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REGULAR AND FUNDAMENTAL RELATIONS ON AN *S*-HYPERSET

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1. INTRODUCTION

The hyperstructure theory was surfaced up in 1934 when Marty defined hypergroups ([6]), started up analysing their properties and applied them to groups, rational algebraic functions. Since then many researchers have studied in this field and developed it. Several papers have been published on the algebraic hyperstructures (for instance one can see [1], [2], [3], [4], [6], [7], [8]). A short review of the theory of hypergroups appears in [3]. Let S be a non-empty set. $P(S)$ denotes the set of all subsets of S. A hyper operation on S is a mapping $\circ : S \times S \mapsto P(S)$ written as $(a, b) \mapsto a \circ b$. The set S together with a hyper operation is called a hypergroupoid. A hypergroupoid $(S, \circ)$ is called a semihypergroup if

$$(a \circ b) \circ c = a \circ (b \circ c), \quad \text{for all} \quad a, b, c \in S,$$

where for any subset A of S

$$a \circ A = \begin{cases} \bigcup\limits_{b \in A} (a \circ b), & \text{if} \quad A \neq \emptyset \\ \emptyset, & \text{if} \quad A = \emptyset \end{cases}$$

and

$$A \circ a = \begin{cases} \bigcup\limits_{b \in A} (b \circ a), & \text{if} \quad A \neq \emptyset \\ \emptyset, & \text{if} \quad A = \emptyset \end{cases}$$

A semihypergroup $(S, \circ)$ is called a hypergroup if $x \circ S = S \circ x = S$, for all $x \in S$. Let S be a semihypergroup. Let R be an equivalence relation on S. Let $A, B \in P(S)$. We say that $A\overline{R}B$ if for each $a \in A$ there exists $b \in B$ such that aRb and for each $y \in B$ there exists $x \in A$ such that xRy. Also we say that $A\overline{\overline{R}}B$ if aRb, for all $a \in A$ and for all $b \in B$.

The equivalence relation R on S is said to be strongly regular if for all $a, b \in S$, aRb implies $(x \circ a)\overline{R}(x \circ b)$ and $(a \circ x)\overline{R}(b \circ x)$, for all $x \in S$.

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In [3] the author describes the smallest strongly regular equivalence relation β^* on S such that the quotient semihypergroup S/β^* becomes a semigroup. This relation is called the fundamental relation on S. The fundamental relation β^*_n is the transitive closure of $\beta = \{(p, q) \in S \times S : \text{for some } r_1, r_2, \ldots, r_n \in S \text{ we have } p, q \in \prod_{i=1} r_i\}$. In [1] it is also shown that if S is a hypergroup, then $\beta^* = \beta$.

Throughout the paper S denotes a monoid and Q denotes a non-empty set.

Definition 1.1: [5] A (left) action of S on Q is a function $f : S \times Q \to Q$ (usually denoted by $f(x, q) \to xq$), for all $x \in S$ and $q \in Q$.

Q is called an S-set if there exists an action of S on Q such that

(i) $(xy)\,q = x\,(yq)$

(ii) $1q = q$, for all $x, y \in S$ and $q \in Q$.

Definition 1.2: A (left) hyperaction of S on Q is a function $\cdot : S \times Q \mapsto P(Q)$ (usually denoted by $\cdot\,(x, q) \mapsto x \cdot q$), for all $x \in S$ and $q \in Q$.

Let $A \in P(Q)$ and $x \in S$. We define $x \cdot A \in P(Q)$ by

$$x \cdot A = \begin{cases} \bigcup\limits_{a \in A} (x \cdot a), & \text{if} \quad A \neq \emptyset \\ \emptyset, & \text{if} \quad A = \emptyset \end{cases}$$

Q is called an S-hyper-set if there exists an hyper action $\cdot$ of S on Q such that

(i) $(xy) \cdot q = x \cdot (y \cdot q)$

(ii) $1 \cdot q = \{q\}$, for all $x, y \in S$ and $q \in Q$.

Example: Let Q be a non-empty set. Let B_Q denote the set of all binary relations on Q. Let $\alpha, \beta \in B_Q$. Now B_Q is a semigroup where the binary operation o is defined by $\alpha \circ \beta = \{(x, y) \in Q \times Q : (x, z) \in \beta, (z, y) \in \alpha \text{ for some } z \in Q\}$. One can show that B_Q is semigroup with identity. We now define $\alpha \circ x \in P(Q)$ by $\alpha \circ x = \{y \in Q : (x, y) \in \alpha\}$ for all $\alpha \in B_Q$ and $x \in Q$. Clearly this is a hyper action of B_Q on Q and Q is a B_Q-hyper-set.

