ASYMPTOTIC AND GEOMETRIC PROPERTIES OF COMPACTLY PERTURBED WIENER PROCESS AND SELF-INTERSECTION LOCAL TIME

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ABSTRACT. The new class of Gaussian processes which are obtained by a compact perturbation in the reproducing kernel of Wiener process is introduced. The finite families of increments of our processes on small time intervals behave as increments of Wiener process. Consequently a lot of asymptotical properties of Wiener process are inherited. The law of iterated logarithm, the analogue of the Levy modulus of continuity and almost uniformity of hitting distribution on the small circles are proved. The renormalized Fourier–Wiener transform of the self-intersection local time for compactly perturbed Wiener process is constructed.

1. Introduction

In this article we consider self-intersection local times for one class of planar Gaussian processes. The interest to the self-intersection local times of planar Brownian motion has a long history since the theorem of Dvoretzky, Erdös, Kakutani [3] which established the existence of multiple self-intersections. The various kinds of renormalization were proposed for the self-intersection local times of planar Brownian motion and certain Levy processes in the articles [4, 5, 10, 15, 16]. Most of these articles essentially use the Markov property of the considered process. The aim of this paper is to present an approach to investigation of planar Gaussian processes which does not use the Markov property. We introduce a new class of Gaussian processes which are obtained with the help of compact perturbation in the reproducing kernel of Wiener process. Such processes inherit many properties of Wiener process. As an example we prove here the law of iterated logarithm, the analogue of the Levy modulus of continuity and almost uniformity of hitting distribution on the small circles. The finite families of small increments of our processes behave like the increments of Wiener process. This allows us to prove that compactly perturbed Wiener process has strong local nondeterminism property, which is a generalization of local nondeterminism property introduced by S.Berman [1]. Finally we present the renormalization for Fourier–Wiener transform of the self-intersection local times for compactly perturbed Wiener process.

2. Compactly Perturbed Wiener Process

Let ξ be the Gaussian white noise in the space $L_2([0;1])$ [9]. Suppose that S is a compact operator in $L_2([0;1])$ with $\|S\| < 1$. Denote by I the identity operator in the same space.

Definition 2.1. The process

$$x(t) = ((I + S)\mathbf{1}_{[0;t]}, \xi)$$

is called by the *compactly perturbed Wiener process.*

In the sequel we will use notation $g^0(t) = \mathbf{1}_{[0;t]}$. Note that in the case $S = 0$ the process x is a Wiener process. Due to the properties of compact operator S the process x inherits properties of Wiener process on the small intervals of time. For example, it satisfies the law of iterated logarithm. To formulate the precise statement we need some additional notations. For any interval $[a; b] \subset [0; 1]$ denote by $Q_{a,b}$ the operator of multiplication on $\mathbf{1}_{[a;b]}$. It is known [7] that the compact operator S has the analog of absolute continuity property. Namely

$$\|SQ_{a,b}\| \to 0, \ b - a \to 0.$$

Denote

$$\varphi(t) = \sup_{b-a \leq t} \|SQ_{a,b}\|.$$

Theorem 2.2. *Let* $\varphi(t) = O(\sqrt{t}), t \to 0$. *Then almost surely,*

$$\varlimsup_{t \to 0} \frac{x(t)}{\sqrt{2t \text{lln} \frac{1}{t}}} = 1.$$

Proof. Since $\varlimsup_{t \to 0} \frac{w(t)}{\sqrt{2t \text{lln} \frac{1}{t}}} = 1$ then it is enough to check that with probability 1

$$\lim_{t \to 0} \frac{(Sg^0(t), \xi)}{\sqrt{2t \text{lln} \frac{1}{t}}} = 0. \tag{2.1}$$

To prove (2.1) we will use the Borel–Cantelli lemma. Put $h(t) = \sqrt{2t \text{lln} \frac{1}{t}}$ and $t_n = \theta^n$ for some $0 < \theta < 1$. Denote by $\eta(t) = (Sg^0(t), \xi)$. We need to estimate

$$P\left\{ \sup_{\theta^{n+1} \leq t \leq \theta^n} \eta(t) > \varepsilon h(\theta^n) \right\}. \tag{2.2}$$

