

LATTICE MODULES OVER RINGS OF BOUNDED RANDOM VARIABLES

KARL-THEODOR EISELE AND SONIA TAIEB

ABSTRACT. In the context of representation theorems for conditional and multi-period risk measures, we will extend the work, started in [9], on locally convex modules over the ring $\lambda = L^\infty(\mathcal{G})$, by adding an order or a lattice structure. The dual spaces of lattice λ -modules turn out to be topologically and Dedekind-complete λ -modules. For Banach lattice modules a version of the Namioka-Klee theorem will be proved, as well as the subdifferentiability of λ -convex functions on the interior of their domains. The completeness of L_λ^p - and $L_\lambda^{(p_2, p_1)}$ -modules will be shown and their dual modules characterized. Similarly, we will establish the duality between Morse and their corresponding Orlicz λ -modules.

1. Introduction

In [9] we studied locally convex modules over the ring $\lambda = L^\infty(\mathcal{G})$ of bounded variables of a sub- σ -algebra \mathcal{G} . This has been done in the understanding that a broader functional analytic base for representation theorems of conditional or multi-period risk measures would be rather useful.

There the dual spaces of these locally convex λ -modules are studied, an adapted version of the Bipolar theorem is established, and new forms of the Krein-Šmulian theorem as well as of the Alaoglu-Bourbaki theorem are given.

Here we are going to continue this analysis of λ -modules by introducing order or lattice structures. These additional tools seem to be necessary since in some important cases topological and order convergences do not coincide. This requires additional properties like the Fatou property to guarantee the (semi-)continuity of those risk measures in the order convergence sense (see [6] or [12]).

Our first result concerns the dual spaces of locally convex lattice λ -modules. We will show that they are topologically and Dedekind-complete (i.e. order-complete) lattice λ -modules. Consequently we are confronted with the question of the Dedekind-semi-continuity for conditional risk measures on dual modules.

Similar to [21], [3], and [17], we will also show a version of the Namioka-Klee theorem for Banach lattice λ -modules where a λ -convex function is continuous and subdifferentiable in the interior of their domain of definition.

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As examples we will treat the λ -modules of L_λ^p and $L_\lambda^{(p_1, p_2)}$ in section 8. In several cases as for example that of entropic risk measures, it is necessary to use Orlicz spaces or their subspaces Orlicz hearts (see [5] and [4]). The subdifferentiability of the entropic risk measure on an Orlicz module has been dealt with in [17]. In section 9 we investigate the dual modules of some Orlicz (resp. Morse) λ -modules.

2. Locally Convex Modules Over $\lambda = L^\infty$

We will continue the research started in [9] and recall the main notations used there.

For a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a sub- σ -algebra $\mathcal{G} \subseteq \mathcal{F}$ we denote by $(\mathcal{G} \cap A)^+$ the set of all $A' \in \mathcal{G}$ with $A' \subseteq A$ and $\mathbb{P}(A') > 0$ (similar for \mathcal{F}^+). The set of all \mathcal{F} -measurable (resp. \mathcal{G} -measurable) real random variables is L^0 (resp. $L^0(\mathcal{G})$), while \overline{L}^0 was the set of all \mathcal{F} -measurable variables with values in $[-\infty, \infty]$. For any space L of random variables its positive orthant was $L_+ := \{X \in L \mid X \geq 0\}$, while we set $L_\# := \{X \in L \mid \exists \varepsilon > 0 \text{ with } X \geq \varepsilon\}$. Equalities and inequalities of random variables are understood to hold \mathbb{P} almost surely, in particular $X > Y$ means that $\mathbb{P}(X > Y) = 1$. The multiplication with 0 is dominant: i.e. $\frac{0}{0} = 0 \cdot \infty = 0 \cdot (-\infty) = 0$.