In [7] the authors have studied some properties of the S-hyper-sets. In this paper we want to find the smallest strongly regular equivalence relation on an S-hyper-set Q such that the quotient S-hyper-set Q/δ is just an S-set. Also we describe δ when Q is a connected S-hyper-set. Introducing the notions of complete part and complete closure of a subset of Q, we give another description of the smallest strongly regular equivalence relation. In the last section we have studied some properties of the hyper-homomorphism on an S-hyper-set.

2. REGULAR AND FUNDAMENTAL RELATION ON AN S-HYPER-SET

Let R be an equivalence relation on a S-hyper-set Q. Let $A, B \in P(Q)$. We say that $A\overline{R}B$ if for each $a \in A$ there exists $b \in B$ such that aRb and for each $y \in B$ there exists $x \in A$ such that xRy. Also we say that $A\overline{\overline{R}}B$ if aRb, for all $a \in A$ and for all $b \in B$.

The equivalence relation R on Q is said to be regular if for all $q, p \in Q$, qRp implies $(x \cdot q)\ \overline{R}\ (x \cdot p)$, for all $x \in S$.

The equivalence relation R on Q is said to be strongly regular if for all $q, p \in Q$, qRp implies $(x \cdot q)\,\overline{\overline{R}}\,(x \cdot p)$, for all $x \in S$.

Definition 2.1: An S-hyper-set Q is said to be connected if for all $p, q \in Q$ there exists $x \in S$ such that $p \in x \cdot q$.

Theorem 2.2: Let Q be an S-hyper-set and R be an equivalence relation on Q.

(i) If R is regular, then Q/R (the set of all equivalence classes modulo R) will be an S-hyper-set with respect to the hyper action

$$r \otimes \bar{x} = \{\bar{y} : y \in r \cdot x\}, \text{ for all } r \in S \text{ and } \bar{x} \in Q/R.$$

(ii) If $(Q/R, \otimes)$ is an S-hyper-set then R is regular.

(iii) If Q is connected, then $(Q/R, \otimes)$ is connected.

Proof: (i) Let $\bar{x} = \bar{z}$ and $\bar{p} \in r \otimes \bar{x}$. Then xRz and $p \in r \cdot x$. Now xRz implies $(r \cdot x)\,\overline{R}\,(r \cdot z)$. Therefore there exists $q \in r \cdot z$ such that pRq and so $\bar{p} = \bar{q}$. This implies $\bar{p} \in r \otimes \bar{z}$. Therefore $r \otimes \bar{x} \subseteq r \otimes \bar{z}$. Similarly we can show that $r \otimes \bar{z} \subseteq r \otimes \bar{x}$. Therefore the hyper product $r \otimes \bar{x} = \{\bar{y} : y \in r \cdot x\}$ is well-defined. Now we show that $(Q/R, \otimes)$ is an S-hyper-set. Let $r, s \in S$ and $\bar{x} \in Q/R$. Let $\bar{y} \in rs \otimes \bar{x}$, then $y \in rs \cdot x \Rightarrow y \in r \cdot (s \cdot x) \Rightarrow y \in r \cdot z$, for some $z \in s \cdot x \Rightarrow \bar{y} \in r \otimes \bar{z}$, where $\bar{z} \in s \otimes \bar{x} \Rightarrow \bar{y} \in r \otimes (s \otimes \bar{x})$. Therefore $rs \otimes \bar{x} \subseteq r \otimes (s \otimes \bar{x})$. Similarly we can show that $r \otimes (s \otimes \bar{x}) \subseteq rs \otimes \bar{x}$. Hence $rs \otimes \bar{x} = r \otimes (s \otimes \bar{x})$. Again $1 \otimes \bar{x} = \{\bar{y} : y \in 1 \cdot x = \{x\}\} = \{\bar{x}\}$. Therefore $(Q/R, \otimes)$ is an S-hyper-set.

(ii) $(Q/R, \otimes)$ is an S-hyper-set. Let pRq, then $\bar{p} = \bar{q}$. Let $x \in r \cdot p$, where $r \in S$, then $\bar{x} \in r \otimes \bar{p}$. This implies $\bar{x} \in r \otimes \bar{q}$ and so there exists $y \in r \cdot q$ such that $\bar{x} = \bar{y}$. Therefore $x \in r \cdot p$, implies there exists $y \in r \cdot q$ such that xRy. This implies that R is regular.

(iii) Q is connected. Let $\bar{p}, \bar{q} \in Q/R$, then there exists $x \in S$ such that $p \in x \cdot q$, implies $\bar{p} \in x \otimes \bar{q}$, implies Q/R is connected.

