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ON INVARIANT SUBMANIFOLDS OF A GENERALIZED KENMOTSU MANIFOLD SATISFYING CERTAIN CONDITIONS

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Abstract

In this paper, we have studied invariant submanifolds of generalized Kenmotsu manifold satisfying conditions like $Q(\sigma, R) = 0$, $Q(S, \sigma) = 0$, $Q(S, \tilde{\nabla}\sigma) = 0$, $Q(S, \tilde{R} \cdot \sigma) = 0$ *and* $Q(g, \tilde{R} \cdot \sigma) = 0$. *Also we have given an example to verify our results.*

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1. INTRODUCTION

The theory on invariant submanifolds of contact manifold was introduced by Yano and Ishihara in [14]. Later the same authors in [15] obtained the condition for an invariant submanifold to be totally geodesic. In [9], Kowalczyk studied some subclass of semisymmetric manifolds in which a semi-Riemannian manifold satisfying $Q(S, \tilde{R}) = 0$, $Q(S, g) = 0$, where $\tilde{R}$, S are the curvature tensor and Ricci tensor respectively. In 2014, De and Majhi studied (see [4]) the invariant submanifolds of Kenmotsu manifold satisfying conditions $Q(\sigma, \tilde{R}) = 0$ and $Q(S, \sigma) = 0$. Hu and Wang [7] also studied Invariant submanifolds of trans-Sasakian manifold satisfying $Q(S, \tilde{\nabla}\sigma) = 0$, $Q(S, \tilde{R} \cdot \sigma) = 0$ and $Q(g, \tilde{R} \cdot \sigma) = 0$. In 1972, Kenmotsu [8] studied a class of almost contact Riemannian manifold. Later Turgut Vanli and Sari [11] introduced and studied generalized Kenmotsu manifold. And in 2014 [12], the same authors studied invariant submanifolds of generalized Kenmotsu manifold.

In this paper we study invariant submanifolds of generalized Kenmotsu manifold satisfying $Q(\sigma, \tilde{R}) = 0$, $Q(S, \sigma) = 0$, $Q(S, \tilde{\nabla}\sigma) = 0$, $Q(S, \tilde{R} \cdot \sigma) = 0$ and

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$Q(g, \tilde{R} \cdot \sigma) = 0$. The paper is organized as follows: In section 2 we give some basic definitions and results. In third section we study invariant submanifolds of generalized Kenmotsu manifold satisfying the above mentioned conditions. In the last section we give an example for invariant submanifold of generalized Kenmotsu manifold which verifies our results.

2. PRELIMINARIES

A $(2n + s)$ dimensional manifold $\tilde{M}$ is said to be *f*-manifold if there exists an (1,1) type tensor field φ, 1-forms $\eta^1, \ldots, \eta^s$, vector fields $\xi^1, \ldots, \xi^s$ and Riemannian metric g on $\tilde{M}$ such that [11]

$$\varphi^2 = -I + \sum_{i=1}^{s} \eta^i \otimes \xi_i, \ \eta^i(\xi_j) = \phi_{ij}, \ \varphi\xi_i = 0 \tag{2.1}$$

$$g(\varphi X, \varphi Y) = g(X, Y) - \sum_{i=1}^{s} \eta^i(X)\,\eta^i(Y), \tag{2.2}$$

$$\eta^i(X) = g(X, \xi_i), \ g(X, \varphi Y) = -g(\varphi X, Y), \ \eta^i \circ \varphi = 0, \tag{2.3}$$

for any $X, Y \in TM$, $i, j \in \{1, \ldots, s\}$.

Definition 2.1: [11] Let $\tilde{M}$ be a s-contact metric manifold of dimension $(2n+s)$, $s \geq 1$, with structure $(\varphi, \xi_i, \eta^i, g)$. $\tilde{M}$ is said to be a generalized almost Kenmotsu manifold if for all $1 \leq i \leq s$, 1-forms η^i are closed and $d\Phi = 2\sum_{i=1}^{s} \eta^i \wedge \Phi$. A normal generalized almost Kenmotsu manifold $\tilde{M}$ is called a generalized Kenmotsu manifold.