We continue to study modules using the space $\lambda := \lambda(\mathcal{G}) := L^\infty(\mathcal{G})$ as ring whose generic element will most of the time be denoted by ζ . The constant element which equals 1 in λ is given by $\mathbb{1}$. As mentioned $\lambda_\# := \{\zeta \in \lambda \mid \exists \varepsilon > 0 \text{ with } \zeta \geq \varepsilon \cdot \mathbb{1}\}$. The space λ is endowed with the topology of the norm

$$\|\zeta\|_\lambda := \|\zeta\|_\infty. \quad (2.1)$$

For $p \in [1, \infty]$ and $\mathcal{G} \subseteq \mathcal{F}_1 \subseteq \mathcal{F}$, we introduce the conditional norms $\|\cdot\|_{p, \mathcal{F}_1}$ on L^0 by

$$\|X\|_{p, \mathcal{F}_1} := \begin{cases} \lim_{n \rightarrow \infty} \mathbb{E} [|X|^p \wedge n | \mathcal{F}_1]^{1/p} & \text{for } p < \infty, \\ \text{ess. inf} \left\{ \zeta \in \overline{L}_+^0(\mathcal{F}_1) \mid \zeta \geq |X| \right\} & \text{for } p = \infty, \end{cases} \quad (2.2)$$

where $X \in L^0$. As it is well known, for $p \in [1, \infty)$, the *conjugate exponent* q to p is defined by

$$q := \begin{cases} p/(p-1) & \text{if } 1 < p < \infty, \\ +\infty & \text{if } p = 1. \end{cases} \quad (2.3)$$

Let's recall some definitions from [9]:

Definitions 2.1.

- (i) A λ -module E is a set with an additive operation “+” and a multiplication “ \cdot ” by the elements of the ring λ :
 - (a) $E \times E \rightarrow E$, $(X_1, X_2) \mapsto X_1 + X_2$ and
 - (b) $\lambda \times E \rightarrow E$, $(\zeta, X) \mapsto \zeta \cdot X$.
- (ii) The λ -module E is a *topological* one if E has a topology \mathcal{T} such that the module operations (i.a) and (i.b) are continuous w.r. to the corresponding product topologies.
- (iii) A subset D of the λ -module E is called:

- (a) λ -absorbent if for all $X \in E$ there is $\zeta \in \lambda_+$ such that $X \in \zeta \cdot D$,
 - (b) λ -balanced if $\zeta \cdot X \in D$ for all $X \in D$ and $\zeta \in \lambda$ with $\|\zeta\|_\lambda \leq 1$,
 - (c) λ -convex if $\zeta \cdot X_1 + (1 - \zeta) \cdot X_2 \in D$ for all $X_1, X_2 \in D$ and $\zeta \in \lambda$ with $0 \leq \zeta \leq 1$.
- (iv) A function $\mathfrak{q} := \mathfrak{q}|\cdot| : E \rightarrow \lambda_+$ is a λ -seminorm on E if for $X, X_1, X_2 \in E$ and $\zeta \in \lambda$:
- (a) $\mathfrak{q}|\zeta \cdot X| = |\zeta| \cdot \mathfrak{q}|X|$,
 - (b) $\mathfrak{q}|X_1 + X_2| \leq \mathfrak{q}|X_1| + \mathfrak{q}|X_2|$.
- In addition,
- (c) if $\mathfrak{q}|X| = 0$ implies $X = 0$, then \mathfrak{q} is a λ -norm on E . In this case (E, \mathfrak{q}) is called a *normed λ -module*.
- (v) For $\eta \in \lambda_\#$, we let

$$B_{\mathfrak{q},\eta}(X) := \{Y \in E \mid \mathfrak{q}|Y - X| \leq \eta\} \quad (2.4)$$

denote the \mathfrak{q} -ball of radius η around $X \in E$. We write $B_{\mathfrak{q},\eta} := B_{\mathfrak{q},\eta}(0)$ and $B_{\mathfrak{q},r}(X) := B_{\mathfrak{q},r\mathbb{1}}(X)$ for $r \in (0, \infty)$.

