IJCTA, 9(28), 2016, pp. 483-489

Generalized Hyers-Ulam Stability of a Quadratic Functional Equation in Quasi-β-Normed Spaces

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Abstract : In this paper, we obtain the general solution and investigate the generalized Hyers-Ulam stability of a new quadratic functional equation

$$f(3x+y)+f(2x+y)+5f(x-y)=18f(x)+7f(y)$$

in quasi-β-normed spaces using fixed point method. A counter-example is also presented for a singular case.

2000 Mathematics Subject Classification. : 39B82, 39B72.

Keywords and phrases : Reciprocal function, generalized Hyers-Ulam stability.

1. INTRODUCTION

The concept of stability for a functional equation arises when one replaces a functional equation by an inequality which acts as a perturbation of the equation. The first stability problem of functional equation was raised by S.M. Ulam [13] about seventy five years ago and in the next year the Ulam's problems was partially answered by D.H. Hyers [6]. In the year 1950, T. Aoki [1] generalized Hyers' theorem for additive mappings. The result of Hyers was generalized by Th.M. Rassias [12] for approximate linear mappings by allowing the Cauchy difference operator $\mathrm{CD}f(x,y)=f(x+y)-[f(x)+f(y)]$ to be controlled by $\varepsilon\left(\|x\|^p+\|y\|^p\right)$. In 1982, J.M. Rassias [10] treated the Ulam-Gavruta-Rassias stability on linear and non-linear mappings and generalized Hyers' result. In 1994, a further generalization of the Th.M. Rassias' theorem was obtained by P. Gavruta [5] who replaced the bound $\varepsilon\left(\|x\|^p+\|y\|^p\right)$ by a general control function $\varphi(x,y)$. The stability problems of several functional equations have been extensively investiaged by a number of mathematicians, posed with creative thinking and critical dissent who have arrived at interesting results (see [3], [5], [11]).

In 1996, G. Isac and Th.M. Rassias [7] were the first to provide applications of stability theory of functional equations fo the proof of new fixed point theorems with applications. The stability problems of several various functional equations have been extensively investigated by a number of authors using fixed point methods (see [2], [9], [14], [16]).

The functional equation
$$f(x+y)+f(x-y) = 2f(x)+2f(y) \tag{1.1}$$
is said to be a quadratic functional equation because the quadratic function $f(x)=ax^2$ is a solution of the functional equation (1.1).

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In this paper, we achieve the general solution of a new quadratic functional equation

$$f(3x+y)+f(2x+y)+5f(x-y) = 18f(x)+7f(y). \tag{1.2}$$

Further, we investigate the generalized Hyers-Ulam stability in quasi-β-normed spaces via fixed point method. Using Gajda's example, we also provide counter-example for a singular case.

2. PRELIMINARIES

For the sake of convenience, we recall some basic concepts concerning quasi-β-normed spaces. Let β be a fixed real number with $0<\beta\le 1$ and let K denote either $\mathbb{R}$ or $\mathbb{C}$.

Definition 2.1. Let X be a linear space over K. A quasi-β-norm $\|\cdot\|$ is a real-valued function on X satisfying the following conditions:

1. $\|x\|\ge 0$ for all $x\in X$ and $\|x\|=0$ if and only if $x=0$.
2. $\|\lambda x\|=|\lambda|^{\beta}\cdot\|x\|$ for all $\lambda\in K$ and all $x\in X$.
3. There is a constant $K\ge 1$ such that $\|x+y\|\le K(\|x\|+\|y\|)\ \forall\ x,y\in X$.

The pair $(X,\|\cdot\|)$ is called quasi-β-normed space if $\|\cdot\|$ is a quasi-β-norm on X. The smallest possible K is called the modulus of concavity of $\|\cdot\|$.

Definition 2.2. A quasi-β-Banach space is a complete quasi-β-normed space.

3. GENERALIZED SOLUTION OF EQUATION (1.2)

In the following theorem, we obtain the general solution of the functional equation (1.2).

Theorem 3.1. Let X, Y be vector spaces. A function $f: X\to Y$ satisfies (1.2) if and only if f satisfies (1.1).

