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THE REFINED THEORY OF THERMOELASTIC BEAMS POSTING INSIDE WINKLER FOUNDATION

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ABSTRACT

Without employing ad hoc assumptions, various equations and solutions for thermoelastic beams posting inside Winkler foundation are deduced systematically and directly from plane problem of thermoelasticity. These equations and solutions can be used to construct the refined theory of deep beams. A method for the solutions of two-dimensional equations is presented, and with the method the refined theory can now be explicitly established from Biot's solution of thermoelasticity and the Lur'e method. The exact and approximate governing differential equations for thermoelastic beams under transverse loadings and temperature distribution are derived directly from the refined beam theory.

***Keywords*:** the refined theory, thermoelastic beams, Winkler foundation

1. INTRODUCTION

Beams posting inside elastic foundation are one of the most well known structures of vital significance in the structural design and therefore received extensive study from scientific workers. The analysis of structures is usually based on a relatively simple model of the foundation's response to applied loads. Winkler foundation [1] is regarded as an infinite number of closely spaced unconnected linear springs, which is uniquely defined by the foundation modulus.

Deep beam analysis is basically a three-dimensional problem. Compared with three-dimensional elasticity theory, the traditional beam theories are merely approximate, since they predict the deformation of the beam structure from that of its centerline. Due to the alteration in approximation methods and the degrees of approximation, various beam theories have been established. Additionally some observations are made on the formulation of the theories of the conventional beam due to Timoshenko [2], Levinson [3], Bickford [4], Tutek and Aganović [5], Lewiński [6], Tullini and Savoia [7] and the original work of Bernoulli and Euler.

Since the publication of the excellent work of Cheng [8] on deducing the plate theory directly from the three-dimensional theory of elasticity, several extensions have been found in the plate theory [9-11]. Moreover, Gao and coauthors indicated that applications of Cheng's method are quite successful in various beams [12-15]. The significance of Cheng's method is that it opens a systematic way of developing the exact and approximate lower-dimensional theory from a higher-dimensional theory with the aid of the general solution of elasticity and symbolic computation.

This paper presents the refined theory for a thermoelastic beam posting inside Winkler foundation by using the thermoelastic beam method developed by Gao and Wang [15]. Based on linear thermoelastic theory, the refined theory of deep beams is derived by using Biot's solution of thermoelasticity [16] and the Lur'e method [17] without ad hoc assumptions. The exact and approximate governing equations for the beams under transverse loadings and temperature distribution are derived from the beam theory.

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2. BASIC EQUATIONS AND THE LUR'E METHOD

We consider a thermoelastic beam of narrow rectangular cross-section as a plane stress problem. In a Cartesian coordinate system (x, y, z), z is the coordinate normal to the neutral surface (x-y plane) of the beam. We assume that the beam length in x-direction is l, the beam width in y-direction is 1, the beam height in z-direction is h, and $l >> h >> 1$.

Some engineering theories for the response of foundation apparently originate with the classical work of Winkler [1]. Now, let us consider the case that the beam is posted inside Winkler foundation and is subjected only to anti-symmetrical transverse surface loadings q and temperature distribution T_0, i.e.,

$$\tau_{xz} = 0,\ \ \sigma_z = \pm q = \mp k u_z,\ \ T = \pm T_0\ \ \left(z = \pm h/2\right), \tag{1}$$

where k is the constant of proportionality referred to the modulus of subgrade reaction.

In the absence of body force and under steady temperature, the equilibrium equations of isotropic thermoelasticity are expressed by displacement u_x and u_z

$$\begin{aligned} &\nabla^2 u_x + \frac{1+\nu}{1-\nu}\frac{\partial}{\partial x}\left(e - 2\alpha T\right) = 0, \\ &\nabla^2 u_z + \frac{1+\nu}{1-\nu}\frac{\partial}{\partial z}\left(e - 2\alpha T\right) = 0, \\ &\nabla^2 T = 0, \end{aligned} \tag{2}$$

where $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial z^2$ is a two-dimensional Laplacian operator, $e = \partial u_x/\partial x + \partial u_z/\partial z$, n denotes Poisson's ratio and α is the coefficient of thermal expansion. Biot [16] obtained a general solution of the equations in the linear theory of thermoelasticity, so the solution of the governing equations (2) takes the same form [15]

