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Design and Analysis of Interval System using Order Reduction Technique

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ABSTRACT

A mixed method is proposed for the order reduction of an interval system using Cross multiplication technique and simple mathematical manipulation process. Kharitonov polynomial is employed in the interval system before the model order reduction technique is come into the approximation process. The Cross multiplication technique is used to obtain the reduced order denominator polynomial and the corresponding numerator polynomial is obtained through cross multiplication of transfer function polynomials. Genetic Algorithm is employed in the model order reduction process by which reduced order system parameters can be adjusted. The stability of the interval system is analyzed through the Routh-Hurwitz stability criterion.

Index Terms: Cross multiplication, Integral Squared Error (ISE), Kharitonov polynomial, Genetic Algorithm

INTRODUCTION

The analysis and design of practical control systems become complex when the order of the system increases. Therefore, to analyze such systems, it is necessary to reduce it to a lower order system, which is a sufficient representation of the higher order system most practical systems, such as flight vehicles, electric motors, and robots, are formulated in continuous-time uncertain settings. The uncertainties in these systems arise from unmodelled dynamics, parameter variation, sensor noises, actuator constraints, etc. These variations do not follow any of the known probability distributions in general, and are most often quantified in terms of amplitude and/or frequency bounds. Hence, practical systems or plants are most suitably represented by continuous-time parametric interval models, instead of deterministic mathematical models.

The Routh-Hurwitz criterion, Nyquist criterion and root locus plot are some of the techniques used to study the system with parameter uncertainty. The characteristic equation or polynomial of such uncertain system becomes a family of characteristic polynomials and the study of stability in the presence of uncertainty has gained its own importance. The fundamental stability problem is to check whether all the polynomials due to plant uncertainty are Hurwitz table. This property is known as robust stability of parameter uncertain systems. Robust stability was largely ignored in the early years.

INTERVAL SYSTEM

The system with parametric uncertainty is called interval system. An interval polynomial is a polynomial, where each coefficient varies in a prescribed interval. The property of interval system satisfying the stability requirements is known as robust stability. Uncertainty in a system will be producing the following two adverse effects of degradation of system performance and loss of stability. By virtue of the cause that produces this uncertainty it can be divided as

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(i) Structured uncertainty

(ii) Unstructured uncertainty

Structured uncertainty can be treated as the uncertainty, which arises from imperfect knowledge of system parameters. Unstructured type of uncertainty can be defined as the uncertainty in which the dynamics of the system in the higher frequency ranges is not incorporated in the system model. Until 1980, no good theory was developed by scientists to study a system with parametric uncertainties.

However, in recent years there has been a huge increase in research involving real parametric uncertainty. A major reason for this is the crucial theorem of Kharitonov (1978), which is in the Russian technical literature.

In many situations, it is also desirable to replace the high-order system by a lower order model. Typical methods for model reduction include aggregation method, moment matching technique, Pade approximation, and Routh approximation, etc. Recent developments of model reduction have been made towards the direction to handle uncertain interval systems based on variants of the Routh approximation methods, where interval arithmetic is performed to derive Routh α-β or γ-δ canonical continued-fraction expansion and inversion. This technique is employed in the methods proposed by Bandyopadhyay *et al.* (1994), Avinash Upadhye *et al.* (1997), Mallikarjuna Rao *et al.* (2000) and Dolgin and Zeheb (2003). It has been shown, however, that some interval Routh approximants may not be robustly stable even though the original interval system is robustly stable.

The same case occurred in the methods subsequently proposed by Lucas (1983 a). Several methods are available for the model reduction of higher order systems. The pade approximation technique employed in Lucas (1983 b) has simple features such as computational simplicity and fitting of time moments. A major disadvantage of this method is that it sometimes leads to an unstable reduced order model.

As far as model reduction of discrete interval systems is concerned, very limited discussions have been found in literature such as reduction methods discussed in Bandyopadhyay *et al.* (1997), Ali Zilouchian and Dali Wang (1998), Azou *et al.* (2000), Aly and O.A. Sebakhy (1998), Lim *et al.* (2003), Dolgin and Zeheb (2004), Do Chang Oh and Hong Bae Park (1997), Jayanta Pal and Somnath Pan (1995) and Mittal (2004). Among them, Bandyopadhyay *et al.* (1997) proposed a method of model order reduction using Pade approximation to retain dominant poles, where the denominator of the reduced order model is formed by retaining the dominant poles of the given discrete interval system, while the numerator is obtained by matching the first *r* moments of the model with that of the system.

The feedback control system with uncertain parameters can be written as

$$\mathrm{P(s,q)} = \mathrm{a}_0(q) + \mathrm{a}_1(q)s + \mathrm{a}_2(q)s^2 + \dots\dots + \mathrm{a}_\mathrm{n}(q)s^n \tag{1}$$

$$Q = \{q : q_i \in \underline{q}_i, \overline{q}_i,\ i = 0, 1, 2 \dots n\} \tag{2}$$

The above equations indicate the box bounding the uncertain parameter vector q.

