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The Best Quintic Approximation of Circular Arcs of Order Ten

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Abstract: In this paper, we present a polynomial approximation of degree 5 for a circular arc. This quintic approximation is set so that the error function is of degree 10; the Chebyshev error function equioscillates 11 times; the approximation order is 10. The method approximates more than the full circle with Chebyshev uniform error of $1/2^9$. The examples show the competence and simplicity of the quintic approximation and that it can not be improved.

Keywords: Bézier curves; quintic approximation; circular arc; high accuracy; approximation order; equioscillation; CAD.

1. INTRODUCTION

Circles appear in the implicit form that is, commonly, not convenient for computer applications' use in computer graphics, computer aided design, and other computer based applications. The other form of symbolizing circles is the trigonometric structure which is a non-typical style for computer based applications. The necessary, viable, and needed structure for circles is the polynomial form.

At the latest, only NURBS (Non Uniform Rational B-Splines) are capable to properly symbolize circles. NURBS have mathematical frame and harnessing them depends on mathematical knowledge and, in significant cases, demands the potential of employing geometric and analytic concepts. However, circles are directed to be used by people of limited and exclusive knowledge of mathematical notions and processes like computer graphics designers, stylers, animators, computer aided designers, and engineers. Drawing a circle in any CAD system is the primary alphabet in any software. So, it is very genuine to have the circle as a primitive and as a built-in-function in the software. A circle can be represented using rational Bézier curves and can be approximated by polynomial curves. Therefore, approximating a circular arc by polynomial curves with highest possible accuracy is a very important matter.

Parametric curves offer the user flexibility in representing, generating, and creating curves. Furthermore, they also set forth additional degrees of freedom which can be accustomed to make the approximating curve bow with the original curve. This property is used in [13] to improve the approximation order by polynomials of degree n from $n + 1$ to $2n$. So, there is a necessity for adequate parametric approximation of the circle. Bézier curves are elucidated parametrically and become the basics for curves in computer applications.

We treat the circular arc $c : t \mapsto (\cos(t), \sin(t)), -\theta \leq t \leq \theta$, see Figure 1, to be approximated by a polynomial curve with superior uniform approximation. To come to this consequence, the geometric symmetries of the circle are used to fairly choose the Bézier points in order to symbolize the quintic Bézier curve that has highest approximation order of 10.

The circle c is approximated in this paper using a quintic parametrically defined polynomial curve $p : t \mapsto (x(t), y(t)), 0 \leq t \leq 1$, where $x(t), y(t)$ are polynomials of degree 5, that approximates c with least deviation. Many researchers have tackled this issue using different degrees, norms, and methods, see [1, 2, 3, 4, 5, 6, 9, 11, 12, 14, 16, 17, 15]. In [7], methods for approximating circular arcs using quintic polynomial curves with different boundary conditions are considered. The results of our method in this paper are optimal and can not be improved.

The proper distance function to measure the error between p and c is the Euclidean error function:

$$\mathrm{E}(t) := \sqrt{x^2(t) + y^2(t)} - 1. \tag{1}$$

$\mathrm{E}(t)$ will be replaced by the following deviation measure:

$$e(t) := x^2(t) + y^2(t) - 1. \tag{2}$$

Since, $e(t) = 0$ is the implicit equation of the unit circle; this implies that the $e(t)$ error function is a suitable measure to test if $x(t)$ and $y(t)$ satisfy this equation and to measure the error.

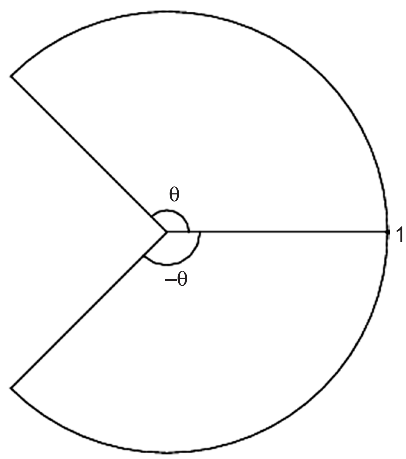

Figure 1: A circular arc

The approximation issue that we consider in this paper is locating a polynomial curve $p : t \mapsto (x(t), y(t))$, $0 \leq t \leq 1$, where $x(t), y(t)$ are quintic polynomials, that "mimic" c and fulfills the following three conditions:

1. p minimizes $\max_{t \in [0, 1]} |e(t)|$,
2. $e(t)$ equioscillates 11 times over [0, 1],
3. p approximates c with order 10.