In the following theorem we show that if the relation R is a strongly regular equivalence relation on an S-hyper-set Q, then the S-hyper-set Q/R becomes an S-set.

Theorem 2.3: Let Q be an S-hyper-set and R be an strongly regular equivalence relation on Q.

(i) Q/R (the set of all equivalence classes modulo R) is an S-set with respect to the hyper action

$$r \otimes \bar{x} = \bar{y}, \forall y \in r \cdot x, \text{ for all } r \in S \text{ and } \bar{x} \in Q/R.$$

(ii) If Q is connected, then $(Q/R, \otimes)$ is a connected S-set.

Proof: (i) $r \otimes \bar{x} = \bar{y}, \forall y \in r \cdot x$.

Let $y, z \in r \cdot x$. Since xRx and R is strongly regular then $(r \cdot x)\overline{\overline{R}}(r \cdot x)$, implies yRz, implies $\bar{y} = \bar{z}$. Therefore $r \otimes \bar{x} = \bar{y}, \forall y \in r \cdot x$ is well-defined and so Q/R is an S-set.

(ii) Straightforward.

Definition 2.4: Let Q be an S-hyper-set. We define a relation β on Q by

$$p\beta q \Leftrightarrow \exists r \in S \text{ and } u \in Q \text{ such that } p, q \in r \cdot u$$

Clearly β is reflexive and symmetric. We denote the transitive closure of β by β^*.

Theorem 2.5: β^* is the smallest strongly regular equivalence relation on an S-hyper-set Q such that Q/β^* is an S-set.

Proof: β^* is an equivalence relation containing β. Let $p\beta^*q$. Then there exist an integer $n \in N$ and $p = x_0, x_1, ..., x_n = q \in Q$ such that $x_0\beta x_1\beta x_2\beta x_3\beta \ldots \beta x_{n-1}\beta x_n$. Therefore there exist $r_1, r_2, ..., r_n \in S$ and $u_1, u_2, ..., u_n \in Q$ such that $x_0, x_1 \in r_1 \cdot u_1, x_1, x_2 \in r_2 \cdot u_2, ..., x_{n-1}, x_n \in r_n \cdot u_n$. Let $r \in S$ and $a \in r \cdot p$. Then $a \in r \cdot (r_1 \cdot u_1)$ i.e. $a \in rr_1 \cdot u_1$. Let $a_1 \in r \cdot x_1$. Then $a_1 \in r \cdot (r_1 \cdot u_1)$ i.e., $a_1 \in rr_1 \cdot u_1$. These together imply $a\beta a_1$. Hence $a\beta^* a_1$. Similarly if $a_2 \in r \cdot x_2$ then $a_1\beta^* a_2$ and so on. Lastly, if $b \in r \cdot q$ then $a_{n-1}\beta^* b$. Therefore for all $a \in r \cdot p$ and $b \in r \cdot q$ we have $a\beta^* b$. This implies that β^* is strongly regular. Therefore by the above theorem Q/β^* is an S-set.

Let R be any strongly regular equivalence relation on Q. Then clearly Q/R is an S-set. Let $p\beta^*q$. Then there exist an integer $n \in N$ and $p = x_0, x_1, ..., x_n = q \in Q$ such that $x_0\beta x_1\beta x_2\beta x_3\beta \ldots \beta x_{n-1}\beta x_n$. Therefore there exist $r_1, r_2, ..., r_n \in S$ and $u_1, u_2, ..., u_n \in Q$ such that $x_0, x_1 \in r_1 \cdot u_1, x_1, x_2 \in r_2 \cdot u_2, ..., x_{n-1}, x_n \in r_n \cdot u_n$.

Since R is strongly regular equivalence relation on Q and $u_i R u_i$, for $i = 1, 2, ..., n$, therefore $(r_i \cdot u_i)\overline{\overline{R}}(r_i \cdot u_i)$, for $i = 1, 2, ..., n$. Therefore $x_{i-1}Rx_i$, for $i = 1, 2, ..., n$. This implies that $x_0 R x_n$ i.e. pRq. Hence $\beta^* \subseteq R$.

Therefore β^* is the smallest strongly regular equivalence relation on Q such that Q/β^* is an S-set.

The relation β^* is called the fundamental relation on Q.

Theorem 2.6: If Q is connected, then $\beta^* = \beta$.