Moreover, if $\tilde{M}$ is a $(2n+s)$-dimensional generalized Kenmotsu manifold with structure $(\varphi, \xi_i, \eta^j, g)$, then the following relations hold:

$$(\tilde{\nabla}_X \varphi)Y = \sum_{i=1}^{s} \left\{ g(\varphi X, Y)\xi_i - \eta^i(Y)\varphi X \right\}, \tag{2.4}$$

$$\tilde{\nabla}_X \xi^j = -\varphi^2 X, \tag{2.5}$$

$$\tilde{R}(X, Y)\,\xi^i = \sum_{j=1}^{s} \left\{ \eta^j(Y)\varphi^2 X - \eta^j(X)\varphi^2 Y \right\}, \tag{2.6}$$

$$\tilde{R}(X, \xi_j)\,\xi_i = \varphi^2 X, \tag{2.7}$$

$$\tilde{R}(\xi_j, X)\,\xi_i = -\varphi^2 X, \tag{2.8}$$

$$\tilde{R}(\xi_k, \xi_j)\,\xi_i = 0, \tag{2.9}$$

$$\tilde{S}(X, \xi_i) = -2n \sum_{j=1}^{s} \eta^j(X), \tag{2.10}$$

$$\tilde{S}(\xi_k, \xi_i) = -2n, \tag{2.11}$$

$$\tilde{S}(\varphi X, \varphi Y) = S(X, Y) + 2n \sum_{i=1}^{s} \eta^i(X)\,\eta^i(Y), \tag{2.12}$$

for all X, Y $\in$ TM, $i, j, k \in \{1, 2, \ldots, s\}$ [11]. Here $\tilde{\nabla}$, $\tilde{R}$ and $\tilde{S}$ are the Riemannian connection, curvature tensor and Ricci tensor on $\tilde{M}$ respectively.

Let M be a submanifold immersed in a $(2n + s)$-dimensional generalized Kenmotsu manifold $\tilde{M}$ with structure $(\varphi, \xi_i, \eta^i, g)$. Let Riemannian metric induced on M is denoted by the same symbol g. The connection induced on tangent bundle TM and normal bundle $T^{\perp}M$ of M are denoted by ∇ and $\nabla^{\perp}$ respectively. Then for any X, Y $\in$ TM and V $\in T^{\perp}M$, the Gauss and Weingarten formulas are given by

$$\tilde{\nabla}_X Y = \nabla_X Y + \sigma(X, Y), \tag{2.13}$$

$$\tilde{\nabla}_X V = -A_V X + \nabla^{\perp}_X V, \tag{2.14}$$

where, σ is the second fundamental form and A_V is the Weingarten map associated with V given by:

$$g(A_V X, Y) = g(\sigma(X, Y), V). \tag{2.15}$$

For any X $\in$ TM and V $\in T^{\perp}M$, we can write

$$\varphi X = TX + NX, \tag{2.16}$$

$$\varphi V = tV + nV, \tag{2.17}$$

where, TX and NX are the tangential and normal parts of φX respectively. Similarly tV and nV are the tangential and normal part of φV respectively.