These conditions are used to locate the Bézier points and to get the values of the parameters that are utilized to satisfy the geometric conditions of the circular arc. For more on these topics, see [8, 10]. To achieve the conditions of the issue of the approximation, the following feature of the Chebyshev polynomials is used. Namely, the monic Chebyshev polynomial $\tilde{\mathrm{T}}_{10}(u), u \in [-1, 1]$, given by:

$$\tilde{T}_{10}(u) = \frac{-1}{512} + \frac{25}{256}u^2 - \frac{25}{32}u^4 + \frac{35}{16}u^6 - \frac{5}{2}u^8 + u^{10}, \quad u \in [-1, 1] \tag{3}$$

is the unique polynomial of degree 10 that equioscillates 11 times between $\pm \frac{1}{2^9}$ for all $u \in [-1, 1]$ and has the least deviation from the x-axis, see [18].

Since the uniform error (for $e(t)$) equals 2^{-9}, so, we allow the angle θ to be as large as possible in order to approximate the largest circular arc with this specified error. Thereafter, this angle θ has to be scaled by a factor that also combined with a reduction in the uniform error.

This paper is organized as follows. Preliminaries are given in section 2. The quintic Bézier curve of least deviation is presented and proved in section 3, and the properties are presented in section 4. Conclusions are summarized in section 5.

2. PRELIMINARIES

In this paper, the curve $p(t)$ is given in Bézier form. The Bézier curve $p(t)$ of degree 5 is given by

$$p(t) = \sum_{i=0}^{5} p_i B_i^5(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}, \quad 0 \le t \le 1 \tag{4}$$

The points p_0, p_1, p_2, p_3, p_4 and p_5 are called the control points, and the polynomials $B_0^5(t) = (1-t)^5$, $B_1^5(t) = 5t(1-t)^4$, $B_2^5(t) = 10t^2(1-t)^3$, $B_3^5(t) = 10t^3(1-t)^2$, $B_4^5(t) = 5t^4(1-t)$ and $B_5^5(t) = t^5$ are the well-known quintic Bernstein polynomials.

Since our purpose is to represent the arc with a polynomial curve with the least possible error, it is not substantial for the errors to take place at the endpoints or elsewhere; it is significant to ensure that this annoyance is as low as conceivable no matter where the error occurs. Our approach considers lessen the wrongdoing over all of the segment [0, 1]. To explore the Bézier form approximation of a circular arc, a careful selection of locations of the Bézier points should be well-done. These locations are substantial to earn the convenient curve that redeems the approximation conditions. Based on the symmetry property of the circle, the right choice for the beginning control point p_0 should obey the following form:

$p_0 = (\alpha_0 \cos(\theta), \beta_0 \sin(\theta))$, where values of α_0 and β_0 could but should not be the same. Similarly, for symmetry reasoning, the valid option for the end control point p_5 is $p_5 = (\alpha_0 \cos(\theta), \beta_0 \sin(\theta))$. Set $p_1 = (a_1, b_1)$, then the point p_4 has to be selected to satisfy the form $p_4 = (a_1, -b_1)$. Set the point $p_2 = (a_2, b_2)$, then the point p_3 has to be selected to satisfy the form $p_3 = (a_2, -b_2)$. Using the substitutions $a_0 = \alpha_0 \cos(\theta)$, $b_0 = \beta_0 \sin(\theta))$, then the convenient choices for the Bézier points have to be, see Figure 2,

$$p_0 = \begin{pmatrix} a_0 \\ b_0 \end{pmatrix}, p_1 = \begin{pmatrix} a_1 \\ b_1 \end{pmatrix}, p_2 = \begin{pmatrix} a_2 \\ b_2 \end{pmatrix}, p_3 = \begin{pmatrix} a_2 \\ -b_2 \end{pmatrix}, p_4 = \begin{pmatrix} a_1 \\ -b_1 \end{pmatrix}, p_5 = \begin{pmatrix} a_0 \\ -b_0 \end{pmatrix}. \tag{5}$$