Proof: Let the S-hyper-set Q be connected. Let $x\beta^*y$, then $\bar{x} = \bar{y}$. Since Q is connected, therefore Q/β^* is connected, there exists $r \in S$ such that $\bar{y} = r \cdot \bar{x}$. This implies $y \in r \cdot x$. Also $\bar{x} = r \cdot \bar{x}$, implies $x \in r \cdot x$. Hence $x\beta y$. Therefore $\beta^* = \beta$.

In the following example we show that if Q is not a connected set then β^* may not be equal to β.

Example 2.7: $A = \{a, b, c\}$ and $Q = \{p, q, r, t\}$. We define a mapping $\cdot : A \times Q \mapsto P(Q)$ by the following table

$\cdot$	p	q	r	t
a	$\{p, q\}$	$\{q\}$	$\{q, r\}$	$\{t\}$
b	$\{q\}$	$\{q\}$	$\{q, r\}$	$\{t\}$
c	$\{p, q\}$	$\{q\}$	$\{q, r\}$	$\{t\}$

From the above table we can show that

$$\begin{array}{llll} a \cdot (a \cdot u) = c \cdot u & b \cdot (a \cdot u) = b \cdot u & c \cdot (a \cdot u) = a \cdot u & \\ a \cdot (b \cdot u) = b \cdot u & b \cdot (b \cdot u) = b \cdot u & c \cdot (b \cdot u) = b \cdot u & \text{for all } u \in Q. \\ a \cdot (c \cdot u) = a \cdot u & b \cdot (c \cdot u) = b \cdot u & c \cdot (c \cdot u) = c \cdot u & \end{array}$$

Let A^* be the free monoid generated by the set A. Now we extend the mapping $\cdot : \mathrm{A} \times Q \mapsto P(Q)$ to the mapping $\cdot : A^* \times Q \mapsto P(Q)$ by

$$\lambda \cdot u = \{u\},\ \lambda \text{ (empty word)} \in A^* \text{ and } \forall u \in Q$$

and

$$(xs) \cdot u = x \cdot (s \cdot u),\ \forall x \in A^*,\ s \in A,\ u \in Q$$

It can be easily verified that

$$(xy) \cdot u = x \cdot (y \cdot u), \forall x,\ y \in A^*,\ u \in Q$$

Therefore the set Q is an A^*-hyper-set.

There exists no $i \in A^*$ such that $p \in i \cdot r$. Therefore Q is not connected.

Now $p, q \in a \cdot p \Rightarrow p\beta q$ and $q, r \in a \cdot r \Rightarrow q\beta r$ together implies $(p, r) \in \beta^*$. But there exist no $i \in A^*$, $u \in Q$ such that $p, r \in i \cdot u$ that is $(p, r) \notin \beta$.

3. COMPLETE PART AND COMPLETE CLOSURE

In this section we describe β^* by the complete part of an S-hyper-set.

Definition 3.1: A subset A of an S-hyper-set Q is said to be a complete part of Q if

$$\forall (x, u) \in S \times Q, \quad x \cdot u \cap A \neq \emptyset \Rightarrow x \cdot u \subseteq A.$$

Theorem 3.2: If R be a strongly regular equivalence relation on a S-hyper-set Q, then $\bar{q}$, the equivalence class containing $q \in Q$ is a complete part of Q.

Proof: Let $r \cdot p \cap \bar{q} \neq \emptyset$. Then there exists $x \in r \cdot p$ such that $x \in \bar{q}$. Therefore $\bar{x} = \bar{q}$. Now for all $y \in r \cdot p$ we have $x\beta^* y$. This implies that xRy and so $\bar{x} = \bar{y} = \bar{q}$ that is $\bar{y} = \bar{q}$. This implies that $y \in \bar{q}$, for all $y \in r \cdot p$ and so $r \cdot p \subseteq q$.

Definition 3.3: Let A be a part of an S-hyper-set Q. Then the intersection of all complete parts of Q containing A is called the complete closure of A and it is denoted by $\mathcal{C}(A)$.

Definition 3.4: Let A be a part of an S-hyper-set Q. Then we define

$$K_1(A) = A,\ K_2(A) = \{x \in Q \mid \exists r_1 \in S, p_1 \in Q : x \in r_1 \cdot p_1, r_1 \cdot p_1 \cap K_1(A) \neq \emptyset\}, \ldots,$$

$$K_{n+1}(A) = \{x \in Q \mid \exists r_n \in S, p_n \in Q : x \in r_n \cdot p_n, r_n \cdot p_n \cap K_n(A) \neq \emptyset\}.\ K(A) = \bigcup_{n \geq 1} K_n(A).$$

Theorem 3.5: $K(A) = \mathcal{C}(A)$.