5.3. Kharitonov Polynomial

The kharitonov theorem is an extension of the Routh stability criterion to interval polynomials. An interval polynomial is polynomial, where each coefficient varies at prescribed interval. The kharitonov theorem states that an interval polynomial family, which has an infinite number of members, is Hurwitz stable if and only if a finite small subset of four polynomials known as the kharitonov polynomial of the family are stable.

Consider a set of monic n^{th} order polynomial of the form,

$$P(s) = a_n s^n + a_{n-1} s^{n-1} + \ldots a_1 s + a_0 \tag{3}$$

with all real coefficients. The polynomial $P(s)$ is said to be Hurwitz stable if all of its zeros belong to left half of the s-plane. Let the coefficients are bounded between upper and lower limits as $\underline{a}_i < a_i < \overline{a}_i$, $i = 0, 1, 2 \ldots n-1$, where $\underline{a}_i$ represents the lower limit and $\overline{a}_i$ represents the upper limit. The parameters associated with $P(s)$ define an 'n' dimensional parameter space, where every point in the space represents the n coefficients of $P(s)$. According to the Kharitonov theorem, the Hurwitz stability of just four polynomials selected from set 'N' guarantees stability of all polynomials belong to 'N' where N is the set of all polynomials whose coefficients belong to specified intervals.

The polynomial $P(s)$ is split into two components $g(s)$ and $h(s)$ where, $g(s)$ is polynomial of even degree and $h(s)$ is polynomial of odd degree.

Defining two even polynomials,

$$K_1^{even,\min}(S) = g_1(s) = \underline{a}_0 + \overline{a}_2\, s^2 + \underline{a}_4\, s^4 + \ldots \tag{4}$$

$$K_1^{even,\max}(S) = g_2(s) = \overline{a}_0 + \underline{a}_2\, s^2 + \overline{a}_4\, s^4 + \ldots \tag{5}$$

Defining two odd polynomials,

$$K_1^{odd,\min}(S) = h_1(s) = \underline{a}_1\, s + \overline{a}_3\, s^3 + \underline{a}_5\, s^5 + \ldots \tag{6}$$

$$K_1^{odd,\max}(S) = h_2(s) = \overline{a}_1\, s + \underline{a}_3\, s^3 + \overline{a}_5\, s^5 + \ldots \tag{7}$$

The four Kharitonov polynomials would be

$$K_{11}(S) = g_1(s) + h_1(s) = \underline{a}_0 + \underline{a}_1\, s + \overline{a}_2\, s^2 + \overline{a}_3\, s^3 + \underline{a}_4\, s^4 + \underline{a}_5\, s^5 + \ldots \tag{8}$$

$$K_{12}(S) = g_1(s) + h_2(s) = \underline{a}_0 + \overline{a}_1\, s + \overline{a}_2\, s^2 + \underline{a}_3\, s^3 + \underline{a}_4\, s^4 + \overline{a}_5\, s^5 + \ldots \tag{9}$$

$$K_{21}(S) = g_2(s) + h_1(s) = \overline{a}_0 + \underline{a}_1\, s + \underline{a}_2\, s^2 + \overline{a}_3\, s^3 + \overline{a}_4\, s^4 + \underline{a}_5\, s^5 + \ldots \tag{10}$$

$$K_{22}(S) = g_2(s) + h_2(s) = \overline{a}_0 + \overline{a}_1\, s + \underline{a}_2\, s^2 + \underline{a}_3\, s^3 + \overline{a}_4\, s^4 + \overline{a}_5\, s^5 + \ldots \tag{11}$$

The main drawback on application of Kharitonov theorem is that it cannot be applied to polynomials with affine linear uncertainty structures. For such systems, the edge theorem described by Bartlett *et al.* (1988) and the generalized Kharitonov theorem provide solution. The edge theorem states that the whole family of polynomial is stable if and only if all the exposed edges of the polytopic family are stable.

5.4. Problem Statement

The mathematical representation of an interval system is given by,

$$G(s) = \frac{[\underline{a}_{n-1}, \overline{a}_{n-1}]s^{n-1} + [\underline{a}_{n-2}, \overline{a}_{n-2}]s^{n-2} + \ldots + [\underline{a}_1, \overline{a}_1]s + [\underline{a}_0, \overline{a}_0]}{[\underline{b}_n, \overline{b}_n]s^n + [\underline{b}_{n-1}, \overline{b}_{n-1}]s^{n-1} + [\underline{b}_{n-2}, \overline{b}_{n-2}]s^{n-2} + \ldots + [\underline{b}_1, \overline{b}_1]s + [\underline{b}_0, \overline{b}_0]} = \frac{N(s)}{D(s)} \tag{12}$$

where, $[\underline{a}_i, \overline{a}_i]$ for $i = 0$ to n–1 and $[\underline{b}_i, \overline{b}_i]$ for $i = 0$ to n are the interval parameters.