It will be apparent that there are more than one solution; the consonant solution of best approximation begins in the second quadrant and ends in the fourth quadrant counter clockwise. Therefore, in order to have the Bézier curve p begin in the second quadrant, go counter clockwise through fourth, third, first, second, and ends in the fourth quadrant as the circular arc c, the following stipulations should be satisfied:

$$a_0, a_1, b_1, b_2 < 0, a_2, b_0 > 0 \tag{6}$$

Employ the Bézier points in (5) in the Bézier curve $p(t)$ in (4) to obtain:

$$p(t)=\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}=\begin{pmatrix} a_0(\mathrm{B}_0^5(t)+\mathrm{B}_5^5(t))+a_1(\mathrm{B}_1^5(t)+\mathrm{B}_4^5(t))+a_2(\mathrm{B}_2^5(t)+\mathrm{B}_3^5(t)) \\ b_0(\mathrm{B}_0^5(t)-\mathrm{B}_5^5(t))+b_1(\mathrm{B}_1^5(t)-\mathrm{B}_4^5(t))+b_2(\mathrm{B}_2^5(t)-\mathrm{B}_3^5(t)) \end{pmatrix}, 0 \le t \le 1. \tag{7}$$

The Bézier curve is settled by the 6 parameters a_0, a_1, a_2, b_0, b_1, b_2. These parameters are hired to get the best uniform approximation. We want to impose the conditions on the polynomial curve p; the polynomials $x(t)$ and $y(t)$ are substituted into $e(t)$. This leads to a polynomail of degree 10 that is solved using a computer algebra system. These proceedings are demonstrated in the next section.

3. THE QUINTIC BÉZIER CURVE OF LEAST DEVIATION

The values of a_0, a_1, a_2, b_0, b_1, b_2 that satisfy the conditions of the approximation problem are specified numerically, rounded fittingly, in the following theorem.

Theorem 1: The Bézier curve (7) together with the Bézier points in (5) and the values of the parameters a_0, a_1, a_2, b_0, b_1, b_2 given by

$$a_0=-0.978179455549407,\ a_1=-1.338058000836784,\ a_2=2.365283682287911, \tag{8}$$

$$b_0=0.21241016393227463,\ b_1=-1.6287837424896061,\ b_2=-2.4356328876380307 \tag{9}$$

achieves the following three conditions: p minimizes the uniform norm of the error function $\max_{t\in[0,1]}|e(t)|$ and approximates c with order 10, and the error function $e(t)$ equioscillates 11 times in [0, 1]. The error functions fulfill:

$$-\frac{1}{2^9}\le e(t)\le\frac{1}{2^9},-\frac{1}{2^9(2-\varepsilon)}\le \mathrm{E}(t)\le\frac{1}{2^9(2+\varepsilon)}, \text{ where } \varepsilon=\max_{0\le t\le 1}|\mathrm{E}(t)|\approx 2^{-10}, \forall t\in[0,1]. \tag{10}$$

Proof: We begin by considering the polynomials $x(t)$ and $y(t)$ in equation (7) and substituting them into the error function $e(t)$ in (2). Disposition the phrase and performing several simplifications gives the following equation:

$$\begin{aligned} e(t) = {} & \left(4b_0^2-40b_0b_1+100b_1^2+80b_0b_2-400b_1b_2+400b_2^2\right)t^{10} \\ & +\left(-20b_0^2+200b_0b_1-500b_1^2-400b_0b_2+2000b_1b_2-2000b_2^2\right)t^9 \\ & +\left(25a_0^2+150a_0a_1+225a_1^2-100a_0a_2-300a_1a_2+100a_2^2+65b_0^2-570b_0b_1+1225b_1^2\right. \\ & \left.+1060b_0b_2-4500b_1b_2+4100b_2^2\right)t^8 \\ & +\left(-100a_0^2-600a_0a_1-900a_1^2+400a_0a_2+1200a_1a_2-400a_2^2-140b_0^2+1080b_0b_1\right. \\ & \left.-1900b_1^2-1840b_0b_2+6000b_1b_2-4400b_2^2\right)t^7 \\ & +\left(200a_0^2+1100a_0a_1+1500a_1^2-700a_0a_2-1900a_1a_2+600a_2^2+220b_0^2-1420b_0b_1\right. \\ & \left.+2000b_1^2+2100b_0b_2-5100b_1b_2+2600b_2^2\right)t^6 \\ & +\left(-250a_0^2-1200a_0a_1-1350a_1^2+700a_0a_2+1500a_1a_2-400a_2^2-254b_0^2+1320b_0b_1\right. \\ & \left.-1450b_1^2-1540b_0b_2+2700b_1b_2-800b_2^2\right)t^5 \\ & +\left(210a_0^2+830a_0a_1+700a_1^2-420a_0a_2-600a_1a_2+100a_2^2+210b_0^2-850b_0b_1+700b_1^2\right. \\ & \left.+700b_0b_2-800b_1b_2+100b_2^2\right)t^4 \\ & +\left(-120a_0^2-360a_0a_1-200a_1^2+140a_0a_2+100a_1a_2-120b_0^2+360b_0b_1-200b_1^2\right. \end{aligned}$$