Proof. Assume that f satisfies the functional equation (1.2). Replacing $x=y=0$ in (1.2), one gets $f(0)=0$. Plugging (x,y) into $(0,x)$ in (1.2), we obtain

$$f(-x) = f(x) \tag{3.1}$$

for all $x\in X$. Substituting (x,y) as $(x,-x)$ in (1.2), by using (3.1) and simplifying further, we find

$$f(2x) = 4f(x) \tag{3.2}$$

for all $x\in X$. Switching (x,y) to $(x,-x+y)$ in (1.2) and using equation (3.1), we get

$$f(2x+y)+5f(2x-y)+f(x+y)-7f(x-y) = 18f(x) \tag{3.3}$$

for all $x,y\in X$. Replacing (x,y) by $\left(\frac{x}{2},0\right)$ in (3.3), on obtains that

$$f\left(\frac{x}{2}\right) = \frac{1}{4}f(x) \tag{3.4}$$

for all $x\in X$. Switching (x,y) to $(x,-y)$ in (3.3), we have

$$f(2x-y)+5f(2x+y)+f(x-y)-7f(x+y) = 18f(x) \tag{3.5}$$

for all $x,y\in X$. Now, summing the equations (3.4) and (3.5) and then dividing by 6 in the resulting equation, we find

$$f(2x+y)+f(2x-y) = 6f(x)+f(x+y)+f(x-y) \tag{3.6}$$

for all $x,y\in X$. Plugging (x,y) into $\left(\frac{x}{2},y\right)$ in (3.6), we obtain

$$4[f(x+y)+f(x-y)] = 6f(x)+f(x+2y)+f(x-2y) \tag{3.7}$$

for all $x, y \in X$. Interchaning x and y in (3.7) and using (3.1), we get

$$4[f(x+y)+f(x-y)] = 6f(y)+f(2x+y)+f(2x-y) \tag{3.8}$$

for all $x, y \in X$. Using (3.6) in (3.8) and then dividing the resulting equation by 3, we arrive at equation (1.1).

Conversely, suppose f satisfies (1.1). Replacing (x, y) by $(2x, x+y)$ in (1.1), we get

$$f(3x+y)+f(x-y) = 8f(x)+2f(x+y) \tag{3.9}$$

for all $x, y \in X$. Plugging (x, y) into $(x, x+y)$ in (1.1), one finds that

$$f(2x+y)+f(y) = 2f(x)+2f(x+y) \tag{3.10}$$

for all $x, y \in X$. Summing the equation (3.9) with the equation (3.10), we have

$$f(3x+y)+f(2x+y)+f(x-y) = 10f(x)-f(y)+4f(x+y) \tag{3.11}$$

for all $x, y \in X$. Multiplying the equation (1.1) by 4, we obtain

$$4f(x+y) = 8f(x)+8f(y)-4f(x-y) \tag{12}$$

for all $x, y \in X$. Using (3.12) in (3.11), we arrive at (1.2), which completes the proof.

4 .GENERALIZED HYERS- ULAM STABILITY OF EQUATION (1.2)

Throughout this section, we assume that X is a linear space, Y is a quasi-β-Banach space with norm $\|\cdot\|_Y$. Let K be the modulus of concavity of $\|\cdot\|_Y$. We will investigate the generalized Hyers-Ulam stability of the functional equation (1.2) in quasi-β-normed spaces. For notational convenience, given a function $f : X \to Y$, we define the difference operator

$$D_q f(x, y) = f(3x+y)+f(2x+y)+5f(x-y)-18f(x)-7f(y), \forall x, y \in X.$$

Lemma 4.1. (see [15]). Let $j \in \{-1,1\}$ be fixed, $s, a \in N$ with $a \geq 2$ and $\Psi : X \to [0,\infty)$ be a function such that there exists an $L < 1$ with $\Psi\left(a^j x\right) \leq a^{js\beta} L \Psi(x)$ for all $x \in X$. Let $f : X \to Y$ be a mapping satisfying

$$\left\| f(ax) - a^s f(x) \right\|_Y \leq \Psi(x) \tag{4.1}$$

for all $x \in X$, then there exists a uniquely determined mapping $F : X \to Y$ such that $F(ax) = a^s F(x)$ and

$$\| f(x) - F(x) \|_Y \leq \frac{1}{a^{s\beta} \left|1 - L^j\right|} \Psi(x) \tag{4.2}$$

for all $x \in X$.