By applying the proposed model order reduction techniques, the corresponding reduced order model is obtained as,

$$G_r(s) = \frac{[\underline{d}_0, \overline{d}_0] + [\underline{d}_1, \overline{d}_1]s + ... + [\underline{d}_{k-1}, \overline{d}_{k-1}]s^{k-1}}{[\underline{e}_0, \overline{e}_0] + [\underline{e}_1, \overline{e}_1]s + ... + [\underline{e}_{k-1}, \overline{e}_{k-1}]s^{k-1} + [\underline{e}_k, \overline{e}_k]s^k} = \frac{N_r(s)}{D_r(s)} \tag{13}$$

Instead of applying the model order reduction procedure to single interval system as a whole the system can be split up to 4 Kharitonov polynomials of the denominator and 4 Kharitonov polynomials of the numerator. After arriving at the reduced denominator and each denominator has 4 possible combinations of the numerator and for the 4 denominator functions totally 16 possible system models are obtained.

In this chapter, the proposed methodologies are employed for the order reduction of an interval system. Where, model order reduction methodologies are applied to any four transfer functions in 16 possible system transfer functions. The interval limits of reduced order system transfer function are chosen from four reduced order system transfer functions obtained from the 16 groups. The stability of the interval system is analyzed with the help of Kharitonov polynomials through Routh-Hurwitz stability criterion. The validity of the proposed methods is investigated with the help of some numerical examples.

5.5. Numerical Illustration

Example-5.1

The 3rd order interval system stated in Rajeswari Mariappan (2004) is considered as

$$G(s) = \frac{[2,3]s^2 + [17.5,18.5]s + [15,16]}{[2,3]s^3 + [17,18]s^2 + [35,36]s + [20.5,21.5]} \tag{14}$$

The given interval system transfer function is in the form of

$$G(s) = \frac{[\underline{a}_2, \overline{a}_2]s^2 + [\underline{a}_1, \overline{a}_1]s + [\underline{a}_0, \overline{a}_0]}{[\underline{b}_3, \overline{b}_3]s^3 + [\underline{b}_2, \overline{b}_2]s^2 + [\underline{b}_1, \overline{b}_1]s + [\underline{b}_0, \overline{b}_0]} = \frac{N(s)}{D(s)} \tag{15}$$

For which, the reduced order interval system is to be derived in the form of

$$G_2(s) = \frac{[\underline{d}_1, \overline{d}_1]s + [\underline{d}_0, \overline{d}_0]}{[\underline{e}_2, \overline{e}_2]s^2 + [\underline{e}_1, \overline{e}_1]s + [\underline{e}_0, \overline{e}_0]} \tag{16}$$

From Equation (14), the following values are noted down and are used to obtain the Kharitonov polynomials.

$\underline{a}_0 = 15$ $\quad$ $\overline{a}_0 = 16$

$\underline{a}_1 = 17.5$ $\quad$ $\overline{a}_1 = 18.5$

$\underline{a}_2 = 2$ $\quad\quad$ $\bar{a}_2 = 3$

$\underline{b}_0 = 20.5$ $\quad\quad$ $\bar{b}_0 = 21.5$

$\underline{b}_1 = 35$ $\quad\quad$ $\bar{b}_1 = 36$

$\underline{b}_2 = 17$ $\quad\quad$ $\bar{b}_2 = 18$

$\underline{b}_3 = 2$ $\quad\quad$ $\bar{b}_3 = 3$

The kharitonov polynomials are obtained separately for the higher order interval system's numerator and denominator polynomials.

Numerator Kharitonov Polynomials

Consider the numerator polynomial,

$$N(s) = [\underline{a}_2, \bar{a}_2]s^2 + [\underline{a}_1, \bar{a}_1]s + [\underline{a}_0, \bar{a}_0] = [2,3]s^2 + [17.5,18.5]s + [15,16] \tag{17}$$

The polynomial N(s) is split into two components g(s) and h(s) where, g(s) is polynomial of even degree and h(s) is polynomial of odd degree.