Proof: First we prove that $K(A)$ is complete. Let $r \cdot p \cap K(A) \neq \emptyset$. Then $r \cdot p \cap K_n(A) \neq \emptyset$, for some $n \in \mathbb{N}$. This implies that $r \cdot p \subseteq K_{n+1}(A) \neq \emptyset$. Hence $r \cdot p \subseteq K(A) \neq \emptyset$. This implies that $K(A)$ is complete.

Let $B \supset A$ and B be complete. Now $B \supset K_1(A)$. Assume that $B \supset K_n(A)$. Let $y \in K_{n+1}(A)$. Then there exist $r \in S$, $p \in Q$ such that $y \in r \cdot p$ and $r \cdot p \cap K_n(A) \neq \emptyset$. Let $x \in r \cdot p$ and $x \in K_n(A) \subseteq B$. Then $r \cdot p \cap B \neq \emptyset$. Therefore $r \cdot p \subseteq B$. This implies that $y \in B$. Hence $K_{n+1}(A) \subseteq B$ i.e., $K(A) \subseteq B$. Therefore $K(A) = \mathcal{C}(A)$.

Theorem 3.6: The relation $xKy \Leftrightarrow x \in \mathcal{C}(y)$ is an equivalence relation.

Proof: For all $x \in Q$, $x \in \mathcal{C}(x)$. Therefore xKx, $\forall x \in Q$. Let xKy and yKz. Then $x \in \mathcal{C}(y)$ and $y \in \mathcal{C}(z)$. Since $\mathcal{C}(z)$ is a complete part containing y, therefore $\mathcal{C}(y) \subseteq \mathcal{C}(z)$. This implies that $x \in \mathcal{C}(z)$ i.e. xKz. Therefore K is reflexive and transitive.

Now we have to show that $K_n(K_2(x)) = K_{n+1}(x)$, for all $n > 1$.

$$K_2(K_2(x)) = \{y \in Q : y \in r \cdot p \text{ and } r \cdot p \cap K_2(x) \neq \emptyset\} = K_3(x).$$

$$K_3(K_2(x)) = \{y \in Q : y \in r \cdot p \text{ and } r \cdot p \cap K_2(K_2(x)) \neq \emptyset\}$$

$$= \{y \in Q : y \in r \cdot p \text{ and } r \cdot p \cap K_3(x) \neq \emptyset\} = K_4(x).$$

Proceeding similarly we obtain $K_n(K_2(x)) = K_{n+1}(x)$.

Now we have to show that $x \in K_n(y) \Leftrightarrow y \in K_n(x)$, for all $n \in \mathbb{N}^*$.

If $x \in K_1(y)$, then $x = y$. Therefore $y \in K_1(x)$.

If $x \in K_2(y)$, then $x \in r \cdot p$ and $r \cdot p \cap K_1(y) \neq \emptyset \Rightarrow x \in r \cdot p, y \in r \cdot p \Rightarrow y \in r \cdot p$ and $r \cdot p \cap K_1(x) \neq \emptyset \Rightarrow y \in K_2(x)$.

If $x \in K_3(y)$, then $x \in r \cdot p$ and $r \cdot p \cap K_2(y) \neq \emptyset \Rightarrow \exists z \in r \cdot p$ such that $z \in K_2(y) \Rightarrow y \in K_2(z), z \in r \cdot p, r \cdot p \cap K_1(x) \neq \emptyset \Rightarrow y \in K_2(z), z \in K_2(x) \Rightarrow y \in K_2(K_2(x)) \Rightarrow y \in K_3(x)$.

Proceeding similarly we obtain $x \in K_n(y) \Rightarrow y \in K_n(x)$. Thus we obtain $x \in K_n(y) \Leftrightarrow y \in K_n(x)$.

Now $xKy \Rightarrow x \in \mathcal{C}(y) \Rightarrow x \in K(y)$

$\Rightarrow x \in K_n(y)$, for some $n \in \mathbb{N}^*$.

$\Rightarrow y \in K_n(x)$, for some $n \in \mathbb{N}^*$.

$\Rightarrow y \in K(x) = \mathcal{C}(x)$.

$\Rightarrow yKx$.

Therefore K is an equivalence relation.

Theorem 3.7: $xKy \Leftrightarrow x\beta^* y$.