Theorem 4.2. Let $j \in \{-1,1\}$ be fixed. Let $\varphi : X \times X \to [0,\infty)$ be a function such that there exists an $L < 1$ with $\varphi\left(2^j x, 2^j y\right) \leq 4^{j\beta} \varphi(x, y)$ for all $x, y \in X$. Let $f : X \to Y$ be a mapping satisfying

$$\left\| D_q f(x, y) \right\|_Y \leq \varphi(x, y) \tag{4.3}$$

for all $x, y \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\| f(x) - Q(x) \|_Y \leq \frac{1}{4^\beta \left|1 - L^j\right|} \tilde{\varphi}(x) \tag{4.4}$$

for all $x \in X$, where
$$\tilde{\varphi}(x) = \frac{K}{24^\beta}\left[\varphi(x, -x) + K\left(\frac{7}{5}\right)^\beta \left(\varphi(0,0) + \varphi(0, x)\right)\right]. \tag{4.5}$$

Proof. Substituting $x = y = 0$ in (4.3), we get

$$\|f(0)\|_Y \leq \frac{1}{18^\beta}\varphi(0,0). \tag{4.6}$$

Replacing (x, y) by $(0, x)$ in (4.3), we obtain

$$\|5f(x)-5f(-x)+18f(0)\|_Y \leq \varphi(0,x) \tag{4.7}$$

for all $x \in X$. Using (4.6) in (4.7), we have

$$\|f(x)-f(-x)\|_Y \leq \frac{K}{5^\beta}\left(\varphi(0,0)+\varphi(0,x)\right) \tag{4.8}$$

for all $x \in X$. Plugging x into $-x$ in (4.3), we get

$$\|6f(2x)-24f(x)+7f(x)-7f(-x)\|_Y \leq \varphi(x,-x) \tag{4.9}$$

for all $x \in X$. Using (8) and (9), we obtain

$$\|6f(2x)-24f(x)\|_Y \leq K\left[\varphi(x,-x)+K\left(\frac{7}{5}\right)^\beta\left(\varphi(0,0)+\varphi(0,x)\right)\right]$$

for all $x \in X$. Therefore, $\left\|f(x)-\frac{1}{4}f(x)\right\|_Y \leq \tilde{\varphi}(x)$ for all $x \in X$. By Lemma 4.1, there exists a unique mapping $Q : X \to Y$ such that $Q(2x) = 4Q(x)$ and

$$\|f(x)-Q(x)\|_Y \leq \frac{1}{4^\beta\left|1-L^j\right|}\tilde{\varphi}(x) \tag{4.10}$$

for all $x \in X$. It remains to show that Q is a quadratic map. By (4.3), we have

$$\left\|4^{-jn}D_Q f\left(2^{jn}x, 2^{jn}y\right)\right\|_Y \leq 4^{-jn}\varphi\left(2^{jn}x, 2^{jn}y\right) \leq 4^{-jn\beta}\left(4^{j\beta}L\right)^n\varphi(x,y) \leq L^n\varphi(x,y)$$

for all $x, y \in X$ and $n \in \mathsf{N}$. So $\|D_Q f(x,y)\|_Y = 0$ for all $x, y \in X$. Thus the mapping $Q : X \to Y$ is quadratic, as desired.

Corollary 4.3. Let X be a quasi-α-normed space with quasi-α-norm $\|\cdot\|_X$, and let Y be a (β, p)-Banach space with (β, p)-norm $\|\cdot\|_Y$. Let δ, λ be positive numbers with $\lambda \neq \frac{2\beta}{\alpha}$ and $f : X \to Y$ be a mapping satisfying

$$\|D_q f(x,y)\|_Y \leq \delta\left(\|x\|_X^\lambda + \|y\|_X^\lambda\right)$$

for all $x, y \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\|f(x)-Q(x)\|_Y \leq \begin{cases} \dfrac{\delta\varepsilon_\lambda}{4^\beta - 2^{\alpha\lambda}}\|x\|_X^\lambda, & \text{for } \lambda < \dfrac{2\beta}{\alpha} \\[2ex] \dfrac{2^{\lambda\alpha}\delta\varepsilon_\lambda}{4^\beta\left(2^{\lambda\alpha}-4^\beta\right)}\|x\|_X^\lambda, & \text{for } \lambda > \dfrac{2\beta}{\alpha} \end{cases}$$

for all $x \in X$, where $\varepsilon_\lambda = \dfrac{K}{24^\beta}\left(2+K\left(\dfrac{7}{5}\right)^\beta\right)$

Proof. The proof is obtained by considering $\varphi(x,y) = \delta\left(\|x\|_X^\lambda + \|y\|_X^\lambda\right)$, for all $x, y \in X$ and $L = \frac{2^{\alpha\lambda}}{4^\beta}$ in Theorem 4.2.