Let us use the Berman inequality [12]. Recall its statement. Suppose that $y(t), t \in T \subset \mathbb{R}^d$ is a centered Gaussian field on a set T. Denote by

$$\bar{p}(r) = \sup_{|s-t| \leq r} \sqrt{E(y_t - y_s)^2}.$$

Define the Fernique integral $I : \mathbb{R}_+ \to [0; \infty]$ by the following formula

$$I(\delta) = \int_0^\infty \bar{p}(\delta e^{-u^2}) du.$$

It is known [12] that

$$P\{\sup_T y(t) > r\} \leq C\Big(1 - \Phi\Big(\frac{r}{\sigma}\Big)\Big)/\Big(I^{-1}\Big(\frac{1}{r}\Big)\Big)^d, \tag{2.3}$$

where

$$\sigma = \sqrt{\sup_T \mathrm{Var}\, y(t)}, \ \ \Phi(u) = \int_{-\infty}^{u} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\, dx.$$

In our case $d = 1, \sigma \leq \sqrt{\theta^n}\varphi(\theta^n)$,

$$E((\eta(t) - \eta(s)))^2 = \|S\mathbf{1}_{[s,t]}\|^2,$$

$$\overline{p}(r) = \sup_{\substack{\theta^{n+1} \leq s < t \leq \theta^n \\ t - s \leq r}} \|S\mathbf{1}_{[s,t]}\| \leq \sqrt{\theta^n - \theta^{n+1}}\varphi(r). \tag{2.4}$$

Due to (2.3)

$$P\Big\{\sup_{\theta^{n+1} \leq t \leq \theta^n} \eta(t) > \varepsilon h(\theta^n)\Big\} \leq c\Big(1 - \Phi\Big(\frac{\varepsilon h(\theta^n)}{\sigma}\Big)\Big)/I^{-1}\Big(\frac{1}{\varepsilon h(\theta^n)}\Big). \tag{2.5}$$

Since

$$1 - \Phi(r) \leq \frac{1}{\sqrt{2\pi}}\frac{1}{r}e^{-\frac{r^2}{2}},$$

then (2.5) is less or equal to

$$c\frac{\sigma}{\varepsilon h(\theta^n)}e^{-\frac{\varepsilon^2 h^2(\theta^n)}{2\sigma^2}}/I^{-1}\Big(\frac{1}{\varepsilon h(\theta^n)}\Big),$$

where c is some positive constant.

Define

$$\alpha_n := \frac{\sigma}{\varepsilon h(\theta^n)}e^{-\frac{\varepsilon^2 h^2(\theta^n)}{2\sigma^2}}/I^{-1}\Big(\frac{1}{\varepsilon h(\theta^n)}\Big)$$

and check that

$$\sum_{n=1}^{\infty} \alpha_n < \infty. \tag{2.6}$$

Notice that for the Fernique integral the following estimate holds

$$I(\delta) \leq \sqrt{\theta^n - \theta^{n+1}}\int_0^{\infty} \varphi(\delta e^{-u^2})du.$$

Put $F(\delta) = \int_0^{\infty} \varphi(\delta e^{-u^2})du$. It is obvious that

$$\frac{1}{I^{-1}\Big(\frac{1}{\varepsilon h(\theta^n)}\Big)} \leq \frac{1}{F^{-1}\Big(\frac{1}{\varepsilon h(\theta^n)}\Big)\sqrt{\theta^n - \theta^{n+1}}}. \tag{2.7}$$

Since $\varphi(t) = O(\sqrt{t}), \ t \to 0$, then

$$F(\delta) \leq c\sqrt{\delta}. \tag{2.8}$$

The estimates (2.4), (2.7), (2.8) imply that

$$\alpha_n \leq c\varepsilon h(\theta^n)(\theta^n)^3 e^{-\frac{\varepsilon^2 h^2(\theta^n)}{2c^2(\theta^n)^2}}. \tag{2.9}$$

It follows from (2.9) that (2.6) holds. The Borel–Cantelli lemma ends the proof of the theorem. $\qquad\square$

Further we discuss the Levy modulus of continuity [11, 14] for compactly perturbed Wiener process.

Theorem 2.3. *Let* $\varphi(t) = O(\sqrt{t}), t \to 0$. *Then almost surely,*

$$\overline{\lim_{h \to 0}} \sup_{0 \le t \le 1-h} \frac{|x(t+h) - x(t)|}{\sqrt{2h \ln \frac{1}{h}}} = 1.$$

Proof. In case of Wiener process Levy proved [11, 14] that almost surely,

$$\overline{\lim_{h \to 0}} \sup_{0 \le t \le 1-h} \frac{|w(t+h) - w(t)|}{\sqrt{2h \ln \frac{1}{h}}} = 1.$$

Here it is sufficient to check that

$$P\left\{ w : \forall \, \varepsilon > 0 \, \exists \, \delta : \forall \, h \in (0; \delta) \, \forall \, t \in [0; 1-h] \quad \frac{|(Sg^0(t, t+h), \xi)|}{\sqrt{2h \ln \frac{1}{h}}} < \varepsilon \right\} = 1,$$

(2.10)

where $g^0(t, t+h) = \mathbf{1}_{[t; t+h]}$.