Since for any $\eta \in \lambda_\#$ we find some real $r > 0$ with $\eta \geq r \cdot \mathbb{1}$, we can replace $B_{\mathfrak{q},\eta}$ by $B_{\mathfrak{q},r}$ almost everywhere in the following.

Let \mathfrak{q} be a λ -seminorm and $\eta \in \lambda_\#$. The \mathfrak{q} -ball $B_{\mathfrak{q},\eta}$ is a λ -convex, -absorbent, and -balanced subset of E . We generalize the simple set $B_{\mathfrak{q},\eta}$ to the following system: Let \mathfrak{Q} be a family of λ -seminorms on E . Then

$$\mathcal{U}_{\mathfrak{Q}} := \left\{ B_{\mathfrak{Q}_f,\eta} \mid \mathfrak{Q}_f \text{ is a finite subset of } \mathfrak{Q} \text{ and } \eta \in \lambda_\# \right\} \quad \text{where} \quad (2.5)$$

$$B_{\mathfrak{Q}_f,\eta} := \left\{ X \in E \mid \sup_{\mathfrak{q} \in \mathfrak{Q}_f} \mathfrak{q}|X| \leq \eta \right\} \quad (2.6)$$

defines a system of λ -convex, -absorbent, and -balanced subsets of E , closed under finite intersections. If we generate a topology $\mathcal{T}_{\mathfrak{Q}}$ on the λ -module E by using the system $\mathcal{U}_{\mathfrak{Q}}$ as a neighborhood base of $0 \in E$, then the properties (iv.a) and (iv.b) of definition 2.1 show that E is a topological λ -module. Therefore (E, \mathfrak{Q}) is a locally convex λ -module with the topology $\mathcal{T}_{\mathfrak{Q}}$ in the sense of the following definition.

Definition 2.2. The λ -module E is *locally convex* if it has a neighborhood base \mathcal{U} of $0 \in E$ with the following properties:

- (i) $0 \in U$ for all $U \in \mathcal{U}$,
- (ii) \mathcal{U} is downward filtrated: for all $U_1, U_2 \in \mathcal{U}$ there exists $U \in \mathcal{U}$ with $U \subseteq U_1 \cap U_2$,
- (iii) for all $U \in \mathcal{U}$ there exists $U' \in \mathcal{U}$ with $U' + U' \subseteq U$,
- (iv) for all $U \in \mathcal{U}$ and $\zeta \in \lambda_\#$ there exists $U' \in \mathcal{U}$ with $\zeta \cdot U' \subseteq U$,
- (v) all $U \in \mathcal{U}$ are λ -absorbent, -balanced, and -convex.

The topology $\mathcal{T}_{\mathcal{U}}$ is defined by the fact that a subset $O \subseteq E$ is open if and only if for all $X \in O$ there exists $U \in \mathcal{U}$ with $X + U \subseteq O$.

Obviously, the properties (i) to (v) imply that $(E, \mathcal{T}_{\mathcal{U}})$ is a topological λ -module in the sense of definition 2.1. In [9] we have shown that the topology $\mathcal{T}_{\mathcal{U}}$ of a locally

convex λ -module E with a neighborhood base \mathcal{U} of 0 is Hausdorff if and only if

$$\bigcap_{U \in \mathcal{U}} U = \{0\}. \quad (2.7)$$

Assumption 2.3. *Whenever we deal with a locally convex λ -module E in the following sections, we always consider condition (2.7) to be complied with.*

For any finite subset $\mathfrak{Q}_f = \{\mathfrak{q}_1, \dots, \mathfrak{q}_n\} \subseteq \mathfrak{Q}$ it evidently holds that

$$\{X \mid \inf_{1 \leq i \leq n} \mathfrak{q}_i |X| \leq \eta\} \subseteq \{X \mid \sum_{1 \leq i \leq n} \mathbb{1}_{A_i} \cdot \mathfrak{q}_i |X| \leq n\eta\}$$

for all $A_i \in \mathcal{G}^+$, and so $\sum_{1 \leq i \leq n} \mathbb{1}_{A_i} \cdot \mathfrak{q}_i |\cdot|$ is a λ -seminorm, too. Therefore, there is no restriction on making the following