Defining two even polynomials,

$$K_1^{even,\min}(S) = g_1(s) = \underline{a}_0 + \bar{a}_2\, s^2 = 15 + 3s^2 \tag{18}$$

$$K_1^{even,\max}(S) = g_2(s) = \bar{a}_0 + \underline{a}_2\, s^2 = 16 + 2s^2 \tag{19}$$

Defining two odd polynomials,

$$K_1^{odd,\min}(S) = h_1(s) = \underline{a}_1\, s = 17.5s \tag{20}$$

$$K_1^{odd,\max}(S) = h_2(s) = \bar{a}_1\, s = 18.5s \tag{21}$$

The four Kharitonov polynomials would be

$$N_1(s) = K_{11}(s) = g_1(s) + h_1(s) = \underline{a}_0 + \underline{a}_1\, s + \bar{a}_2\, s^2 = 15 + 17.5s + 3s^2 \tag{22}$$

$$N_2(s) = K_{12}(s) = g_1(s) + h_2(s) = \underline{a}_0 + \bar{a}_1\, s + \bar{a}_2\, s^2 = 15 + 18.5s + 3s^2 \tag{23}$$

$$N_3(s) = K_{21}(S) = g_2(s) + h_1(s) = \bar{a}_0 + \underline{a}_1\, s + \underline{a}_2\, s^2 = 16 + 17.5s + 2s^2 \tag{24}$$

$$N_4(s) = K_{22}(S) = g_2(s) + h_2(s) = \bar{a}_0 + \bar{a}_1\, s + \underline{a}_2\, s^2 = 16 + 18.5s + 2s^2 \tag{25}$$

Denominator Kharitonov Polynomials

Consider the denominator polynomial,

$$\begin{aligned} D(s) &= [\underline{b}_3, \overline{b}_3]s^3 + [\underline{b}_2, \overline{b}_2]s^2 + [\underline{b}_1, \overline{b}_1]s + [\underline{b}_0, \overline{b}_0] \\ &= [2,3]s^3 + [17,18]s^2 + [35,36]s + [20.5, 21.5] \end{aligned} \tag{26}$$

The polynomial $D(s)$ is split into two components g(s) and h(s) where, g(s) is polynomial of even degree and h(s) is polynomial of odd degree.

Defining two even polynomials,

$$K_1^{even,\min}(s) = g_1(s) = \underline{b}_0 + \overline{b}_2\, s^2 = 20.5 + 18s^2 \tag{27}$$

$$K_1^{even,\max}(s) = g_2(s) = \overline{b}_0 + \underline{b}_2\, s^2 = 21.5 + 17s^2 \tag{28}$$

Defining two odd polynomials,

$$K_1^{odd,\min}(s) = h_1(s) = \underline{b}_1\, s + \overline{b}_3\, s^3 = 35s + 3s^3 \tag{29}$$

$$K_1^{odd,\max}(s) = h_2(s) = \overline{b}_1\, s + \underline{b}_3\, s^3 = 36s + 2s^3 \tag{30}$$

The four Kharitonov polynomials would be

$$D_1(s) = K_{11}(s) = g_1(s) + h_1(s) = \underline{b}_0 + \underline{b}_1\, s + \overline{b}_2\, s^2 + \overline{b}_3\, s^3 = 20.5 + 35s + 18s^2 + 3s^3 \tag{31}$$

$$D_2(s) = K_{12}(s) = g_1(s) + h_2(s) = \underline{b}_0 + \overline{b}_1\, s + \overline{b}_2\, s^2 + \underline{b}_3\, s^3 = 20.5 + 36s + 18s^2 + 2s^3 \tag{32}$$

$$D_3(s) = K_{21}(s) = g_2(s) + h_1(s) = \overline{b}_0 + \underline{b}_1\, s + \underline{b}_2\, s^2 + \overline{b}_3\, s^3 = 21.5 + 35s + 17s^2 + 3s^3 \tag{33}$$

$$D_4(s) = K_{22}(s) = g_2(s) + h_2(s) = \overline{b}_0 + \overline{b}_1\, s + \underline{b}_2\, s^2 + \underline{b}_3\, s^3 = 21.5 + 36s + 17s^2 + 2s^3 \tag{34}$$

From the kharitonov polynomials available for numerator and denominator of higher order interval system, the following four interval system transfer functions may be obtained.

$$G_1(s) = \frac{N_1(s)}{D_1(s)} = \frac{\underline{a}_0 + \underline{a}_1\, s + \overline{a}_2\, s^2}{\underline{b}_0 + \underline{b}_1\, s + \overline{b}_2\, s^2 + \overline{b}_3\, s^3} = \frac{15 + 17.5s + 3s^2}{20.5 + 35s + 18s^2 + 3s^3} \tag{35}$$

$$G_2(s) = \frac{N_2(s)}{D_2(s)} = \frac{\underline{a}_0 + \overline{a}_1\, s + \overline{a}_2\, s^2}{\underline{b}_0 + \overline{b}_1\, s + \overline{b}_2\, s^2 + \underline{b}_3\, s^3} = \frac{15 + 18.5s + 3s^2}{20.5 + 36s + 18s^2 + 2s^3} \tag{36}$$

$$G_3(s) = \frac{N_3(s)}{D_3(s)} = \frac{\overline{a}_0 + \underline{a}_1\, s + \underline{a}_2\, s^2}{\overline{b}_0 + \underline{b}_1\, s + \underline{b}_2\, s^2 + \overline{b}_3\, s^3} = \frac{16 + 17.5s + 2s^2}{21.5 + 35s + 17s^2 + 3s^3} \tag{37}$$

$$G_4(s)=\frac{N_4(s)}{D_4(s)}=\frac{\overline{a}_0+\overline{a}_1\,s+\underline{a}_2\,s^2}{\overline{b}_0+\overline{b}_1\,s+\underline{b}_2\,s^2+\underline{b}_3\,s^3}=\frac{16+18.5s+2s^2}{21.5+36s+17s^2+2s^3} \quad (38)$$