$$-180b_0b_2+100b_1b_2\big)t^3$$
$$+\left(45a_0^2+90a_0a_1+25a_1^2-20a_0a_2+45b_0^2-90b_0b_1+25b_1^2+20b_0b_2\right)t^2$$
$$+\left(-10a_0^2-10a_0a_1-10b_0^2+10b_0b_1\right)t+\left(a_0^2+b_0^2-1\right).$$

The approximation conditions are satisfied if the error function is equalized with the polynomial of least deviation among all monic polynomials of degree 10. So, the last equation which exemplifies the error function has to be equalized with the Chebyshev polynomial of first kind of degree 10, $T_{10}(2t-1)/512$. We know that $T_{10}(u) = \cos(10 \arccos(u))$, $u \in [-1, 1]$ is the unique monic polynomial of degree 10 that has the least deviation from the origin. It is given by equation (3), see [18]. Comparing the coefficients of equal powers of both sides and using the utilities of the computer algebra system in Mathematica, the solution that fulfills the conditions in (6) is established. Unfortunately, the solution is a collection of lengthy fractions and radicals that is impractical to write down the values of the parameters in this paper, so, we write them in decimal forms in equations (8) and (9). This shows that r fulfills the three conditions of the approximation problem. To prove the error formula for $E(t)$, the relation to $e(t)$ is established. The error function $e(t)$ minimized is linked to the Euclidean error $E(t)$ by the formulation:

$$e(t)=x^2(t)+y^2(t)-1=\left(\sqrt{x^2(t)+y^2(t)}+1\right)\left(\sqrt{x^2(t)+y^2(t)}-1\right)=\left(2+E(t)\right)E(t).$$

Thus

$$E(t)=\frac{e(t)}{2+E(t)}.$$

Substituting the bounds for $e(t)$ gives

$$-\frac{1}{2^9(2-\varepsilon)}\le E(t)\le\frac{1}{2^9(2+\varepsilon)},\text{ where } \varepsilon=\max_{0\le t\le 1}|E(t)|\approx 2^{-10},\ t\in[0,1].$$

This completes the proof of Theorem 1.

The circular arc and the approximating Bézier curve are plotted in Figure 2. The resulting error between the curve and the approximation is not identified by the human eyes which is clear from figure of the corresponding error plotted in Figure 3.

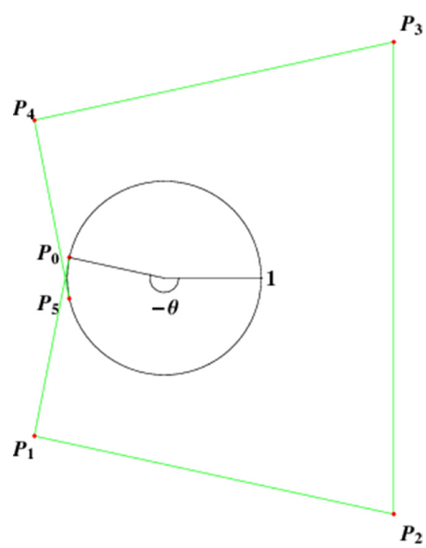

Figure 2: Circular arc and it's quintic Bézier curve in Theorem 1

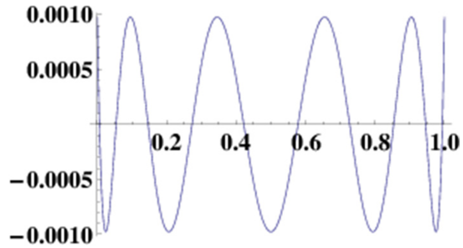

Figure 3: Euclidean Error of the quintic Bézier curve in Theorem 1

The resulting Bézier curve reveals a brilliant positioning of the Bézier points to embrace more than a whole circle whilst possessing the Chebyshev error. We could not foresee a quintic polynomial to approximate more than the full circle further accurately than this approximation. The characteristics of the approximating Bézier curve are specified in the following section.