$$\begin{aligned} &u_x = P_1 - \frac{1+\nu}{4}\frac{\partial}{\partial x}\left(P_0 + xP_1 + zP_3\right), \\ &u_z = P_3 - \frac{1+\nu}{4}\frac{\partial}{\partial z}\left(P_0 + xP_1 + zP_3\right), \\ &T = -\frac{1}{4\alpha}\nabla^2 P_0, \end{aligned} \tag{3}$$

in which the displacement function P_i (i = 1, 3) and P_0 is two two-dimensional harmonic functions and a two-dimensional biharmonic function, respectively,

$$\begin{aligned} &\nabla^2 P_i = \frac{\partial^2 P_i}{\partial z^2} + \partial_x^2 P_i = 0, \\ &\nabla^4 P_0 = \frac{\partial^4 P_0}{\partial z^4} + 2\partial_x^2\frac{\partial^2 P_0}{\partial z^2} + \partial_x^4 P_0 = 0. \end{aligned} \tag{4}$$

Since this paper is to deal with the bending of beams, we will only consider the asymmetric deformation of beams caused by a set of anti-symmetric surface loadings with respect to the x-y plane. Thus only even functions of z are required for u_x and T, and odd functions of z for u_z. For the Lur'e method [17], satisfying these requirements and treating Eq. (4) as an ordinary differential equation in z with constant coefficients, one obtains the following symbolic solution of Eq. (4):

$$\begin{aligned} &P_1(x,z) = \frac{\sin(z\partial_x)}{\partial_x} g_1(x), \\ &P_3(x,z) = \cos(z\partial_x) g_3(x), \\ &P_0(x,z) = \frac{\sin(z\partial_x)}{\partial_x} g_4(x) + z\cos(z\partial_x) g_0(x), \end{aligned} \tag{5}$$

where g_0, g_1, g_3 and g_4 are unknown functions of x yet to be determined, in which the trigonometric functions $\sin(z\partial_x) / \partial_x$ and $\cos(z\partial_x)$ have the following symbolic expressions:

$$\begin{aligned}\frac{\sin(z\partial_x)}{\partial_x} &= z\left(1-\frac{1}{3!}z^2\partial_x^2+\frac{1}{5!}z^4\partial_x^4-\cdots\right),\\ \cos(z\partial_x) &= \left(1-\frac{1}{2!}z^2\partial_x^2+\frac{1}{4!}z^4\partial_x^4-\cdots\right).\end{aligned} \tag{6}$$

From Appendix A of Ref. [15], we can know that the following expression always can be satisfied without loss in generality:

$$P_0 + xP_1 + zP_3 = -z\cos(z\partial_x) f, \tag{7}$$

with

$$f = \int_0^x g_1(t)\mathrm{d}t - (g_0 + g_3).$$

3. THE REFINED THEORY

Substituting Eqs. (5) and (7) into Eq. (3), one obtains

$$\begin{aligned}u_x &= \frac{\sin(z\partial_x)}{\partial_x}g_1 + \frac{1+\nu}{4}z\cos(z\partial_x) f',\\ u_z &= \cos(z\partial_x)g_3 + \frac{1+\nu}{4}\left[\cos(z\partial_x) - z\partial_x\sin(z\partial_x)\right] f,\\ T &= \frac{1}{2\alpha}\sin(z\partial_x)g_0'.\end{aligned} \tag{8}$$

The angle of rotation and the deflection of the mid-plane can be found to be

$$\begin{aligned}\psi &= -\left.\frac{\partial u_x}{\partial z}\right|_{z=0} = -g_1 - \frac{1+\nu}{4}f',\\ w &= u_z\big|_{z=0} = g_3 + \frac{1+\nu}{4}f.\end{aligned} \tag{9}$$