Assumption 2.4. *If $\mathfrak{q}_i \in \mathfrak{Q}$ and $A_i \in \mathcal{G}^+$ for $1 \leq i \leq n$, then $\sum_{i=1}^n \mathbb{1}_{A_i} \cdot \mathfrak{q}_i |\cdot| \in \mathfrak{Q}$.*

The locally convex topology on the λ -module E is now defined by the neighborhood base

$$\mathcal{U}_{\mathfrak{Q}} = \{B_{\mathfrak{q}, \eta} := \{X \in E \mid \mathfrak{q}|X| \leq \eta\} \mid \mathfrak{q} \in \mathfrak{Q}, \eta \in \lambda_{\#}\}. \quad (2.8)$$

Let (E, \mathfrak{Q}) be a locally convex λ -module satisfying assumptions 2.3 and 2.4. From [9] we quote the following definitions and the characterization of continuous λ -linear functions on E :

Definitions 2.5.

- (i) A net $(X_{\iota})_{\iota \in I}$ in E is a *Cauchy net* if for all $\eta \in \lambda_{\#}$ and $\mathfrak{q} \in \mathfrak{Q}$ there exists $\iota_{\mathfrak{q}, \eta} \in I$ such that for all $\iota_1, \iota_2 \geq \iota_{\mathfrak{q}, \eta}$

$$X_{\iota_1} - X_{\iota_2} \in B_{\mathfrak{q}, \eta}. \quad (2.9)$$

- (ii) The λ -module E is *complete* if every Cauchy sequence has a limit in E .
- (iii) In the case $\mathfrak{Q} = \{\|\cdot\|\}$ with a λ -norm $\|\cdot\|$, a complete λ -module $(E, \|\cdot\|)$ is called a *Banach λ -module*.

If (E, \mathfrak{Q}) is a locally convex λ -module with a set \mathfrak{Q} of λ -seminorms and the topology $\mathcal{T}_{\mathfrak{Q}}$, then we denote the λ -dual space of E by E' , i.e. E' is the λ -module of all continuous λ -linear functions $Z : E \ni X \mapsto \langle X, Z \rangle \in \lambda$ with

$$\langle \zeta_1 \cdot X_1 + \zeta_2 \cdot X_2, Z \rangle = \zeta_1 \cdot \langle X_1, Z \rangle + \zeta_2 \cdot \langle X_2, Z \rangle \quad (2.10)$$

for all $\zeta_i \in \lambda$ and $X_i \in E$, $i = 1, 2$.

Proposition 2.6. *On the locally convex λ -module (E, \mathfrak{Q}) with assumption 2.4, a λ -linear function $Z : E \rightarrow \lambda$ is continuous if and only if there exist $\mathfrak{q} \in \mathfrak{Q}$ and $\eta \in \lambda_{\#}$ such that for all $X \in E$*

$$|\langle X, Z \rangle| \leq \eta \cdot \mathfrak{q}|X|. \quad (2.11)$$

Let us recall the following definitions from [9]. For $\mathfrak{q} \in \mathfrak{Q}$, the *conjugate λ -seminorm* is given by

$$\mathfrak{q}'|Z| := \text{ess. sup} \{|\langle X, Z \rangle| \mid X \in B_{\mathfrak{q}, 1}\} \quad (2.12)$$

for $Z \in E'$. It can easily be seen that $\mathfrak{q}'|\cdot|$ is indeed a λ -seminorm on E' . By $\mathfrak{Q}' := \{\mathfrak{q}' \mid \mathfrak{q} \in \mathfrak{Q}\}$ we denote the set of conjugate λ -seminorms on E' . The \mathfrak{q}' -ball of radius η around $Z \in E'$ is

$$B_{\mathfrak{q}',\eta}(Z) := \{Z' \in E' \mid \mathfrak{q}'|Z' - Z| \leq \eta\} \quad (2.13)$$

where $\mathfrak{q}' \in \mathfrak{Q}'$ and $\eta \in \lambda_{\sharp}$.