In [13], a $(0, k + 2)$-type tensor field Q(E, T) on a Riemannian manifold $\tilde{M}$ is defined as follows:

$$Q(E, T)(X_1, X_2, \ldots, X_k; X, Y) = -T((X \wedge_E Y) X_1, X_2, \ldots, X_k) - T(X_1, (X \wedge_E Y) X_2, \ldots, X_k) - \ldots T(X_1, X_2, \ldots, X_{k-1}, (X \wedge_E Y)X_k), \tag{2.18}$$

where, T is a $(0, k)$-type tensor field $(k \geq 1)$, E is a (0, 2)-type tensor field and $(X \wedge_E Y)Z = E(Y, Z)X - E(X, Z)Y$. Moreover, for any vector fields X and Y tangent to a submanifold M of $\tilde{M}$ and a real valued function f on M, M is said to be pseudo-parallel [2] if

$$\tilde{R}(X, Y) \cdot \sigma = f Q(g, \sigma).$$

Similarly, a submanifold M is said to be 2-pseudo-parallel and Ricci generalized pseudo-parallel if [10] $\tilde{R}(X, Y) \cdot \nabla\sigma = fQ(S, \nabla\sigma)$ and $\tilde{R}(X, Y) \cdot \sigma = fQ(S, \sigma)$ for any $X, Y \in TM$ respectively. From [12], we have the following results.

Theorem 2.1: Let M be an invariant submanifold of a generalized Kenmotsu manifold $\tilde{M}$. Then the following equalities hold on M.

$$(\nabla_X \varphi)Y = \sum_{i=1}^{s}\left\{g(\varphi X, Y)\xi_i - \eta^i(Y)\varphi X\right\}, \tag{2.19}$$

$$\nabla_X \xi_j = -\varphi^2 X, \tag{2.20}$$

$$\sigma(X, \xi_j) = 0, \tag{2.21}$$

$$\sigma(X, \varphi Y) = \varphi\sigma(X, Y), \tag{2.22}$$

$$R(X, Y)\,\xi_i = \sum_{j=1}^{s}\left\{\eta^j(Y)\varphi^2 X - \eta^j(X)\varphi^2 Y\right\}, \tag{2.23}$$

$$S(X, \xi_i) = -2n\sum_{j=1}^{s}\eta^j(X), \tag{2.24}$$

Theorem 2.2: An invariant submanifold M of a generalized Kenmotsu manifold $\tilde{M}$ in which ξ_i are tangent to M for all $i \in 1, 2, \ldots, s$ is also a generalized Kenmotsu manifold.

3. INVARIANT SUBMANIFOLDS OF GENERALIZED KENMOTSU MANIFOLD SATISFYING Q(σ, R) = 0 AND Q(S, σ) = 0

Theorem 3.3: An invariant submanifold of a generalized Kenmotsu manifold is totally geodesic if and only it satisfies $Q(\sigma, R) = 0$.

Proof: Let us consider an invariant submanifold of generalized Kenmotsu manifold satisfying $Q(\sigma, R) = 0$. Then,

$$0 = Q(\sigma, R)(X, Y, Z; U, V),$$

for any vector fields $X, Y, Z, U, V \in TM$. From (2.18) and the definition of $(X \wedge_E Y)Z$, we get

$$0 = -\sigma(V, X)R(U, Y)Z + \sigma(U, X)R(V, Y)Z - \sigma(V, Y)R(X, U)Z + \sigma(U, Y)R(X, V)Z - \sigma(V, Z)R(X, V)U + \sigma(U, Z)R(X, Y)V. \tag{3.1}$$

By taking $Z = \xi_i$ and $V = \xi_i$ in (3.1), we get

$$\sigma(U, X)Y - \sigma(U, X)\sum_{i=1}^{s}\eta^i(Y)\xi_i - \sigma(U, Y)X + \sigma(U, Y)\sum_{i=1}^{s}\eta^i(X)\xi_i = 0. \tag{3.2}$$

Taking inner product of (3.2) with W and contracting Y and W by putting Y = W = e_j, we get

$$(2n + s - 1)\sigma(U, X) = 0.$$

Thus M is totally geodesic. Converse part is directly follows from (3.1).

Theorem 3.4: An invariant submanifold of a generalized Kenmotsu manifold is totally geodesic if and only it satisfies Q(S, σ) = 0.