Define $\psi(t) = \sqrt{2t \ln \frac{1}{t}}$. For $0 < \theta < 1$ put $h^n = \theta^n$ and estimate

$$P\left\{ \sup_{\theta^{n+1} \le h \le \theta^n} \sup_{0 \le t \le 1-h} \frac{(Sg^0(t, t+h), \xi)}{\psi(\theta^n)} > \varepsilon \right\}.$$

Let us apply the Berman inequality to the Gaussian field

$$y(v) = (Sg^0(t, t+h), \xi), v \in T,$$

where

$$T = \{(t, h) : 0 \le t \le 1-h, \quad \theta^{n+1} \le h \le \theta^n\} \subset \mathbb{R}^2.$$

Put $\mathfrak{M} = \{0 \le t_1 \le 1-h_1, \ 0 \le t_2 \le 1-h_2, \ \theta^{n+1} \le h_i \le \theta^n,$
$i = 1, 2 : \ \|(t_1, h_1) - (t_2, h_2)\| \le r\}$. Notice that

$$\begin{aligned}
\overline{p}(r) &= \sup_{\mathfrak{M}} \sqrt{E(S(g^0(t_1, t_1+h_1) - g^0(t_2, t_2+h_2)), \xi)^2} \\
&= \sup_{\mathfrak{M}} \|S(g^0(t_1, t_1+h_1) - g^0(t_2, t_2+h_2)\| \\
&\le \sup_{\mathfrak{M}} \|S\| \|g^0(t_1, t_1+h_1) - g^0(t_2, t_2+h_2))\| \le c\sqrt{r},
\end{aligned}$$

with some positive constant c. It implies that in our case $I(\delta) \le c\sqrt{\delta}$. By using the Berman inequality [12] with $d = 2$ we conclude that

$$P\left\{ \sup_T y(u) > \varepsilon \psi(\theta^n) \right\} \le c \frac{\sigma}{\varepsilon \psi(\theta^n)} e^{-\frac{\varepsilon^2 \psi^2(\theta^n)}{2\sigma^2}} / I^{-1}\left(\frac{1}{\varepsilon \psi(\theta^n)}\right)^2.$$

Since $\sigma \le \varphi(\theta^n) \le c\theta^n$ then the last expression less or equal to $\alpha_n = c(\varepsilon \psi(\theta^n))^3 \theta^n e^{-\frac{\varepsilon^2 \psi^2(\theta^n)}{2(\theta^n)^2}}$. Since $\sum_{n+1}^{\infty} \alpha_n < +\infty$ then applying Borel–Cantelli lemma one can get the statement of the theorem. $\square$

The next statement shows that the planar compactly perturbed Wiener process hits the small circle from its center with the almost uniform distribution. Here we consider planar process $x(t) = (x_1(t), x_2(t))$, where x_1 and x_2 are independent copies of the compactly perturbed Wiener process.

For process x define $\tau_r^x = \inf\{t : \|x(t)\| = r\}$ ($\tau_r^x = 1$ if $\max_{[0;1]} \|x(t)\| < r$).

Theorem 2.4. *Let $\varphi(t) = o(\ln t)^{-3}, t \to 0$. Then the distribution of the random vector $\frac{1}{r} x(\tau_r^x)$ converges weakly to the uniform distribution on the unit circle, when $r \to 0$.*

Proof. Since for $r > 0, t \in [0;1]$

$$P\{\tau_r^w < t\} = P\Big\{ \max_{[0;1]} \|w(s)\| \geq \frac{r}{\sqrt{t}} \Big\}.$$

Consequently

$$P\{\tau_r^w < t\} \to 1, \ \frac{r}{\sqrt{t}} \to 0.$$

In the future we will suppose, that $t \to 0$ in a such way that

$$\frac{r}{\sqrt{t}} \to 0. \tag{2.11}$$

Put $\eta(t) = ((Sg^0(t), \xi), (Sg^0(t), \xi_2))$, where ξ_1, ξ_2 are independent white noises. Then $x(t) = w(t) + \eta(t)$. Let us recall the Slepian comparison inequality [13].

