2\,|\lambda_1 \cap \lambda_2|$$

Now $|\lambda_3|$ is even. It is verified from the above equation because $|\lambda_1|$ and $|\lambda_2|$ are both even or we can say that, the degree of vertex v in C_3 is even. Since this should be true for all vertices in C_3. So C_3 belong to W_c. Hence W_c is a subspace of W_G in G.

3. CONCLUSIONS

From above discussion it is clear that for any simple graph can always associate a group. Thus every graph comes equipped with a group; the algebraic property which measures symmetry of the simple graph. The group action of a graph is useful for studying both the group and the graph with symmetric difference operation. We investigate the group axioms on the graph, and we study the simple graph by the properties of its group. This is an important mathematical method which produced various important mathematics results. It is shown that all the group theoretic properties can be studied for any simple graph also. So we can use these properties to have better understanding of graphs and hence it open new area for applications of graph.

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