Case-1: Order Reduction by Cross Multiplication of Polynomials Method

The proposed model order reduction method-1 is applied for the system interval functions $G_1(s)$, $G_2(s)$, $G_3(s)$ and $G_4(s)$. From their corresponding reduced order interval systems, the reduced order interval systems of given higher order system can be constituted.

Order Reduction of System $G_1(s)$

Consider the system transfer function

$$G_1(s)=\frac{N_1(s)}{D_1(s)}=\frac{\underline{a}_0+\underline{a}_1\,s+\overline{a}_2\,s^2}{\underline{b}_0+\underline{b}_1\,s+\overline{b}_2\,s^2+\overline{b}_3\,s^3}=\frac{15+17.5s+3s^2}{20.5+35s+18s^2+3s^3} \quad (39)$$

Corresponding second order system transfer function is to derived in the form of

$$G_{r1}(s)=\frac{\underline{d}_1\,s+\underline{d}_0}{\overline{e}_2\,s^2+\underline{e}_1\,s+\underline{e}_0}$$

$G_1(s)$ can be extended into power series about $s = 0$ of the form,

$$G_1(s)=c_0+c_1s+c_2s^2 \quad (40)$$

where

$$c_0=\frac{\underline{a}_0}{\underline{b}_0}=\frac{15}{20.5}=0.7317 \quad (41)$$

and the values of c_1, c_2,… are calculated by using the following formula,

$$c_k=\frac{1}{\underline{b}_0}\left[a_k-\sum_{j=1}^{k}b_jc_{k-j}\right],\ k>0$$

$$c_1=\frac{1}{\underline{b}_0}\left[\underline{a}_1-\underline{b}_1\,c_0\right]=\frac{1}{20.5}[17.5-(35)\times 0.7317]=0.3956$$

$$c_2=\frac{1}{\underline{b}_0}\left[\overline{a}_2-(\underline{b}_1\,c_1+\overline{b}_2\,c_0)\right]=\frac{1}{20.5}[3-(35\times 0.3956+18\times 0.7317)]=-1.1715$$

Then for $G_{r1}(s)$ to be Pade approximation of $G_1(s)$, the following equations are obtained.

$$\underline{d}_0=\underline{e}_0\cdot c_0 \quad (42)$$

$$\underline{d}_1 = \underline{e}_0 \cdot c_1 + \underline{e}_1 \cdot c_0$$
$$0 = \underline{e}_0 \cdot c_2 + \underline{e}_1 \cdot c_1 + \bar{e}_2 \cdot c_0 \quad (43)$$

From Equation (42), the following expression can be obtained.

$$c_0 = \frac{\underline{d}_0}{\underline{e}_0} \quad (44)$$

From Equations (41) and (44),

$$c_0 = \frac{\underline{a}_0}{\underline{b}_0} = \frac{\underline{d}_0}{\underline{e}_0} \quad (45)$$

Since in this example $\underline{a}_0 = 15$ and $\underline{b}_0 = 20.5$, it can be presumed as,

$$\underline{d}_0 = 15 \text{ and } \underline{e}_0 = 20.5 \quad (46)$$

The reduced order numerator and denominator polynomials constant terms $\underline{d}_0$ and $\underline{e}_0$ obtained in this step are used to obtain the unknown parameters $\underline{d}_1$, $\underline{e}_1$ and $\bar{e}_2$.

The given higher order system transfer function is compared with the general second order transfer function arrangement as,

$$\frac{3s^2 + 17.5s + 15}{3s^3 + 18s^2 + 35s + 20.5} = \frac{\underline{d}_1 s + \underline{d}_0}{\bar{e}_2 s^2 + \underline{e}_1 s + \underline{e}_0} \quad (47)$$

By cross multiplying the above equation, the following condition can be obtained.

$$(3s^2 + 17.5s + 15)(\bar{e}_2 s^2 + \underline{e}_1 s + \underline{e}_0) = (3s^3 + 18s^2 + 35s + 20.5)(\underline{d}_1 s + \underline{d}_0) \quad (48)$$

$$3\bar{e}_2 s^4 + (3\underline{e}_1 + 17.5)s^3 + (3+17.5+15)s^2 + (17.5+15)s + 15 = 3s^4 + (3+18)s^3 + (18+35)s^2 + (35+20.5)s + 20.5 \quad (49)$$

On comparing the coefficients of same power of 's' term on both sides, the following equations are obtained.