Lemma (Slepian). *Let ξ and ζ be mean zero $\mathbb{R}^n$-valued Gaussian random variables such that for an arbitrary $0 \leq j, k \leq n$*

$$E(\zeta_j - \zeta_k)^2 \leq E(\xi_j - \xi_k)^2.$$

Then $E \sup_{j,k} |\zeta_j - \zeta_k| \leq E \sup_{j,k} |\xi_j - \xi_k|$.

Note that

$$E \max_{[0;t]} \|\eta(s)\| \leq E(\max_{[0;t]} |\eta_1(s)| + \max_{[0;t]} |\eta_2(s)|)$$
$$= 2E \max_{[0;t]} |\eta_1(s)| \leq 2E \max_{s_1, s_2 \in [0;t]} |\eta_1(s_1) - \eta_1(s_2)|. \tag{2.12}$$

Applying the Slepian inequality one can obtain that (2.12) is less or equal to

$$2\varphi(t)E \max_{s_1, s_2 \in [0;t]} |w(s_1) - w(s_2)| = 2\sqrt{t}\varphi(t)E \max_{s_1, s_2 \in [0;t]} |w(s_1) - w(s_2)| = c\sqrt{t}\varphi(t).$$

Consequently it is possible to take $\delta \to 0$ in a such way that

$$P\{\max_{[0;t]} \|\eta(s)\| < \delta\} \to 1,$$

and $\frac{\delta}{r} \to 0$. For example it is enough to take $\delta = \sqrt{t}\varphi(t)^{1/2}, r = \sqrt{t}\varphi(t)^{1/4}$. On the set $\{\tau_r^w < t\} \cap \{\max_{[0;t]} \|\eta(s)\| < \delta\}$ the following relations holds

$$\tau_{r-2\delta}^w \leq \tau_{r-\delta}^x \leq \tau_r^w.$$

Consequently on this set

$$\left\| \frac{1}{r-\delta} x(\tau^x_{r-\delta}) - \frac{1}{r} w(\tau^w_r) \right\|$$

$$\leq \left\| \left(\frac{1}{r} - \frac{1}{r-\delta} \right) w(\tau^w_r) \right\| + \frac{1}{r-\delta} \left\| x(\tau^x_{r-\delta}) - w(\tau^w_r) \right\|$$

$$\leq \left(\frac{1}{r} - \frac{1}{r-\delta} \right) r + \frac{1}{r-\delta} \delta + \frac{1}{r-\delta} \cdot 2 \cdot \max_{[\tau^w_{r-2\delta};\tau^w_r]} \| w(s) - w(\tau^w_{r-2\delta}) \|.$$

To get the desired convergence it remains to estimate

$$\max_{[\tau^w_{r-2\delta};\tau^w_r]} \| w(s) - w(\tau^w_{r-2\delta}) \|$$

on the set $\{\tau^w_r < t\}$. Using the strong Markov property of w and independence of its coordinates w_1 and w_2 we conclude that

$$\max_{[\tau^w_{r-2\delta};\tau^w_r]} \| w(s) - w(\tau^w_{r-2\delta}) \|$$

$$\leq \{ \max_{[0;\zeta]} \widetilde{w}_1(s) + \min(r, \max_{[0;\zeta]} \widetilde{w}_2(s) \} + \delta.$$

Here $\widetilde{w}_1$ and $\widetilde{w}_2$ are independent Wiener processes, $\widetilde{w}_1(0) = \delta$ and ζ is the moment of the first hitting zero by $\widetilde{w}_1$ abridged by 1. Using independence one can see, that

$$E\{\min(r, \max_{[0;\zeta]} |\widetilde{w}_2(s)|)\} \leq EE(\max_{[0;\zeta]} \widetilde{w}_2(s)|\zeta) \leq cE\sqrt{\zeta}.$$

One can check that

$$M\sqrt{\zeta} \sim \delta \ln \frac{1}{\delta^2}, \ \delta \to 0.$$

Now using Levy inequality [8] one can get the following estimate

$$\varepsilon P\{ \max_{[0;\zeta]} \widetilde{w}_1(s) > \varepsilon \} \leq cE\widetilde{w}_1(\min(1,\zeta)) = c\delta,$$

where c is some positive constant. Hence

$$P\text{-}\lim_{r \to 0} \frac{1}{r-\delta} \max_{[\tau^w_{r-2\delta};\tau^w_r]} \| w(s) - w(\tau^w_{r-2\delta}) \| = 0.$$