Proof: Let us consider an invariant submanifold of generalized Kenmotsu manifold satisfying Q(S, σ) = 0. Then,

$$0 = Q(S, \sigma)(X, Y; U, V),$$

for any vector fields X, Y, U, V ∈ TM. From (2.19) and the definition of $(X \wedge_E Y)$ Z, we get

$$0 = -S(V, X)\sigma(U, Y) + S(U, X)\sigma(V, Y) - S(V, Y)\sigma(X, U) + S(U, Y)\sigma(X, V). \quad (3.3)$$

Taking U = Y = ξ_i in the above equation, we arrive at

$$-2n\sigma(X, V) = 0.$$

Thus M is totally geodesic. Converse part directly follows from (3.3).

Corollary 3.1: In an invariant submanifold of generalized Kenmotsu manifold Q(σ, R) = 0 if and only if Q(S, σ) = 0.

4. INVARIANT SUBMANIFOLDS OF GENERALIZED KENMOTSU MANIFOLD SATISFYING $Q(S, \tilde{\nabla}\sigma) = 0$ AND $Q(S, \tilde{R} \cdot \sigma) = 0$.

Theorem 4.5: An invariant submanifold of generalized Kenmotsu manifold is totally geodesic if and only if $Q(S, \tilde{\nabla}\sigma) = 0$.

Proof: Let us consider an invariant submanifold of generalized Kenmotsu manifold Satisfying $Q(S, \tilde{\nabla}\sigma) = 0$. Then,

$$0 = Q(S, \tilde{\nabla}_X \sigma)(Y, W; U, V),$$

for any vector fields X, Y, W, U, V ∈ TM. From (2.18) and the definition of $(X \wedge_E Y)Z$, we get

$$0 = -(\tilde{\nabla}_X \sigma)(S(V, Y)U, W) + (\tilde{\nabla}_X \sigma)(S(U, Y)V, W) - (\tilde{\nabla}_X \sigma)(Y, S(V, W)U) + (\tilde{\nabla}_X \sigma)(Y, S(U, W)V).$$

Continuing with simple calculations, we get

$$\begin{aligned}0 = &-\nabla^{\perp}_{X}\,\sigma(S(V, Y)U, W) + \sigma(\nabla_X\, S(V, Y)U, W) + \sigma(S(V, Y)U, \nabla_X\, W)\\ &+\nabla^{\perp}_{X}\,\sigma(S(U, Y)V, W) - \sigma(\nabla_X\, S(U, Y)V, W) - \sigma(S(U, Y)V, \nabla_X\, W)\\ &-\nabla^{\perp}_{X}\,\sigma(S(V, W)U, Y) + \sigma(\nabla_X\, S(V, W)U, Y) + \sigma(S(V, W)U, \nabla_X\, Y)\\ &-\nabla^{\perp}_{X}\,\sigma(S(U, W)V, Y) - \sigma(\nabla_X\, S(U, W)V, Y) - \sigma(S(U, W)V, \nabla_X\, Y). \quad (4.1)\end{aligned}$$

Now by taking $Y = W = V = \xi_j$ in (4.1) and then using (2.20), we arrive at

$$-2n\sigma(U, -\varphi^2\, X) = 0.$$

In view of (2.1), we get

$$2n\sigma(U, X) = 0.$$

Thus M is totally geodesic. Converse part follows directly from (4.1).

Theorem 4.6: An invariant submanifold of generalized Kenmotsu manifold is totally geodesic if and only if $Q(S, \tilde{R} \cdot \sigma) = 0$.

