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Modulus of Continuity & Approximation

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Abstract: The present paper altogether concentrates upon the study of the quantitative version of convergence of the partial sums $S_n(f, x)$ of the Fourier series to $f(x)$. In point of fact, we have essentially determined the rate at which $S_n(f, x)$ converge to $f(x)$ in terms of modulus of continuity.

1. Introduction and Main Result

It is well-known that the partial sums $S_n(f, x)$ of the Fourier series:

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) \tag{1.1}$$

of the function $f(x)$ converge to $f(x)$ subject to suitable conditions on $f(x)$ as $n \to \infty$.

As a matter of fact, we are interested here to concentrate upon the RATE at which $S_n(f, x)$ converge to $f(x)$. So far as the study of this rate of convergence is concerned, Stečkin[6] in 1952, Kominar[2] in 1961, Natanson[4] in 1972, Bojanić[1] in 1979 and Kumar and Dubey[3] in 1982 have thoroughly contemplated upon this aspect.

In order to stretch the above-mentioned thread of ideas; the main object of this paper is determine the rate of this convergence phenomenon in terms of modulus of continuity of the function $f(x)$.

More precisely, we solemnly wish to establish the following result:

Theorem: Let $\omega(t)$ be the modulus of continuity of a 2π-periodic continuous function defined over $[0, \pi]$. Then the approximation of $f(x)$ by the partial sums $S_n(f, x)$ of its Fourier series is given by

$$|S_n(f,x) - f(x)| \le \left\{\left(\frac{3}{2} - \frac{1}{n}\right)\omega\left(\pi/n\right) + \frac{1}{n}\sum_{k=1}^{n}\omega\left(\pi/k\right)\right\}. \tag{1.2}$$

2. Proof of the Theorem

We begin with the following formula as shown by Zygmund {[7], p. 50 (5.6) – ii}:

$$S_n(f, x) - f(x) = \frac{1}{\pi}\int_0^{\pi} \phi_x(t)\,\frac{\sin(n+1/2)t}{2\sin 1/2\,t}\,dt \tag{2.1}$$

where $\phi_x(t) = f(x+t) + f(x-t) - 2f(x)$. We further write

$$S_n(f, x) - f(x) = \frac{1}{\pi}\left(\int_0^{\pi/n} + \int_{\pi.n}^{\pi}\right)\phi_x(t)\,\frac{\sin(n+1/2)t}{2\sin 1/2\,t}\,dt = A + B, \text{ say.}$$

Let $\omega(t)$ be the modulus of continuity of the function $f(x)$; then it is well- known that

$$\left|\frac{1}{2}f(x+t) + \frac{1}{2}f(x-t) - f(x)\right| = |\phi_x(t)| \le \omega(t),$$

for the detailed study of this concept, the reference may be made Rivlin [5].

Now we proceed to estimate A.

By virtue of the fact that $\left|\dfrac{\sin(n+1/2)t}{\sin \frac{1}{2} t}\right| \le \left(n + \frac{1}{2}\right)$ we have, therefore

$$|A| \le \frac{1}{\pi}\int_0^{\pi/n} \omega(t)\left\{n + \tfrac{1}{2}\right\} dt$$

$$\le \frac{\left(n+\frac{1}{2}\right)}{\pi}\,\omega(\pi/n)\,\frac{\pi}{n} \le \left(1 + \frac{1}{2n}\right)\omega\left(\tfrac{\pi}{n}\right)$$

$$\le \frac{3}{2}\,\omega(\pi/n)\,. \tag{2.1}$$

Now we estimate B. Let us assume that

$$\mu_n(t) = \int_t^{\pi} \frac{\sin(n+1/2)y}{\sin \frac{1}{2} y}\,dy \quad \text{and} \quad |\mu_n(t)| \le \frac{\pi}{nt},$$

for $0 < t \le \pi$; for this estimate the reference may be made to Bojanić[1]. We have

$$B = \frac{1}{\pi}\int_{\pi.n}^{\pi} \phi_x(t)\,\frac{\sin(n+1/2)t}{2\sin 1/2\,t}\,dt\,.$$

On integrating by parts, we find that

$$B = \frac{1}{\pi}\left[\omega(t)\int_t^{\pi}\frac{\sin(n+1/2)y}{\sin \frac{1}{2} y}\,dy\right]_{\pi/n}^{\pi} - \frac{1}{\pi}\int_{\pi/n}^{\pi}\left\{\int_t^{\pi}\frac{\sin(n+1/2)y}{\sin \frac{1}{2} y}\,dy\right\}d\omega(t)$$

$$= \frac{1}{\pi}[\omega(t)\mu_n(t)]_{\pi/n}^{\pi} - \frac{1}{\pi}\int_{\pi/n}^{\pi}\mu_n(t)\,d\omega(t)$$

$$= \frac{1}{\pi}[\omega(\pi)\mu_n(\pi)-\omega(\pi/n)\mu_n(\pi/n)] - \frac{1}{\pi}[\mu_n(t)\omega(t)]_{\pi/n}^{\pi} + \frac{1}{\pi}\int_{\pi/n}^{\pi}\frac{d}{dt}\mu_n(t)\omega(t)dt$$

$$= \frac{1}{\pi}\int_{\pi/n}^{\pi}\frac{d}{dt}\mu_n(t)\omega(t)\,dt\,.$$

Therefore, we have

$$|B| \le \frac{1}{\pi}\int_{\pi/n}^{\pi}\frac{d}{dt}\left\{\frac{\pi}{nt}\right\}\omega(t)\,dt$$

$$\le \frac{1}{n}\int_{\pi/n}^{\pi}\frac{\omega(t)}{t^2}\,dt\,.$$

Substituting (π/t) for t, we find that

$$|B| \le \frac{1}{n}\int_1^{n}\omega(\pi/t)\,dt \le \frac{1}{n}\sum_{k=1}^{n-1}\int_k^{k+1}\omega(\pi/t)\,dt$$

$$\le \frac{1}{n}\sum_{k=1}^{n-1}\omega(\pi/k)\,. \qquad (2.2)$$

On adding the estimates (2.1) and (2.2), we obtain

$$|S_n(f,x)-f(x)| \le \left\{\frac{3}{2}\,\omega\left(\pi/n\right) + \frac{1}{n}\sum_{k=1}^{n-1}\omega\left(\pi/k\right)\right\}$$

$$\le \left\{\left(\frac{3}{2}-\frac{1}{n}\right)\omega\left(\pi/n\right) + \frac{1}{n}\sum_{k=1}^{n}\omega\left(\pi/k\right)\right\}.$$

This complete the proof of the Theorem.

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