It finishes the proof. $\qquad\qquad\square$

The compactly perturbed Wiener processes have another property which is important for consideration of the self-intersection local times. This is the strong local nondeterminism property. The property of local nondeterminism was introduced by S.Berman in [1]. It reflects the independence between the increments of the process. If the process is defined as an inner product with white noise

$$x(t) = (g(t), \xi), \ t \in [0;1],$$

then the strong local nondeterminism property can be formulated as a condition on the increments of g. Suppose, that the function g is such, that
$\forall\, 0 \leq t_1 < \ldots < t_n \leq 1:$

$$\Gamma_{t_1 \ldots t_n} = G(\Delta g(t_1), \ldots, \Delta g(t_{n-1})) > 0.$$

Here $G(e_1, \ldots, e_m)$ is the Gram determinant for the vectors $e_1, \ldots, e_m$.

Definition 2.5. The process x (or the function g) is *strongly locally nondetermined* if for arbitrary $k \geq 2$ and subset $M \subset \{1, \ldots, k-1\}$

$$\Gamma_{t_1 \ldots t_k} \sim G(\Delta g(t_i), i \notin M) \prod_{i \in M} \|\Delta g(t_i)\|^2,$$

when $\max_{i \in M} \Delta t_i \to 0$.

The condition of this definition means that the small increments of x are in some sense uniformly independent. Let us check that compactly perturbed Wiener process satisfies Definition 2.5.

Lemma 2.6. *Suppose that* $\|S\| < 1$, *then the function* $g(t) = (I + S)g_0(t)$ *has the strong local nondeterminism property.*

This lemma was proved in [2] but here we present more straightforward proof.

Proof. Firstly, note that for an arbitrary $\delta > 0$

$$\inf\{G(g(t_1'') - g(t_1'), \ldots, g(t_n'') - g(t_n')) : 0 \leq t_1' < t_1'' \leq \ldots \leq t_n' < t_n'' \leq 1,$$

$$\min_{i=\overline{1,n}}(t_i'' - t_i') \geq \delta\} > 0.$$

This relation obviously follows from the existence of $(I+S)^{-1}$. Also it follows from the compactness of operator S that

$$\|g(t'') - g(t')\| \sim \sqrt{t'' - t'}, \ t'' - t' \to 0.$$

Now consider the sequence $\{0 \leq t_1^k < \ldots < t_n^k \leq 1, \ k \geq 1\}$ and the subset $M' \subset \{1, \ldots, n-1\}$ such that

$$\max_{i \in M'} t_{i+1}^k - t_i^k \to 0, \ k \to \infty,$$

$$\lim_{k \to \infty} \min_{i \notin M'} t_{i+1}^k - t_i^k > 0.$$

Again using compactness of S, one can easily check that

$$\Gamma_{t_1^k, \ldots, t_n^k} \sim G(g(t_{i+1}) - g(t_i), \ i \notin M'), \ \prod_{i \in M'} (t_{i+1}^k - t_i^k), \ k \to \infty.$$

The statement of the lemma follows from this relation by the standard arguments. Lemma is proved. $\qquad\square$

It is interesting to compare the strong local nondeterminism with the local nondeterminism [1]. In our terms (using representation with the function g) the local nondeterminism property can be formulated as follows.

Definition 2.7. The process x (or the function g) is *locally nondermined* if for every $n \geq 2$

$$\lim_{t_n - t_1 \to 0} \frac{\Gamma_{t_1 \ldots t_n}}{\Gamma_{t_1 \ldots t_{n-1}} \|\Delta g(t_{n-1})\|^2} > 0.$$

Evidently this condition follows from the condition of Definition 2.1. The backward statement is not true.

It was mentioned that the local nondeterminism property was useful condition for existence of local times. In the next section we will discuss the renormalization of the self-intersection local times for compactly perturbed Wiener process. May be it can be built for strongly nondetermined processes, but at the present moment we can not prove this.