Proof: Let us consider an invariant submanifold of generalized Kenmotsu manifold satisfying $Q(S, \tilde{R} \cdot \sigma) = 0$. Then,

$$0 = Q(S, \tilde{R}(X, Y) \cdot \sigma)(Z, W; U, V),$$

for any vector fields $X, Y, Z, W, U, V \in TM$. From (2.18) and the definition of $(X \wedge_E Y)Z$, we get

$$\begin{aligned}0 = &-(\tilde{R}(X, Y) \cdot \sigma)(S(V, Z)U, W) + (\tilde{R}(X, Y) \cdot \sigma)(S(U, Z)V, W)\\ &-(\tilde{R}(X, Y) \cdot \sigma)(Z, S(V, W)U) + (\tilde{R}(X, Y) \cdot \sigma)(Z, S(U, W)V).\end{aligned}$$

Using the definition of $\tilde{R}$, we get

$$\begin{aligned}0 = &-S(V, Z)[R^{\perp}(X, Y)\sigma(U, W) + \sigma(R(X, Y)U, W) - \sigma(U, R(X, Y)W)]\\ &+S(U, Z)[R^{\perp}(X, Y)\sigma(V, W) - \sigma(R(X, Y)V, W) - \sigma(V, R(X, Y)W)]\\ &-S(V, W)[R^{\perp}(X, Y)\sigma(Z, U) - \sigma(R(X, Y)Z, U) + \sigma(Z, R(X, Y)U)]\\ &+S(U, W)[R^{\perp}(X, Y)\sigma(Z, V) - \sigma(R(X, Y)Z, V) - \sigma(Z, R(X, Y)V)]. \quad (4.2)\end{aligned}$$

Now by taking $Y = Z = V = \xi_j$ in (4.2) we arrive at

$$-2n\sigma(R(X, \xi_j)\,\xi_j, U) = 0.$$

In view of (2.23) and (2.1), we get

$$2n\sigma(U, X) = 0.$$

Thus M is totally geodesic. Converse part follows directly from (4.2).

5. EXAMPLE

In this section we construct an example for invariant submanifold of a generalized Kenmotsu manifold and verify our results. Let take $n = 2$ and $s = 3$ and consider

$\tilde{M} = \{(x_1, x_2, y_1, y_2, z_1, z_2, z_3) \in \mathbb{R}^7 : z_i \neq 0 \text{ for } i = 1, 2, 3\}$, a seven dimensional manifold, where $(x_1, x_2, y_1, y_2, z_1, z_2, z_3)$ are standard coordinates in $\mathbb{R}^7$. Choose the vector fields as [11].

$$E_1 = f\left[\frac{\partial}{\partial x_1} + \frac{\partial}{\partial y_1}\right], E_2 = f\left[-\frac{\partial}{\partial x_1} + \frac{\partial}{\partial y_1}\right],$$

$$E_3 = f\left[\frac{\partial}{\partial x_2} + \frac{\partial}{\partial y_2}\right], E_4 = f\left[-\frac{\partial}{\partial x_2} + \frac{\partial}{\partial y_2}\right],$$

$$E_5 = \frac{\partial}{\partial z_1}, E_6 = \frac{\partial}{\partial z_2}, E_7 = \frac{\partial}{\partial z_3},$$

where, f is defined as $f = e^{-(z_1 + z_2 + z_3)}$ are linearly independent at each point of $\tilde{M}$.

Let g be the Riemannian metric defined by

$$\frac{1}{f^2}\left[\sum_{i=1}^{2} dx_i \otimes dx_i + dy_i \otimes dy_i\right] + \sum_{j=1}^{3} dz_j \otimes dz_j.$$

Then we have

$$g(E_i, E_j) = \begin{cases} 1, i = j \\ 0, i \neq j, \end{cases}$$

for all $i, j \in \{1, \ldots, 7\}$. Let for any vector field X, 1-forms are defined by $\eta^1(X) = g(X, E_5)$, $\eta^2(X) = g(X, E_6)$ and $\eta^3(X) = g(X, E_7)$. Then $\{E_1, \ldots, E_7\}$ is an orthonormal basis of $\tilde{M}$.