Corollary 4.4. Let X be a quasi-α-normed space with quasi-α-norm $\|\cdot\|_{X}$, and let Y be a (β, p)-Banach space with (β, p)-norm $\|\cdot\|_{Y}$. Let δ, r, s be positive numbers with $\lambda = r+s \neq \frac{2\beta}{\alpha}$ and $f : X \to Y$ be a mapping satisfying

$$\left\|D_q f(x,y)\right\|_{Y} \le \delta \|x\|_{X}^{r} \|y\|_{X}^{s}$$

for all $x, y \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\|f(x) - Q(x)\|_{Y} \le \begin{cases} \dfrac{\delta\varepsilon_{r,s}}{4^{\beta} - 2^{\alpha\lambda}} \|x\|_{X}^{\lambda}, & \text{for } \lambda < \dfrac{2\beta}{\alpha} \\ \dfrac{2^{\lambda\alpha}\delta\varepsilon_{r,s}}{4^{\beta}\left(2^{\lambda\alpha} - 4^{\beta}\right)} \|x\|_{X}^{\lambda}, & \text{for } \lambda > \dfrac{2\beta}{\alpha} \end{cases}$$

for all $x \in X$, where

$$\varepsilon_{r,s} = \frac{K}{24^{\beta}}$$

Proof. By choosing $\varphi(x,y) = \delta\|x\|_{X}^{r}\|y\|_{X}^{s}$, for all $x, y \in X$ and $L = \frac{2^{\alpha\lambda}}{4^{\beta}}$ in Theorem 4.2, we obtain the required results.

Corollary 4.5. Let X be a quasi-α-normed space with quasi-α-norm $\|\cdot\|_{X}$, and let Y be a (β, p)-Banach space with (β, p)-norm $\|\cdot\|_{Y}$. Let δ, r, s be positive numbers with $\lambda = r+s \neq \frac{2\beta}{\alpha}$ and $f : X \to Y$ be a mapping satisfying

$$\left\|D_q f(x,y)\right\|_{Y} \le \delta\left[\|x\|_{X}^{r}\|y\|_{X}^{s} + \left(\|x\|_{X}^{r+s} + \|y\|_{X}^{r+s}\right)\right]$$

for all $x, y \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\|f(x) - Q(x)\|_{Y} \le \begin{cases} \dfrac{\delta\varepsilon_{r,s}}{4^{\beta} - 2^{\alpha\lambda}} \|x\|_{X}^{\lambda}, & \text{for } \lambda < \dfrac{2\beta}{\alpha} \\ \dfrac{2^{\lambda\alpha}\delta\varepsilon_{r,s}}{4^{\beta}\left(2^{\lambda\alpha} - 4^{\beta}\right)} \|x\|_{X}^{\lambda}, & \text{for } \lambda > \dfrac{2\beta}{\alpha} \end{cases}$$

for all $x \in X$, where

$$\varepsilon_{r,s} = \frac{K}{24^{\beta}}\left(3 + K\left(\frac{7}{5}\right)^{\beta}\right).$$

Proof. By taking $\varphi(x,y) = \delta\left[\|x\|_{X}^{r}\|y\|_{X}^{s} + \left(\|x\|_{X}^{r+s} + \|y\|_{X}^{r+s}\right)\right]$, for all $x, y \in X$ and $L = \frac{2^{\alpha\lambda}}{4^{\beta}}$ in Theorem 4.2, we arrive at the desired results.

The following example shows that the assumption $\lambda \neq \frac{2\beta}{\alpha}$ cannot be omitted in Corollary 4.3. This example is a modification of the example of Gajda [4] for the additive functional inequality.