4. PROPERTIES OF THE QUINTIC BÉZIER CURVE

The most important characteristics of the error functions are the roots and the extrema. These properties characterize the approximating quintic Bézier curve. The first characteristic concerns the roots of the error functions $e(t)$ and $\mathrm{E}(t)$ that are specified in the following proposition.

Proposition I: The roots of the error functions $e(t)$ and $\mathrm{E}(t)$ are:

$$t_1 = \frac{1}{2}\left(1+\cos\left(\frac{\pi}{20}\right)\right) = 0.993844, \quad t_2 = \frac{1}{2}\left(1+\cos\left(\frac{3\pi}{20}\right)\right) = 0.945503$$

$$t_3 = t_3 = \frac{1}{2}\left(1+\frac{1}{\sqrt{2}}\right) = 0.853553, t_4 = \frac{1}{2}\left(1+\sin\left(\frac{3\pi}{20}\right)\right) = 0.726995, t_5 = \frac{1}{2}\left(1+\sin\left(\frac{\pi}{20}\right)\right) = 0.578217,$$

$$t_6 = \frac{1}{2}\left(1-\sin\left(\frac{\pi}{20}\right)\right) = 0.421783, t_7 = \frac{1}{2}\left(1-\sin\left(\frac{3\pi}{20}\right)\right) = 0.273005,$$

$$t_8 = \frac{1}{2}\left(1-\frac{1}{\sqrt{2}}\right) = 0.146447, t_9 = \frac{1}{2}\left(1-\cos\left(\frac{3\pi}{20}\right)\right) = 0.0544967, t_{10} = \frac{1}{2}\left(1-\cos\left(\frac{\pi}{20}\right)\right) = 0.00615583.$$

They also satisfy

$$t_i + t_j = 1, \quad \text{for} \quad i + j = 11.$$

Proof: Immediate substitution of the values of t_i in $e(t)$ gives $e(t_i) = 0$, $i = 1, ..., 10$. Since $e(t)$ is a polynomial of degree 10 and has 10 roots, therefore, these ones are all the roots. The error function $\mathrm{E}(t)$ has the same roots as $e(t)$ because $\mathrm{E}(t) = 0$ iff $\sqrt{x^2(t)+y^2(t)} = 1$ iff $x^2(t) + y^2(t) = 1$ iff $e(t) = 0$.

The approximating quintic Bézier curve p in Theorem 1 and the circular arc c intersect at the points $p(t_i) = c(t_i)$, $i = 1, ..., 10$.

Regarding the extreme values, we have the following proposition.

Proposition II: The extreme values of $e(t)$ and $\mathrm{E}(t)$ occur at the parameters:

$$\tilde{t}_0 = 1, \quad \tilde{t}_1 = \frac{1}{2}\left(1+\cos\left(\frac{\pi}{10}\right)\right) = 0.975528, \quad \tilde{t}_2 = \frac{1}{2}\left(1+\cos\left(\frac{\pi}{5}\right)\right) = 0.904508,$$

$$\tilde{t}_3 = \frac{1}{2}\left(1+\cos\left(\frac{3\pi}{10}\right)\right) = 0.793893, \quad \tilde{t}_4 = \frac{1}{2}\left(1+\cos\left(\frac{2\pi}{5}\right)\right) = 0.654508$$

$$\tilde{t}_5 = \frac{1}{2}, \quad \tilde{t}_6 = \frac{1}{2}\left(1-\cos\left(\frac{2\pi}{5}\right)\right) = 0.345492, \quad \tilde{t}_7 = \frac{1}{2}\left(1-\cos\left(\frac{3\pi}{10}\right)\right) = 0.206107,$$

$$\tilde{t}_8 = \frac{1}{2}\left(1-\cos\left(\frac{\pi}{5}\right)\right) = 0.0954915, \quad \tilde{t}_9 = \frac{1}{2}\left(1-\cos\left(\frac{\pi}{10}\right)\right) = 0.0244717, \quad \tilde{t}_{10} = 0.$$