Coefficient of s^4: $3\bar{e}_2 = 3\underline{d}_1$ (50)

Coefficient of s^3: $3\underline{e}_1 + 17.5\bar{e}_2 = 3\underline{d}_0 + 18\underline{d}_1$ (51)

Coefficient of s^2: $3\underline{e}_0 + 17.5\underline{e}_1 + 15\bar{e}_2 = 18\underline{d}_0 + 35\underline{d}_1$ (52)

Coefficient of s^1: $17.5\underline{e}_0 + 15\underline{e}_1 = 35\underline{d}_0 + 20.5\underline{d}_1$ (53)

Coefficient of s^0: $15\underline{e}_0 = 20.5\underline{d}_0$ (54)

The above equations are solved with the values of $\underline{d}_0$, $\underline{e}_0$ as obtained in step-1. In which last Equation (54) gives $\underline{d}_0 = 0.7317\underline{e}_0$. From Equation (50), $\underline{d}_1$ value is replaced by $\overline{e}_2$ in Equations (51)-(53). $\underline{d}_0$ and $\underline{e}_0$ are replaced by known values of $\underline{a}_0$ and $\underline{b}_0$ respectively given in higher order system Equation (39). With the known value of $\underline{d}_0$ and $\underline{e}_0$, last but two Equation (52) and last but one Equation (53) are solved to obtain the values of $\overline{e}_2$, $\underline{e}_1$ and $\underline{d}_1$. The ISE value involved by choosing last three equations is less.

The values of $\overline{e}_2$, $\underline{e}_1$ and $\underline{d}_1$ are determined by using any other pair of equations which will give large error in the system approximation. Therefore, the corresponding reduced order model is obtained as,

$$G_{r1}(s) = \frac{\underline{d}_1 s + \underline{d}_0}{\overline{e}_2 s^2 + \underline{e}_1 s + \underline{e}_0} = \frac{3.713s + 15}{3.713s^2 + 16.157s + 20.5} \tag{55}$$

Order Reduction of System $G_2(s)$, $G_3(s)$ and $G_4(s)$

The reduced order model transfer functions of $G_2(s)$, $G_3(s)$ and $G_4(s)$ are determined in the similar fashion as in the case of $G_1(s)$. The corresponding reduced order system transfer functions are,

$$G_{r2}(s) = \frac{\overline{d}_1 s + \underline{d}_0}{\overline{e}_2 s^2 + \overline{e}_1 s + \underline{e}_0} = \frac{1.3725s + 15}{0.915s^2 + 12.592s + 20.5} \tag{56}$$

$$G_{r3}(s) = \frac{\underline{d}_1 s + \overline{d}_0}{\underline{e}_2 s^2 + \underline{e}_1 s + \overline{e}_0} = \frac{2.26s + 16}{3.374s^2 + 14.522s + 21.5} \tag{57}$$

$$G_{r4}(s) = \frac{\overline{d}_1 s + \overline{d}_0}{\underline{e}_2 s^2 + \overline{e}_1 s + \overline{e}_0} = \frac{2.243s + 16}{2.243s^2 + 14.318s + 21.5} \tag{58}$$

From the reduced order system transfer functions available in Equations (55)-(58), the following conditions are obtained.

$$\underline{d}_0 = \min\{15, 16\} = 15$$

$$\underline{d}_1 = \min\{1.3725, 2.243, 2.26, 3.713\} = 1.3725$$

$$\underline{e}_0 = \min\{20.5, 21.5\} = 20.5$$

$$\underline{e}_1 = \min\{12.592, 14.318, 14.522, 16.157\} = 12.592$$

$$\underline{e}_2 = \min\{0.915, 2.243, 3.374, 3.713\} = 0.915$$

$$\bar{d}_0 = \max\{15, 16\} = 16$$

$$\bar{d}_1 = \max\{1.3725, 2.243, 2.26, 3.713\} = 3.713$$

$$\bar{e}_0 = \max\{20.5, 21.5\} = 21.5$$

$$\bar{e}_1 = \max\{12.592, 14.318, 14.522, 16.157\} = 16.157$$

$$\bar{e}_2 = \max\{0.915, 2.243, 3.374, 3.713\} = 3.713$$

From the above resulting values, the reduced order interval system transfer function can be obtained as,

$$G_2(s) = \frac{[\underline{d}_1, \bar{d}_1]s + [\underline{d}_0, \bar{d}_0]}{[\underline{e}_2, \bar{e}_2]s^2 + [\underline{e}_1, \bar{e}_1]s + [\underline{e}_0, \bar{e}_0]} = \frac{[1.3725, 3.713]s + [15, 16]}{[0.915, 3.713]s^2 + [12.592, 16.157]s + [20.5, 21.5]} \quad (59)$$

The unit step responses of higher order interval system and reduced order interval system of Cross multiplication of polynomials method are shown in Figure 5.1.