Proof: $xKy \Rightarrow x \in \mathcal{C}(y) \Rightarrow x \in K(y) \Rightarrow x \in K_n(y)$, for some $n \in \mathbb{N}^*$. This implies $\exists r \in S, p \in Q$ such that $x \in r \cdot p$ and $r \cdot p \cap K_{n-1}(y) \neq \emptyset$. Let $x_1 \in r \cdot p$ and $x_1 \in K_{n-1}(y)$, implies that there exists $x_1 \in Q$ such that $x\beta x_1$ and $x_1 \in K_{n-1}(y)$. Proceeding similarly we obtain $x_1, x_2, \ldots, x_{n-1} \in Q$ such that $x\beta x_1 \beta x_2 \beta \ldots \beta x_{n-1}$ and $x_{n-1} \in K_1(y) = \{y\}$. This implies that $x\beta^* y$.

Therefore $xKy \Rightarrow x\beta^* y$.

Conversely, let $x\beta^* y$, then $x\beta^n y$, for some $n \in \mathbb{N}^*$. Then there exist $x_1, x_2, \ldots, x_{n-1} \in Q$ such that $x\beta x_1 \beta x_2 \beta \ldots \beta x_{n-1} \beta y$. Therefore there exist $r_1, r_2, \ldots, r_n \in S$ and $u_1, u_2, \ldots, u_n \in Q$ such that $x, x_1 \in r_1 \cdot u_1; x_1, x_2 \in r_2 \cdot u_2; \ldots; x_{n-1}, y \in r_n \cdot u_n$.

$\Rightarrow x \in r_1 \cdot u_1, r_1 \cdot u_1 \cap K_1(x_1) \neq \emptyset; x_1 \in r_2 \cdot u_2, r_2 \cdot u_2 \cap K_1(x_2) \neq \emptyset; \ldots;$

$x_{n-1} \in r_n \cdot u_n, r_n \cdot u_n \cap K_1(y) \neq \emptyset$.

$\Rightarrow x \in K_2(x_1), x_1 \in K_2(x_2), \ldots, x_{n-1} \in K_2(y)$.

$\Rightarrow x \in K(x_1), x_1 \in K(x_2), \ldots, x_{n-1} \in K(y)$.

$\Rightarrow x \in \mathcal{C}(x_1), x_1 \in \mathcal{C}(x_2), \ldots, x_{n-1} \in \mathcal{C}(y)$.

$\Rightarrow xKx_1, x_1Kx_2, \ldots, x_{n-1}Ky$.

$\Rightarrow xKy$.

Therefore $x\beta^* y \Leftrightarrow xKy$.

Theorem 3.8: If B is non-empty part of a S-hyper-set Q, then

$$\mathcal{C}(B) = \bigcup_{b \in B} \mathcal{C}(b).$$

Proof: $b \in B \Rightarrow \mathcal{C}(b) \subseteq \mathcal{C}(B) \Rightarrow \bigcup_{b \in B} \mathcal{C}(b) \subseteq \mathcal{C}(\mathrm{B})$.

Let $x \in \mathcal{C}(B) = K(B) \Rightarrow x \in K_n(B)$, for some $n \in \mathbb{N}^*$. Then there exist $x_1, x_2, ..., x_{n-1} \in Q$ such that xKx_{n-1}, where $x_{n-1} \in B$. This implies that $x \in \mathcal{C}(b)$, for some $b \in B \Rightarrow x \in \bigcup_{b \in B} \mathcal{C}(b)$.

Therefore $\mathcal{C}(B) = \bigcup_{b \in B} \mathcal{C}(b)$.

4. HYPER HOMOMORPHISM

Definition 4.1: Let P and Q be two S-hyper-sets. A mapping $f : P \mapsto Q$ is said to be a hyper homomorphism if

$$f(r \cdot p) = r \cdot f(p), \text{ for all } r \in S \text{ and } p \in P.$$

where $f(r \cdot p) = \{f(t) : t \in r \cdot p\}$.

If f is surjective then we say that f is an epimorphism. If f is bijective then we say that f is an isomorphism.

Definition 4.2: Let $f : P \mapsto Q$ be a mapping from a set P to a set Q and R be a relation on P. Then we define a relation $f(R)$ on Q by

$$f(R) = \{(f(q), f(q)) \in Q \times Q : (p, q) \in R\}.$$

Theorem 4.3: Let P and Q be two S-hyper-sets. Let $f : P \mapsto Q$ be an epimorphism and R be an equivalence relation on P, then

(i) $f(R)$ is also an equivalence relation on Q.

(ii) If R is regular, then $f(R)$ is also regular.

(iii) If R is strongly regular, then $f(R)$ is also strongly regular.

Proof: Here we prove only (iii).