Now define (1,1) tensor field by

$$\varphi(E_1) = E_2; \varphi(E_2) = -E_1; \varphi(E_3) = E_4; \varphi(E_4) = -E_3;$$

$$\varphi(E_5 = \xi_1) = 0; \varphi(E_6 = \xi_2) = 0; \varphi(E_7 = \xi_3) = 0.$$

The linearity property of g and φ yields (2.1) and (2.2) for any vector fields on $\tilde{M}$. Hence $\tilde{M}(\varphi, \xi_i, \eta^i, g)$ defines almost s-contact metric manifold.

The Riemannian connection $\tilde{\nabla}$ of the metric g is given by the Koszul's formula

$$2g(\tilde{\nabla}_X Y, Z) = Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) - g(X, [Y, Z]) - g(Y, [X, Z]) + g(Z, [X, Y]).$$

Using Koszul's formula we get:

$$\tilde{\nabla}_{E_i} E_j = \begin{cases} -(\xi_1 + \xi_2 + \xi_3), i = j \\ 0, i \neq j \text{ for } i, j = 1, \ldots, 4, \end{cases} \tag{5.1}$$

$$\tilde{\nabla}_{E_i} \xi_j = E_i, \text{ and } \tilde{\nabla}_{\xi_j} \xi_j = 0 \text{ for } i = 1, \ldots, 4 \text{ and } j = 1, 2, 3.$$

Let $f : M \to \tilde{M}$ be an isometric immersion defined by $(x, y, v_1, v_2, v_3) = (x, y, v_1, 0, 0, v_2, v_3)$.

It can be easily proved that $M = \{(x, y, v_1, v_2, v_3) \in \mathbb{R}^5 : (x, y, v_1, v_2, v_3) \neq 0\}$, where (x, y, v_1, v_2, v_3) are standard coordinates in $\mathbb{R}^5$ is a 5-dimensional submanifold of $\tilde{M}$. Choose the vector fields as:

$$E_1 = f\left[\frac{\partial}{\partial x_1} + \frac{\partial}{\partial y_1}\right], E_2 = f\left[-\frac{\partial}{\partial x_1} + \frac{\partial}{\partial y_1}\right],$$

$$E_5 = \frac{\partial}{\partial z_1}, E_6 = \frac{\partial}{\partial z_2}, E_7 = \frac{\partial}{\partial z_3},$$

where, f is defined as $f = e^{-(z_1 + z_2 + z_3)}$ are linearly independent at each point of M. Let the Riemannian metric be defined as same as in the manifold $\tilde{M}$. Now, for any vector field X, 1-forms are defined by $\eta^1(X) = g(X, E_5)$, $\eta^2(X) = g(X, E_6)$ and $\eta^3(X) = g(X, E_7)$. Then $\{E_1, E_2, E_5, E_6, E_7\}$ is an orthonormal basis of M. Now define (1,1) tensor field by:

$$\varphi(E_1) = E_2;\ \varphi(E_2) = -E_1;\ \varphi(E_5 = \xi_1) = 0;\ \varphi(E_6 = \xi_2) = 0;\ \varphi(E_7 = \xi_3) = 0.$$

Using Koszul's formula, we can easily calculate

$$\nabla_{E_1} E_1 = \nabla_{E_2} E_2 = -(\xi_1 + \xi_2 + \xi_3), \tag{5.2}$$

$$\nabla E_i E_j = E_i, \text{ for } i = 1, 2 \text{ and } j = 5, 6, 7.$$

and the remaining $\nabla_{E_i} E_j = 0$, for $1 \leq i, j \leq 7$ and $i, j \neq 3, 4$.

It can be easily verified that M is an invariant submanifold of $\tilde{M}$. Now from (5.1) and (5.2), we see that $\sigma(E_i, E_j) = 0$, for all $i, j = 1, 2, 5, 6, 7$. Thus M is totally geodesic. Hence theorems 3.3, 3.4, 4.5 and 4.6 are verified.

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Example is constructed and verified using Maple.

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