The closeness between the higher order model and reduced order lower limit system is analyzed with the help of ISE value and is listed in Table 5.1.

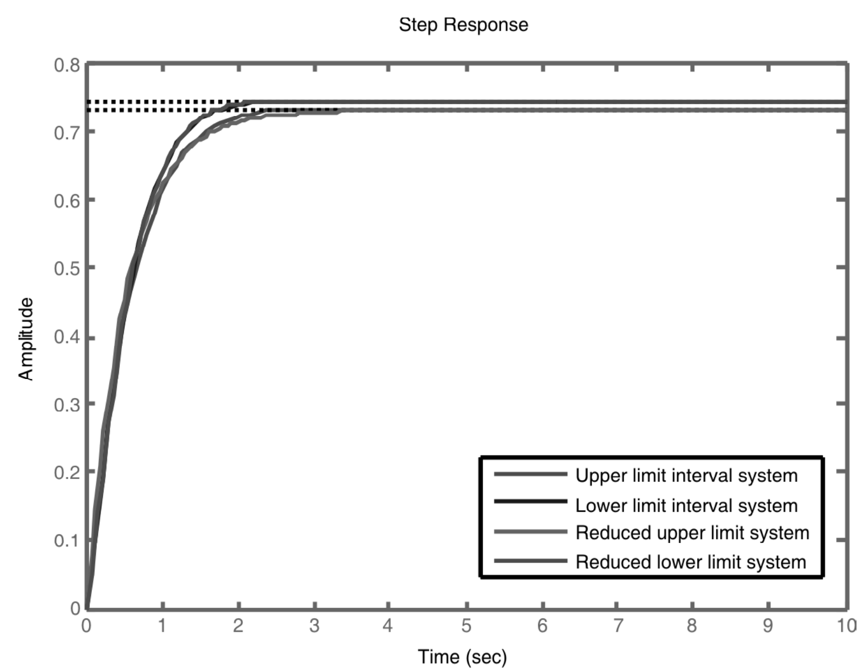

Figure 5.1: Step Responses of Higher order and Reduced order Interval Systems through Cross Multiplication of Polynomials Method in Example-5.1

Table 5.1
Comparison of Error Index Values with Existing Methods for Example 5.1

Reduction Method	*ISE*	
	Lower limit	*Upper limit*
Rajeswari Mariappan (2004)	0.4132	0.5825
Sastry G. V. K. *et al.* (2000)	0.2256	0.0095
Kranthikumar *et al.* (2011)	0.0124	0.0169
Kranthikumar *et al.* (2011 a)	0.0089	0.0113
Cross multiplication of polynomials method	0.00012989	0.000003469

5.6. Stability Analysis of Interval System

The Kharitonov theorem states that the stability of a polynomial can be determined by testing just the four Kharitonov polynomials which are obtained by using upper and lower bounds of the unknown parameters. The Kharitonov theorem is based on Mikhailov criterion which states that for a control system to be stable if it is necessary and sufficient that the locus of its characteristic equation of order '*n*' starts on for $s = j\omega$ and move successfully in the counter clockwise direction through '*n*' quadrants as ù increases from 0 to ∞.

The value set of an interval polynomial at a fixed frequency is a rectangle called Kharitonov triangle whose sides are parallel to the real and imaginary axes. The inclusion or exclusion of the origin from this rectangle value set can be known easily from the corner points, which corresponds to the Kharitonov polynomials. The movement of Mikhailov loci of K_{11}(s), K_{12}(s), K_{21}(s) and K_{22}(s) should not include its origin in the value sets and it is possible to find minimum gain and phase margins of an interval plant from a subset of fixed transfer function.

The stability of an interval system also can be determined by applying the Routh-Hurwitz criterion to the selected Kharitonov polynomials of given interval system. If the elements in the first column of the Routh array are positive then the interval system is stable, otherwise not. The nth order interval system is considered as,

$$G(s) = \frac{[\underline{a}_{n-1}, \overline{a}_{n-1}]s^{n-1} + [\underline{a}_{n-2}, \overline{a}_{n-2}]s^{n-2} + \ldots + [\underline{a}_1, \overline{a}_1]s + [\underline{a}_0, \overline{a}_0]}{[\underline{b}_n, \overline{b}_n]s^n + [\underline{b}_{n-1}, \overline{b}_{n-1}]s^{n-1} + [\underline{b}_{n-2}, \overline{b}_{n-2}]s^{n-2} + \ldots + [\underline{b}_1, \overline{b}_1]s + [\underline{b}_0, \overline{b}_0]}$$