3. Renormalization of the Fourier–Wiener Transform for the Self-intersection Local Times

In this section we consider

$$T_k^x = \int_{\Delta_k} \prod_{i=1}^{k-1} \delta_0(x(s_{i+1}) - x(s_i)) d\vec{s},$$

where $\Delta_k = \{0 \le s_1 \le \ldots \le s_k \le 1\}$. This formal expression can be considered as a self-intersection local time for process x. It is well known that for the Wiener process the self-intersection local time needs renormalization to be properly defined [4, 16]. Here we propose the renormalization for the formal Fourier–Wiener transform of the self-intersection local time for x. Let us recall that for random variable α which has a finite second moment and is measurable with respect to the white noise (ξ_1, ξ_2) its Fourier–Wiener transform [17] is

$$\mathcal{T}(\alpha)(h_1, h_2) = E\alpha \mathcal{E}(h_1, h_2) = E\alpha \exp\{(h_1, \xi_1) + (h_2, \xi_2) - \frac{1}{2}\|h_1\|^2 - \frac{1}{2}\|h_2\|^2\}.$$

It is well-known [9], that Fourier–Wiener transform uniquely determines random variable α. To consider the Fourier–Wiener transform of $\prod_{i=1}^{k-1} \delta_0(x(s_{i+1}) - x(s_i))$ one can substitute δ_0 by the two-dimensional Gaussian density

$$f_\varepsilon(u) = \frac{1}{2\pi\varepsilon} e^{-\frac{\|u\|^2}{2\varepsilon}}$$

and pass to the limit when $\varepsilon \to 0$.

It can be checked that the formal Fourier–Wiener transform is

$$\mathcal{T}\left(\prod_{i=1}^{k-1} \delta_0(x(s_{i+1}) - x(s_i))\right)(h_1, h_2) = \frac{e^{-\frac{1}{2}\left(\|P_{t_1 \ldots t_k} h_1\|^2 + \|P_{t_1 \ldots t_k} h_2\|^2\right)}}{\Gamma_{t_1 \ldots t_k}},$$

where $\Gamma_{t_1 \ldots t_k}$ is a Gram determinant constructed on $\Delta g(t_1), \ldots, \Delta g(t_{k-1})$ (we suppose that for any $0 \le t_1 < t_2 < \ldots < t_k \le 1$, $\Gamma_{t_1 \ldots t_k} \ne 0$), $g(t) = (I + S)g^0(t)$, $P_{t_1 \ldots t_k}$ – is a projection on the linear span of $(\Delta g(t_1), \ldots, \Delta g(t_{k-1}))$. We will construct regularization [6] for

$$\int_{\Delta_k} \frac{e^{-\frac{1}{2}(\|P_{t_1 \ldots t_k} h_1\|^2 + \|P_{t_1 \ldots t_k} h_2\|^2)}}{\Gamma_{t_1 \ldots t_k}} d\vec{t}$$

because this integral is divergent on every diagonal of the simplex Δ_k.

Denote by $\widetilde{\Delta g}(t_1), \ldots, \widetilde{\Delta g}(t_{k-1})$ the orthonormal system which is obtained from $\Delta g(t_1), \ldots, \Delta g(t_{k-1})$ via the orthogonalization procedure. Since the elements $\Delta g(t_1), \ldots, \Delta g(t_{k-1})$ are linearly independent all the elements $\widetilde{\Delta g}(t_1),$

$\ldots, \widetilde{\Delta g}(t_{k-1})$ are non-zero. For $M \subset \{1, \ldots, k-1\}$ denote by P_M the projection on $\widetilde{\Delta g}(t_i), i \in M$. The main result of this section is the following theorem.

Theorem 3.1 ([2]). *For an arbitrary $h \in L_2([0;1])$ the following integral converges*

$$\int_{\Delta_k} \Gamma_{t_1 \ldots t_k}^{-1} \left(\sum_{M \subset \{1, \ldots, k-1\}} (-1)^{|M|} e^{-\frac{1}{2}\|P_M h\|^2} \right) d\vec{t}.$$

Proof. Let $\Delta_1, \ldots, \Delta_n$ be disjoint intervals $(t_1^1, t_2^1), \ldots, (t_1^n, t_2^n)$ of $[0;1]$. Suppose that $\Delta_i g = g(t_2^i) - g(t_1^i)$, $G(\Delta_1 g, \ldots, \Delta_n g)$ is a Gram determinant constructed on $\Delta_1 g, \ldots, \Delta_n g$. Since for any $n \geq 1$ (see Lemma 2.6)

$$c_n = \inf_{t_1^1, t_2^1, \ldots, t_1^n, t_2^n \in [0;1]} \frac{G(\Delta_1 g, \ldots, \Delta_n g)}{\prod_{j=1}^n (t_2^j - t_1^j)} > 0,$$

then

$$\int_{\Delta_k} \Gamma_{t_1 \ldots t_k}^{-1} \left| \sum_{M \subset \{1, \ldots, k-1\}} (-1)^{|M|} e^{-\frac{1}{2}\|P_M h\|^2} \right| d\vec{t}$$

$$\leq c \int_{\Delta_k} \frac{1}{\prod_{i=1}^{k-1}(t_{i+1} - t_i)} \left| \sum_{M \subset \{1, \ldots, k-1\}} (-1)^{|M|} e^{-\frac{1}{2}\|P_M h\|^2} \right| d\vec{t}.$$