Similar to (2.11), we have

$$|\langle X, Z \rangle| \leq \mathfrak{q}|X| \cdot \mathfrak{q}'|Z| \quad (2.14)$$

for all $X \in E$ and $Z \in E'$.

Proposition 2.7. *For a locally convex λ -module (E, \mathfrak{Q}) the dual λ -module (E', \mathfrak{Q}') is complete.*

The proof is given in [9].

3. Hahn-Banach-Theorems for Locally Convex λ -modules

The following theorems of Hahn-Banach type for locally-convex λ -modules (E, \mathfrak{Q}) have been shown in [9]. For the first purely algebraic theorem, a proof using a scalarization method for module-linear functions taking values in a dual module space of a normed vector space can be found in [14]. In [23] a direct proof is given taking order complete lattice modules as image spaces.

As usual, a function $\varphi : E \rightarrow \lambda$ is called λ -sublinear if $\varphi(\zeta \cdot X) = \zeta \cdot \varphi(X)$ and $\varphi(X_1 + X_2) \leq \varphi(X_1) + \varphi(X_2)$ for all $\zeta \in \lambda_+$, $X, X_1, X_2 \in E$.

Theorem 3.1. *Consider a λ -sublinear function $\varphi : E \rightarrow \lambda$, a λ -submodule C of E and a λ -linear function $Z : C \rightarrow \lambda$ such as*

$$\langle X, Z \rangle \leq \varphi(X) \quad \text{for all } X \in C.$$

Then Z extends to a λ -linear function $Z : E \rightarrow \lambda$ such as $\langle X, Z \rangle \leq \varphi(X)$ for all $X \in E$.

The second Hahn-Banach type theorem concerns the separation of a compact convex set C and a disjoint closed convex set D . In the classical separation theorem, the continuous linear separating functional shows a uniform positive distance between C and D , also called strong separation (see [1]). In the λ -module case we have to pay a price to get this strong separation: If for some $A \in \mathcal{G}^+$ we have $\mathbb{1}_B \cdot C \cap \mathbb{1}_B \cdot D = \emptyset$ for all $B \in (\mathcal{G} \cap A)^+$, then for any $\varepsilon > 0$ we find some $A_\varepsilon \subseteq A$ with $\mathbb{P}(A \setminus A_\varepsilon) < \varepsilon$ such that $\mathbb{1}_{A_\varepsilon} \cdot C$ can be strongly separated from $\mathbb{1}_{A_\varepsilon} \cdot D$ by a continuous linear functional. It turns out that this separation property is sufficient to get the results we are looking for.

Theorem 3.2. *Let C and D be λ -convex non-empty subsets of a locally convex λ -module (E, \mathfrak{Q}) with C compact and D closed. For some $A \in \mathcal{G}^+$, suppose that we have*

$$\mathbb{1}_B \cdot C \cap \mathbb{1}_B \cdot D = \emptyset \quad (3.1)$$

for all $B \in (\mathcal{G} \cap A)^+$. Then for $\varepsilon > 0$, there exist a set $A_\varepsilon \subseteq A$ with $\mathbb{P}(A \setminus A_\varepsilon) < \varepsilon$, a λ -linear continuous function $Z : E \rightarrow \lambda$, and $\eta \in \lambda_{\sharp}$ such that

$$\mathbb{1}_{A_\varepsilon} \cdot (\langle Y, Z \rangle + \eta) \leq \mathbb{1}_{A_\varepsilon} \cdot \langle X, Z \rangle \quad (3.2)$$

for all $X \in C$ and $Y \in D$.

4. Ordered λ -Modules

Now, we will introduce a λ -module E with an order structure.

Definition 4.1. The λ -module E is *ordered*, if E is endowed with a partial order \leq such that for all $X_1, X_2, X_3 \in E$ and all $\zeta \in \lambda_+$

- (i) $X_1 \leq X_2$ implies $X_1 + X_3 \leq X_2 + X_3$,
- (ii) $0 \leq X_1$ implies $0 \leq \zeta \cdot X_1$.