These parameters satisfy the equality:

$$\tilde{t}_i + \tilde{t}_j = 1, \quad \text{for} \quad i + j = 10.$$

Proof: The derivative of $e(t)$ is a polynomial of degree 9. Substituting the 9 parameters $\tilde{t}_1, \ldots, \tilde{t}_9$ into this derivative gives $e'(\tilde{t}_i) = 0, \forall i = 1, \ldots, 9$. The polynomial $e'(t)$ has degree 9 and consequently these are all internal critical points. Inspecting the end points adds $\tilde{t}_0 = 1, \tilde{t}_{10} = 0$ to the critical points. For all $t \in [0, 1]$, we have $1 - \frac{1}{512} \le x^2(t) + y^2(t) \le 1 + \frac{1}{512}$, thence $\sqrt{x^2(t) + y^2(t)} \ne 0, \forall t \in [0, 1]$. Differentiate E($t$) and counter equate to 0 to acquire $\frac{e'(t)}{\sqrt{x^2(t) + y^2(t)}} = 0$ iff $e'(t) = 0$. Therefore, $e(t)$ and E(t) reach the extrema at the same values. This finishes the proof of the proposition.

The disagreement in the values of $\mathrm{E}(\tilde{t}_i)$ for odd and even i's occurs because $e(t)$ equioscillates between $\pm\frac{1}{512}$ and $\frac{1}{2^9(2-\varepsilon)} \le \mathrm{E}(t) \le \frac{1}{2^9(2+\varepsilon)}$, where $\varepsilon = \max_{0 \le t \le 1}|\mathrm{E}(t)|$.

Proposition III: The values of the error functions $e(t)$ and E(t) at $\tilde{t}_i$'s are specified by:

$$e(\tilde{t}_{2i}) = \frac{1}{512}, i = 0, \ldots, 5, \quad e(\tilde{t}_{2i+1}) = \frac{-1}{512}, i = 0, \ldots, 4.$$

$$\mathrm{E}(\tilde{t}_{2i}) = \frac{1}{1024}, i = 0, \ldots, 5, \quad \mathrm{E}(\tilde{t}_{2i+1}) = \frac{-1}{1024}, i = 0, \ldots, 4.$$

Therefore,

$$\frac{-1}{512} \le e(t) \le \frac{1}{512}, \frac{-1}{1024} \le \mathrm{E}(t) \le \frac{1}{1024}, t \in [0, 1].$$

Proof: Substituting the parameters in the error functions confer to the parities. The specifics of the proof of the proposition are left to the reader.

The following proposition is a conclusion of Theorem 1 concerning the error at any $t \in [0, 1]$.

Proposition IV: The errors of approximating the circular arc using the quintic Bézier curves in Theorem 1 at any $t \in [0, 1]$ are given by:

$$e(t) = \frac{1}{512} - \frac{25}{64}t + \frac{825}{64}t^2 - 165t^3 + \frac{2145}{2}t^4 - 4004t^5 + 9100t^6 - 12800t^7$$
$$+ 10880t^8 - 5120t^9 + 1024t^{10}, \forall t \in [0, t].$$

Proof: This is a forthright conclusion of Theorem 1. The specifics of the proof of the proposition are left to the reader. W

Employ the relation between E(t) and $e(t)$ to obtain:

$$\mathrm{E}(t) \cong \frac{1}{1024} - \frac{25t}{128} + \frac{825t^2}{128} - \frac{165t^3}{2} + \frac{2145t^4}{4} - 2002t^5 + 4550t^6 - 6400t^7$$
$$+ 5440t^8 - 2560t^9 + 512t^{10}, \forall t \in [0, t].$$

5. CONCLUSIONS

In this article, quintic approximation of the circle is investigated. The approximation in this paper fulfills extremely conclusive circumstances. Unlike the classical approximation that awards order of approximation of 6, this approximation has order of approximation of 10; this is an superb acquisition. Moreover, in the significance of the Chebyshev norm, this approximation is the best and can not be improved. The error function equioscillates 11 times. The numerical examples reveal how efficient this method is. The approximation intersects the circular arc 10 times with maximum error 2^{-9} and thus outperforming any other approximation.

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