From Eqs. (9) and (8) , the final expressions for the displacements and temperature distribution are

$$\begin{aligned}u_x &= -\frac{\sin(z\partial_x)}{\partial_x}\psi + \frac{1+\nu}{4}\left[z\cos(z\partial_x) - \frac{\sin(z\partial_x)}{\partial_x}\right] f',\\ u_z &= \cos(z\partial_x)w - \frac{1+\nu}{4}z\partial_x\sin(z\partial_x) f,\\ T &= \frac{1}{2\alpha}\sin(z\partial_x)g_0',\end{aligned} \tag{10}$$

with

$$f = -\int_0^x \psi(t)\mathrm{d}t - g_0 - w.$$

By using the generalized Hooke's law, expressions (10) can be used to determine the stress components as

$$\sigma_x = -\frac{E}{4}\left\{\left[\frac{1-\nu}{1+\nu}\frac{\sin(z\partial_x)}{\partial_x} - z\cos(z\partial_x)\right]f + \frac{4}{1+\nu}\frac{\sin(z\partial_x)}{\partial_x}\psi' + \frac{2}{1+\nu}\frac{\sin(z\partial_x)}{\partial_x}g_0''\right\},$$
$$\tau_{xz} = -\mu\left[\cos(z\partial_x)(\psi - w') + \frac{1+\nu}{2}z\partial_x\sin(z\partial_x)f'\right],$$
$$\sigma_z = -\frac{E}{4}\left\{\left[\frac{1-\nu}{1+\nu}\frac{\sin(z\partial_x)}{\partial_x} + z\cos(z\partial_x)\right]f'' + \frac{4}{1+\nu}\frac{\sin(z\partial_x)}{\partial_x}w'' + \frac{2}{1+\nu}\frac{\sin(z\partial_x)}{\partial_x}g_0''\right\}. \tag{11}$$

where E and $\mu = E / 2(1+\nu)$ are the Young's modulus and the shear modulus of the beam, respectively.

Substituting the stress expressions in Eq. (11) into the boundary conditions (1) of the beam, we get the following equations:

$$\left(\frac{4}{1+\nu}\tilde{C} - h\partial_x\tilde{S}\right)\psi - \left(\frac{4}{1+\nu}\tilde{C} + h\partial_x\tilde{S}\right)w' - h\partial_x\tilde{S}g_0' = 0,$$
$$\left(\frac{h\partial_x}{2}\tilde{C} + \frac{1-\nu}{1+\nu}\tilde{S} + \frac{1+\nu}{2E}kh\tilde{S}\right)\psi + \left(\frac{h\partial_x}{2}\tilde{C} - \frac{3+\nu}{1+\nu}\tilde{S} + \frac{4k}{E}\frac{\tilde{C}}{\partial_x} + \frac{1+\nu}{2E}kh\tilde{S}\right)w'$$
$$+\left(\frac{h\partial_x}{2}\tilde{C} - \tilde{S} + \frac{1+\nu}{2E}kh\tilde{S}\right)g_0' = 0,$$
$$\tilde{S}g_0' = 2\alpha T_0, \tag{12}$$

where the differential operators $\tilde{S}$ and $\tilde{C}$ are defined by $\tilde{S} = \sin(h\partial_x/2)$ and $\tilde{C} = \cos(h\partial_x/2)$, respectively. Taking the operator $S\left(\frac{4}{1+\nu}\tilde{C} - h\partial_x\tilde{S}\right)$ on both sides of the second expression of Eq. (12) and then using the first and third expressions of Eq. (12), one obtains

$$\frac{E\tilde{S}}{2\partial_x}\left(1 - \frac{2\tilde{S}\tilde{C}}{h\partial_x} + \frac{4k}{Eh\partial_x^2}\tilde{C}^2\right)w'' = -\frac{\alpha E}{2}\left(1 - \frac{2\tilde{S}\tilde{C}}{h\partial_x} + \frac{1+\nu}{2E\partial_x}k\tilde{S}\right)T_0. \tag{13}$$

Taking the operator $S\left(\frac{4}{1+\nu}\tilde{C} + h\partial_x\tilde{S}\right)$ on both sides of the second expression of Eq. (12) and with the use of the first and third expressions of Eq. (12), one obtains

$$\frac{E\tilde{S}}{2\partial_x}\left(1 - \frac{2\tilde{S}\tilde{C}}{h\partial_x} + \frac{4k}{Eh\partial_x^2}\tilde{C}^2\right)\psi' = -\frac{\alpha E}{2}\left(1 - \frac{2\tilde{S}\tilde{C}}{h\partial_x} - \frac{1+\nu}{2E\partial_x}k\tilde{S}\right)T_0. \tag{14}$$