The characteristic equation of the system 1 + $G(s)$ = 0 is obtained as,

$$1 + \frac{[\underline{a}_{n-1}, \overline{a}_{n-1}]s^{n-1} + [\underline{a}_{n-2}, \overline{a}_{n-2}]s^{n-2} + \ldots + [\underline{a}_1, \overline{a}_1]s + [\underline{a}_0, \overline{a}_0]}{[\underline{b}_n, \overline{b}_n]s^n + [\underline{b}_{n-1}, \overline{b}_{n-1}]s^{n-1} + [\underline{b}_{n-2}, \overline{b}_{n-2}]s^{n-2} + \ldots + [\underline{b}_1, \overline{b}_1]s + [\underline{b}_0, \overline{b}_0]} = 0 \quad (60)$$

which results in the polynomial,

$$P(s) = [\underline{q}_n, \overline{q}_n]s^n + [\underline{q}_{n-1}, \overline{q}_{n-1}]s^{n-1} + [\underline{q}_{n-2}, \overline{q}_{n-2}]s^{n-2} + \ldots + [\underline{q}_1, \overline{q}_1]s + [\underline{q}_0, \overline{q}_0] \quad (61)$$

The 3rd order interval system stated in Rajeswari Mariappan (2004) is considered as

$$G(s) = \frac{[2,3]s^2 + [17.5,18.5]s + [15,16]}{[2,3]s^3 + [17,18]s^2 + [35,36]s + [20.5,21.5]} \quad (62)$$

The corresponding characteristic polynomial $P(s)$ is obtained as,

$$1+\frac{[2,3]s^2+[17.5,18.5]s+[15,16]}{[2,3]s^3+[17,18]s^2+[35,36]s+[20.5,21.5]}=0$$

$$P(s)=[2,3]s^3+[19,21]s^2+[52.5,54.5]s+[35.5,37.5] \tag{63}$$

Routh-Hurwitz criterion is easily used to check the stability of a simple polynomial which becomes difficult to apply to families of polynomials because it gives nonlinear equations with the unknown parameters. The polynomial P(s) is Hurwitz stable, if and only if the following polynomials are Hurwitz stable according to the Kharitonov theorem. The Kharitonov polynomials of polynomial obtained in Equation (5.85) are,

$$K_{11}(S)=g_1(s)+h_1(s)=3s^3+21s^2+52.5s+35.5 \tag{64}$$

$$K_{12}(S)=g_1(s)+h_2(s)=2s^3+21s^2+54.5s+35.5 \tag{65}$$

$$K_{21}(S)=g_2(s)+h_1(s)=3s^3+19s^2+52.5s+37.5 \tag{66}$$

$$K_{22}(S)=g_2(s)+h_2(s)=2s^3+19s^2+54.5s+37.5 \tag{67}$$

These polynomials are called Kharitonov polynomials. The nominal polynomial of Equation (63), which is the polynomial with coefficients at the midpoint of the interval and is given by,

$$K(S)=2.5s^3+20s^2+53.5s+36.5 \tag{68}$$

The stability of the above nominal system is determined by applying the Routh-Hurwitz criterion and is depicted in Table 5.2. Since there is no sign change in the first column of Routh array, there are no roots in the right half of the s-plane. Hence, the system is stable. For each of the four polynomials stated in Equations (18) to (21), Routh array can be obtained and there is no sign change in the elements of first column of the array. This indicates the interval system is stable. The same procedure may be repeated to check the stability of reduced order model.

Table 5.2
Routh-Hurwitz **Table for Nominal Kharitonov Polynomial**

S^3	2.5	53.5
S^2	20	36.5
S^1	48.9375	0
S^0	36.5	

Consider the reduced order system transfer function obtained in Equation (77) as,

$$G_{r2}(s)=\frac{d_1s+d_0}{e_2s^2+e_1s+e_0}=\frac{1.337s+1.4137}{s^2+2.8533s+1.9319}$$

The corresponding characteristic equation is obtained as,

$$s^2+2.8533s+1.9319=0 \tag{69}$$

The Routh-Hurwitz table for the above characteristic equation is given in Table 5.3. In this table, there is no sign change in the elements of first column of the array. This indicates the reduced order system is stable. In a similar fashion, the stability can be analyzed for remaining three possible groups of kharitonov polynomial transfer functions. Thus the reduced order interval systems obtained through the proposed model order reduction methods are stable.

Table 5.3
Routh-Hurwitz Table for Reduced order System

S^2	1	1.9319
S^1	2.8533	0
S^0	1.9319	

5.7. Summary

The unit step response of original interval system and reduced order interval system using proposed method were plotted for different illustrations. The step responses of the original and reduced order model interval systems are closer to each other. Hence, depending upon the closeness of approximation desired, the order of reduction in modeling is chosen. The proposed methods are versatile and simple. The time response pattern is excellently preserved in reduced order systems even for lower order approximations. The simulation results show the flexibility of extending the proposed scenarios to the reduction of interval systems.

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