Let $(q, q') \in f(R)$ and $r \in S$. Then there exists $(p, p') \in R$ such that $f(p) = q$ and $f(p') = q'$. Let $t \in r \cdot q$ and $t' \in r \cdot q'$. Then $t \in r \cdot f(p) = f(r \cdot p)$ and $t' \in r \cdot f(p') = f(r \cdot p')$. This implies that there exist $s, s' \in P$ such that $s \in r \cdot p$, $s' \in r \cdot p'$ and $f(s) = t, f(s') = t'$. Since R is strongly regular and $(p, p') \in R$, therefore $(r \cdot p) \overline{\overline{R}} (r \cdot p')$. Since $s \in r \cdot p, s' \in r \cdot p'$, therefore $(s, s') \in R$. This implies that $(f(\mathrm{s}), f(s')) \in f(R)$, where $f(s) \in r \cdot f(p), f(s') \in r \cdot f(p')$ i.e., $t \in r \cdot q$, $t' \in r \cdot q'$. Thus if $t \in r \cdot q$ and $t' \in r \cdot q'$, then $(t, t') \in f(R)$. Hence $f(R)$ is strongly regular.

Theorem 4.4: Let P and Q be two S-hyper-sets. Let $f : P \mapsto Q$ be an epimorphism and if β^* is the fundamental relation on P, then $f(\beta^*)$ is the fundamental relation on Q.

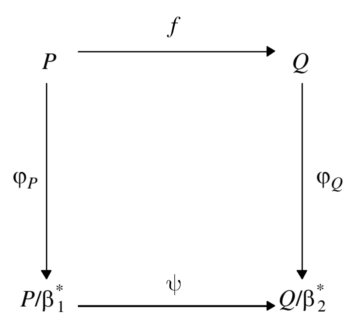
Proof: From the above theorem it follows that $f(\beta^*)$ is a strongly regular equivalence relation on Q. Let R be any strongly regular equivalence relation on Q and let $(q, q') \in f(\beta^*)$. Then there exists $(p, p') \in \beta^*$ such that $f(p) = q, f(p') = q'$. Now $(p, p') \in \beta^*$ implies that there exist $p = p_0, p_1, ..., p_n = p' \in P$ such that $(p_{i-1}, p_i) \in \beta$, $i = 1, 2, ..., n$. Therefore there exist $r_1, r_2, ..., r_n \in S$ and $u_1, u_2, ..., u_n \in P$ such that $p_0, p_1 \in r_1 \cdot u_1; p_1, p_2 \in r_2 \cdot u_2; ...; p_{n-1}, p_n \in r_n \cdot u_n$. This implies that $f(p_0), f(p_1) \in r_1 \cdot f(u_1); f(p_1), f(p_2) \in r_2 \cdot f(u_2); ...; f(p_{n-1}), f(p_n) \in r_n \cdot f(u_n)$. Since R is strongly regular on Q and $(f(u_i), f(u_i)) \in R$, for all $i = 1, \in, ..., n$, therefore $(r_i \cdot f(u_i)) \overline{\overline{R}} (r_i \cdot f(u_i))$, for all $i = 1, \in, ..., n$. This implies that $(f(p_{i-1}), f(p_i)) \in R$, for all $i = 1, 2, ..., n$ and so $(f(p_0), f(p_n)) \in R$ i.e., $(q, q') \in R$. Hence $f(?)$ is the smallest strongly regular equivalence relation on Q and so $f(?)$ is the fundamental relation on Q.

Theorem 4.5: Let P and Q be two S-hyper-sets. Let $f : P \mapsto Q$ be an epimorphism and if β^* is the fundamental relation on P, then $f(\beta^*) = (f(\beta))^*$.

Proof: Let $(q, q') \in f(\beta^*)$. Then there exists $(p, p') \in \beta^*$ such that $f(p) = q, f(p') = q'$. Now $(p, p') \in \beta^*$ implies that there exist $p = p_0, p_1, ..., p_n = p' \in P$ such that $(p_{i-1}, p_i) \in \beta$, $i = 1, 2, ..., n$. This implies that $(f(p_{i-1}), f(p_i)) \in f(\beta)$, for all $i = 1, 2, ..., n$. This implies that $(f(p_0), f(p_n)) \in (f(\beta))^*$ i.e., $(q, q') \in (f(\beta))^*$. Therefore $f(\beta^*)\ (f(\beta))^*$.