Example 4.6. Let $\varphi : \mathbb{R} \to \mathbb{R}$ be defined by

$$\phi(x) = \begin{cases} x^2, \text{ for } |x| < 1, \\ 1, \text{ for } |x| \ge 1. \end{cases} \tag{4.11}$$

Consider that the function $f : \mathbb{R} \to \mathbb{R}$ is defined by

$$f(x) = \sum_{n=0}^{\infty} 2^{-2n} \phi\left(2^n x\right) \tag{4.12}$$

for all $x \in \mathbb{R}$. Then f satisifes the functional inequality

$$| f(3x+y)+ f(2x+y)+5 f(x-y)-18 f(x)-7 f(y)| \leq \frac{2048}{3}\left(|x|^2+|y|^2\right) \tag{4.13}$$

for all $x, y \in \mathbb{R}$ but there do not exist a quadratic mapping $\varphi : \mathbb{R} \to \mathbb{R}$ and a constant $d > 0$ such that

$$\left|f(x)-\mathrm{Q}(x)\right| \leq d\,|x|^2, \quad \text{for all } x \in \mathbb{R}.$$

Proof. It is clear that f is bounded by $\frac{4}{3}$ on $\mathbb{R}$. If $|x|^2+|y|^2=0$ or $|x|^2+|y|^2 \geq \frac{1}{2}$, then

$$\left|\mathrm{D}_q f(x,y)\right| \leq \frac{2048}{3} \leq \frac{1024}{3}\left(|x|^2+|y|^2\right). \tag{4.14}$$

Now, suppose that $0<|x|^2+|y|^2<\frac{1}{2}$. Then there exists a nonnegative integer k such that

$$\frac{1}{2^{k+2}} \leq |x|^2+|y|^2<\frac{1}{2^{k+1}}. \tag{4.15}$$

Hence $2^k|x|^2<\frac{1}{2}$, $2^k|y|^2<\frac{1}{2}$, and $2^n(3x+y)$, $2^n(2x+y)$, $2^n(x-y)$, $2^n x$, $2^n y \in (-1,1)$ for all $n=0,1,\ldots,k-1$. Hence, for $n=0,1,\ldots,k-1$,

$$\phi\left(2^n(3x+y)\right)+\phi\left(2^n(2x+y)\right)+5\phi\left(2^n(x-y)\right)-18\phi\left(2^n x\right)-7\phi\left(2^n y\right) = 0. \tag{4.16}$$

From the definition of f and the inequality (4.15), we obtain that

$$\begin{aligned}\left|\mathrm{D}_q f(x,y)\right| &= \Big|\sum_{n=0}^{\infty}2^{-2n}\phi\left(2^n(3x+y)\right)+\sum_{n=0}^{\infty}2^{-2n}\phi\left(2^n(2x+y)\right)\\ &\quad +5\sum_{n=0}^{\infty}2^{-2n}\phi\left(2^n(x-y)\right)-18\sum_{n=0}^{\infty}2^{-2n}\phi\left(2^n x\right)-7\sum_{n=0}^{\infty}2^{-2n}\phi\left(2^n y\right)\Big|\\ &\leq \sum_{n=k}^{\infty}2^{-2n}\cdot 32=\frac{32\cdot 2^{2(1-k)}}{3}\leq\frac{2048}{3}\left(|x|^2+|y|^2\right).\end{aligned} \tag{4.17}$$

Therefore, f satisfies (4.13) for all $x, y \in \mathrm{R}$. Now, we claim that the functional equation (1.2) is not stable for $\lambda = 2$ in Corollary 4.3 $(\alpha=\beta=p=1)$. Suppose on the contrary that there exist a quadratic mapping $Q:\mathbb{R}\to\mathbb{R}$ and a constant $d>0$ such that $|f(x)-Q(x)|\leq d\,|x|^2$ for all $x\in\mathbb{R}$. Then there exists a constant $c\in\mathbb{R}$ such that $Q(x)=cx^2$ for all rational numbers x (see [8]). So we obtain that

$$|f(x)| \leq (d+|c|)\,|x|^2 \tag{4.18}$$

for all $x \in \mathrm{Q}$. Let $m \in \mathrm{N}$ with $m+1>d+|c|$. If x is a rational number in $(0,4^{-m})$, then $4^n x \in (0,1)$ for all $n=0,1,\ldots,m$, and for this x, we get

$$f(x) = \sum_{n=0}^{\infty}\frac{\phi\left(4^n x\right)}{4^{2n}} \geq \sum_{n=0}^{m}\frac{\left(4^n x\right)^2}{4^{2n}} = (m+1)x^2 > (d+|c|)x^2, \tag{4.19}$$

which contradicts (4.18).

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