One can check that

$$\int_{\Delta_k} \frac{1}{\prod_{i=1}^{k-1}(t_{i+1} - t_i)} \left| \sum_{M \subset \{1, \ldots, k-1\}} (-1)^{|M|} e^{-\frac{1}{2}\|P_M h\|^2} \right| d\vec{t}$$

$$= \int_{\Delta_k} \prod_{j=1}^{k-1} \frac{1 - e^{-\frac{1}{2}(h, \widetilde{\Delta g}(t_j))^2}}{t_{j+1} - t_j} d\vec{t} \leq \int_{\Delta_k} \prod_{j=1}^{k-1} \frac{(h, \widetilde{\Delta g}(t_j))^2}{t_{j+1} - t_j} d\vec{t}.$$

Consider the last integral

$$\int_{t_{k+1}}^1 \frac{(h, \widetilde{\Delta g}(t_{k-1}))^2}{t_k - t_{k-1}} dt_k.$$

Notice that

$$\widetilde{\Delta g}(t_{k-1}) = \frac{\Delta g(t_{k+1}) - P_{t_1 \ldots t_{k-1}} \Delta g(t_{k-1})}{\|\Delta g(t_{k-1}) - P_{t_1 \ldots t_{k-1}} \Delta g(t_{k-1})\|}.$$

Since the process x is strongly locally nondeterministic, then

$$\|\Delta g(t_{k-1}) - P_{t_1 \ldots t_{k-1}} \Delta g(t_{k-1})\|^2 = \frac{\Gamma_{t_1 \ldots t_k}}{\Gamma_{t_1 \ldots t_{k-1}}} \sim \|\Delta g(t_{k-1})\|^2, t_k - t_{k-1} \to 0.$$

Also,

$$\|\Delta g(t_{k-1})\|^2 \sim t_k - t_{k-1}, \ t_k - t_{k-1} \to 0.$$

That is why

$$\frac{\|\Delta g(t_k) - P_{t_1 \ldots t_{k-1}} \Delta g(t_k)\|^2}{t_k - t_{k-1}} \to 1, \ t_k - t_{k-1} \to 0.$$

It implies that there exists the positive constant c such that

$$\frac{\|\Delta g(t_{k-1}) - P_{t_1\ldots t_{k-1}}\Delta g(t_{k-1})\|^2}{t_k - t_{k-1}} \geq c.$$

Consequently

$$\int_{t_{k-1}}^1 \frac{(h, \widetilde{\Delta g}(t_{k-1}))^2}{t_k - t_{k-1}} dt_k$$

$$\leq c \int_{t_{k-1}}^1 \frac{(h, \Delta g(t_{k-1}) - P_{t_1\ldots t_{k-1}}\Delta g(t_{k-1}))^2}{(t_k - t_{k-1})^2} dt_k$$

$$\leq 2c \left[\int_{t_{k-1}}^1 \frac{(h, \Delta g(t_{k-1}) - P_{t_1\ldots t_{k-1}}\Delta g(t_{k-1}))^2}{(t_k - t_{k-1})^2} \right] dt_k$$

$$\leq 2c \left[\int_{t_{k-1}}^1 \frac{(h, \Delta g(t_{k-1}))^2}{(t_k - t_{k-1})^2} dt_k + \int_{t_{k-1}}^{t_k} \frac{(h, P_{t_1\ldots t_{k-1}}\Delta g(t_{k-1}))^2}{(t_k - t_{k-1})^2} dt_k \right].$$

Consider in $L_2([t_{k-1}; 1])$ integral operator with the kernel

$$k(s_1, s_2) = \frac{1}{s_2 - t_{k-1}} \mathbf{1}_{\{s_2 > s_1\}}.$$

Let us check that k defines bounded operator in $L_2([t_{k-1}; 1])$ using the Shur test [7]. If there exists positive functions $p, q : [t_{k-1}; 1] \to (0, \infty)$ and α, β such that

$$\int_{t_{k-1}}^1 k(s_1, s_2) q(s_2) ds_2 \leq \alpha p(s_1),$$

$$\int_{t_{k-1}}^1 k(s_1, s_2) p(s_1) ds_1 \leq \beta q(s_2),$$

then k corresponds to bounded operator with the norm less or equal to $\alpha\beta$.