Conversely, assume that $(q, q') \in (f(\beta))^*$. Then there exist $q = q_0, q_1, ..., q_n = q' \in Q$ such that $(q_{i-1}, q_i) \in f(\beta)$, $i = 1, 2, ..., n$. Therefore there exist $p = p_0, p_1, ..., p_n = p' \in P$ such that $(p_{i-1}, p_i) \in \beta$, for all $i = 1, 2, ..., n$ and $f(p_j) = q_j$, for all $j = 0, 1, 2, ..., n$. This implies that $(p_0, p_n) \in \beta^*$. This implies that $(f(p_0), f(p_n)) \in f(\beta^*)$ i.e., $(q, q') \in f(\beta^*)$. Therefore $(f(\beta))^* \subseteq f(\beta^*)$. Hence $f(\beta^*) = (f(\beta))^*$.

Theorem 4.6: Let P and Q be two S-hyper-sets and $f : P \mapsto Q$ be an hyper homomorphism and β_1^* and β_2^* be the fundamental relations on P and Q respectively, then there is a homomorphism $\psi : P/\beta_1^* \mapsto Q/\beta_2^*$ such that the following diagram commutes



where φ_P and φ_Q are the natural homomorphisms. Moreover if f is an isomorphism then ψ is an isomorphism.

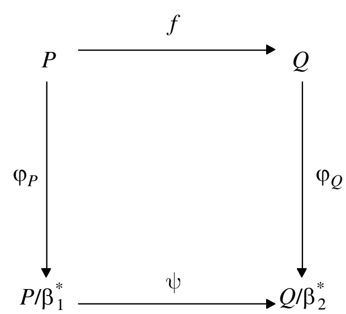
Proof: We define a mapping $\psi : P/\beta_1^* \mapsto Q/\beta_2^*$ by

$$\psi(p\beta_1^*) = f(p)\,\beta_2^*, \quad \text{for all} \quad p\beta_1^* \in P/\beta_1^*$$

First we have to show that the mapping ψ is well-defined. Let $\bar{x} = \bar{y}$. Then $x\beta_1^* y$. This implies that there exist $y_1, y_2, \ldots, y_{n-1} \in P$ such that $x\beta y_1\beta_1 y_2\beta_1 \ldots \beta_1 y_{n-1}\beta_1 y$. Therefore there exist $r_1, r_2, \ldots, r_n \in S$ and $u_1, u_2, \ldots, u_n \in P$ such that $x, y_1 \in r_1 \cdot u_1$; $y_1, y_2 \in r_2 \cdot u_2$; ...; $y_{n-1}, y \in r_n \cdot u_n$. This implies that $f(x), f(y_1) \in r_1 \cdot f(u_1)$; $f(y_1), f(y_2) \in r_2 \cdot f(u_2)$; ...; $f(y_{n-1})$, $f(y) \in r_n \cdot f(u_n)$. This implies that $f(x)\beta_2^* f(y)$ i.e., $\overline{f(x)} = \overline{f(y)}$. Therefore the mapping ψ is well-defined.

Let $\otimes$ and $\odot$ be the actions of S on P/β_1^* and Q/β_2^* respectively. Let $r \in S$ and $\bar{x} \in P/\beta_1^*$ and let $r \otimes \bar{x} = \bar{y}$, where $y \in r \cdot x$. We have $y \in r \cdot x$, therefore $f(y) \in r \cdot f(x)$ and so $r \odot \overline{f(x)} = \overline{f(y)}$. Therefore $\psi(r \otimes \bar{x}) = \psi(\bar{y}) = \overline{f(y)} = r \odot \overline{f(x)} = r \odot \psi(\bar{x})$. This implies that the mapping $\psi : P/\beta_1^* \mapsto Q/\beta_2^*$ is a homomorphism.

Let $x \in P$, then $f(x) \in Q$. Now $\psi\varphi_P(x) = \psi(\bar{x}) = \overline{f(x)} = \varphi_Q(f(x)) = \varphi_Q f(x)$, for all $x \in P$. Therefore the diagram



commutes.

Let f be an isomorphism. Now we show that ψ is one-one and onto.

Let $\psi(\bar{x}) = \psi(\bar{y})$, then $\overline{f(x)} = \overline{f(y)}$, implies that $f(x)\,\beta_2^*\,f(y)$. Therefore there exist $z_1, z_2, \ldots, z_n \in Q$ such that $f(x)\beta_2 z_1\beta_2 z_2\beta_2 \ldots \beta_2 z_n\beta_2 f(y)$. Since f is one-one and onto there exist $x_1, x_2, \ldots, x_n \in P$ such that $x\beta_1 x_1\beta_1 x_2\beta_1 \ldots. \beta_1 x_n\beta_1 y$, where $f(x_i) = z_i$, $i = 1, 2, \ldots, n$. This implies that $x\,\beta_1^*\,y$ and so $\bar{x} = \bar{y}$. This implies ψ is one-one. Also since f is onto ψ is onto.

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