Eqs. (13) and (14) are the exact governing differential equations for the deflection w and the angle of rotation ψ at the neutral surface of beams subjected to the transverse surface loadings and temperature distribution. Since this equation is of infinite order, however, it is not applicable in most cases. To develop a practical thermoelastic beam theory, we need to make certain simplifications. Using Taylor series of the trigonometric functions in Eq. (6) and then dropping all the terms associated with the h^3 or the higher order terms, we arrive at the following equations:

$$D\left(\frac{1}{2} - \frac{1}{48}h^2\partial_x^2\right)\left(1 - \frac{1}{20}h^2\partial_x^2\right)w'''' + 2k\left(\frac{1}{2} - \frac{1}{48}h^2\partial_x^2\right)\left(1 - \frac{1}{4}h^2\partial_x^2\right)w$$
$$= -\frac{E\alpha}{12}\left(1 - \frac{1}{20}h^2\partial_x^2\right)h^2T_0'' - \frac{1+\nu}{4}\alpha kh\left(1 - \frac{1}{6}h^2\partial_x^2\right)T_0, \tag{15}$$

$$D\left(\frac{1}{2}-\frac{1}{48}h^2\partial_x^2\right)\left(1-\frac{1}{20}h^2\partial_x^2\right)\psi'''' + 2k\left(\frac{1}{2}-\frac{1}{48}h^2\partial_x^2\right)\left(1-\frac{1}{4}h^2\partial_x^2\right)\psi$$
$$=-\frac{E\alpha}{12}\left(1-\frac{1}{20}h^2\partial_x^2\right)h^2T_0''' + \frac{1+\nu}{4}\alpha kh\left(1-\frac{1}{6}h^2\partial_x^2\right)T_0'. \tag{16}$$

where $D = Eh^3 / 12$ is the flexural rigidity of beams. Taking the operator $\left(2+h^2\partial_x^2/12\right)\left(1+h^2\partial_x^2/20\right)$ on both sides of Eqs. (15) and (16), and then omitting the h^3 or the higher order terms, the results turn out to be

$$Dw'''' + 2k\left(1-\frac{1}{5}h^2\partial_x^2\right)w = -\frac{E\alpha}{6}h^2T_0'' - \frac{1+\nu}{2}\alpha khT_0, \tag{17}$$

$$D\psi'''' + 2k\left(1-\frac{1}{5}h^2\partial_x^2\right)\psi = -\frac{E\alpha}{6}h^2T_0''' + \frac{1+\nu}{2}\alpha khT_0'. \tag{18}$$

Eqs. (17) and (18) constitute basic equations of the approximate theory of thermoelastic beams posting inside Winkler foundation, and all the expressions about the displacements and stresses can be obtained from these equations. With the h^2 order term omitted from Eq. (17), one obtains

$$Dw'''' + 2kw = -\frac{1+\nu}{2}\alpha khT_0. \tag{19}$$

The governing differential equation (19) has the same structure as that of Bernoulli-Euler thermoelastic beam theory posting inside Winkler foundation.

As a special case, not taking into account the foundation modulus, i.e. $k = 0$, the governing differential equations for elastic beam are obtained directly from Eqs. (17) and (18), the results described above reduce to the corresponding results by Ref. [15]. If temperature distribution is absent, i.e., let $T_0 = 0$, the results described above degenerate to the corresponding results by Ref. [18].

By omitting the higher order terms, the foundation modulus or temperature distribution, the exact thermoelastic beam theory posting inside Winkler foundation can be degenerated into other well-known beam theories. Hence, the results obtained here are considered reliable as a basis for more general applications.

5. CONCLUSION

Based on the linear thermoelastic theory, the refined theory for thermoelastic beams posting inside Winkler foundation has been deduced systematically and directly by using Biot's solution of thermoelasticity and the Lur'e method. It is shown that the displacements and stresses of the beam can be represented by the angle of rotation and the deflection of the neutral surface. The exact governing equations for the beams under transverse loadings and temperature distribution are derived from the beam theory. The approximate governing equations are accurate up to the second-order terms with respect to beam thickness, and they are almost the same as the governing equations of other well-known beam theories.

Because the present theory is derived without requirement of any ad hoc assumptions concerning the deformation or the stress state, results based on them are of high accuracy, appeal to application and help to describe problems in an incisive way. Results show that the theory developed in this paper is reliable and can serve as a basis for further applications.

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