Put

$$p(s_1) = \frac{1}{\sqrt{s_1 - t_{k-1}}},$$

$$q(s_2) = \frac{1}{\sqrt{s_2 - t_{k-1}}}.$$

Then

$$\int_{t_{k-1}}^1 k(s_1, s_2) q(s_2) ds_2 = \int_{s_1}^1 \frac{1}{(s_2 - t_{k-1})^{\frac{3}{2}}} ds_2 \leq \frac{1}{\sqrt{s_1 - t_{k-1}}},$$

$$\int_{t_{k-1}}^1 k(s_1, s_2) p(s_1) ds_1 = \int_{t_{k-1}}^{s_2} \frac{1}{\sqrt{s_1 - t_{k-1}}} ds_1 \cdot \frac{1}{s_2 - t_{k-1}} = \frac{2}{\sqrt{s_2 - t_{k-1}}}.$$

So we get the following estimate

$$2 \int_{t_{k-1}}^1 h(s_1) \int_{s_1}^1 \frac{h(s_2)}{s_2 - t_{k-1}} ds_2 ds_1 \leq 8\|h\|^2.$$

Consequently

$$\int_{t_{k-1}}^1 \frac{(h, \Delta g(t_{k-1}))^2}{(t_k - t_{k-1})^2} dt_k = \int_{t_{k-1}}^1 \frac{((I + S^*)h, \Delta g^0(t_{k-1}))^2}{(t_k - t_{k-1})^2} dt_k$$

$$\leq c \cdot \|(I + S^*)h\|^2 \leq c' \cdot \|h\|^2 \tag{3.1}$$

and

$$\int_{t_{k-1}}^{1} \frac{(h, P_{t_1 \ldots t_{k-1}} \Delta g(t_{k-1}))^2}{(t_k - t_{k-1})^2} dt_k = \int_{t_{k-1}}^{1} \frac{((I + S^*)P_{t_1 \ldots t_{k-1}} h, \Delta g^0(t_{k-1}))^2}{(t_k - t_{k-1})^2} dt_k$$

$$\leq c \cdot \|(I + S^*)P_{t_1 \ldots t_{k-1}} h\|^2 \leq c' \cdot \|h\|^2.$$

The estimates (3.1), (3) imply that

$$\int_{t_{k-1}}^{1} \frac{(h, \widetilde{\Delta g}(t_{k-1}))^2}{t_k - t_{k-1}} dt_k \leq c'' \cdot \|h\|^2.$$

By returning to the integral

$$\int_{\Delta_k} \prod_{j=1}^{k-1} \frac{(h, \widetilde{\Delta g}(t_j))^2}{t_{j+1} - t_j} d\vec{t}$$

we have the following an equality

$$\int_{\Delta_k} \prod_{j=1}^{k-1} \frac{(h, \widetilde{\Delta g}(t_j))^2}{t_{j+1} - t_j} d\vec{t}$$

$$= \int_{\Delta_{k-1}} \prod_{j=1}^{k-2} \frac{(h, \widetilde{\Delta g}(t_j))^2}{t_j - t_{j-1}} \cdot \int_{t_{k-1}}^{1} \frac{(h, \widetilde{\Delta g}(t_{k-1}))^2}{t_k - t_{k-1}} dt_k dt_1 \ldots dt_{k-1}$$

$$\leq c'' \|h\|^2 \int_{\Delta_{k-1}} \prod_{j=1}^{k-2} \frac{(h, \widetilde{\Delta g}(t_j))^2}{t_{j+1} - t_j} d\vec{t}.$$

Repeating the analogous procedure $k - 1$ times we get the following estimate

$$\int_{\Delta_k} \prod_{j=1}^{k-1} \frac{(h, \widetilde{\Delta g}(t_j))^2}{t_{j+1} - t_j} d\vec{t} \leq c''' \|h\|^{2(k-1)}$$

which ends the theorem. Theorem is proved. $\square$

It is easy to see, that in case $S = 0$ (Wiener process) the proposed renormalization coincides with the Rosen renormalization [16].

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