The partial order \leq induces the *positive orthant* E_+ , resp. the *negative orthant* E_- in E

$$E_+ := \{X \mid 0 \leq X \in E\} \quad \text{resp.} \quad E_- := -E_+. \quad (4.1)$$

Obviously, $X_1 \leq X_2$ if and only if $X_2 - X_1 \in E_+$, i.e. the partial order " \leq " is characterized by E_+ .

The triplet (E, E_+, \mathfrak{Q}) denotes a locally convex ordered λ -module where the order is given by the positive orthant E_+ and the topology by the set \mathfrak{Q} of λ -seminorms. In this case, we denote by $\overline{E_+}$ the closure of E_+ in E .

Definition 4.2. A continuous λ -linear function $Z \in E'$ is called *positive* if $Z(E_+) \subseteq \lambda_+$.

By E'_+ we denote the *positive orthant* of the dual module E' , i.e. the cone of all continuous positive λ -linear functions $Z \in E'$. The cone $E'_- := -E'_+$ is the negative orthant in E' . Furthermore, we set

$$E'_\pm := E'_+ - E'_+, \quad (4.2)$$

the space of continuous λ -linear functions Z for which continuous positive λ -linear Z_1 and Z_2 with $Z = Z_1 - Z_2$ exist.

An application of theorem 3.2 shows the existence of continuous positive λ -linear functions (see also [19] for the scalar case).

Proposition 4.3. *Let (E, E_+, \mathfrak{Q}) be a locally convex ordered λ -module and $X_0 \in E$. Then there exist a continuous positive λ -linear function $Z \in E'$, $\eta \in \lambda_\#$, and $A \in \mathcal{G}^+$ with $\mathbb{1}_A \cdot (\langle X_0, Z \rangle + \eta) \leq 0$ if and only if $X_0 \notin \overline{E_+}$.*

Remark 4.4. The proof below makes use of the following mathematical tool: For $X \in E$ and a subset $C \subseteq E$, we define

$$X \sqcap C := \text{ess. sup}\{A \in \mathcal{G}, \mathbb{1}_A \cdot X \in \mathbb{1}_A \cdot C\}. \quad (4.3)$$

Proof. The necessity of the last condition is clear. Conversely, set $D := \overline{E_+}$ and assume $X_0 \notin D$. With $A' := (X_0 \sqcap \overline{E_+})^c \in \mathcal{G}^+$ the condition (3.1) is met, i.e. $\mathbb{1}_B \cdot X_0 \notin \mathbb{1}_B \cdot D$ for all $B \in (\mathcal{G} \cap A')^+$. By theorem 3.2, we find $A \in (\mathcal{G} \cap A')^+$, a continuous λ -linear function Z' and $\eta \in \lambda_\#$ with $\mathbb{1}_A \cdot (\langle Y, Z' \rangle + \eta) \leq \mathbb{1}_A \cdot \langle X_0, Z' \rangle$ for all $Y \in \overline{E_+}$. Setting $Z := -\mathbb{1}_A \cdot Z'$ we find $\mathbb{1}_A \cdot (\langle X_0, Z \rangle + \eta) \leq \langle Y, Z \rangle$ for all $Y \in \overline{E_+}$. Since $\overline{E_+}$ is a cone, we get $Z(E_+) \geq 0$ and $\mathbb{1}_A \cdot (\langle X_0, Z \rangle + \eta) \leq 0$ in particular which shows that Z is positive and meets the asserted inequality for X_0 . \square

Corollary 4.5. *Let (E, E_+, Ω) again be a locally convex ordered λ -module with $E_+ \cap E_- = \{0\}$.*

(i) *The positive orthant E_+ is closed in (E, Ω) if and only if*

$$E_+ = \{X \in E \mid \langle X, Z \rangle \geq 0 \text{ for all } Z \in E'_+\}. \quad (4.4)$$

(ii) *If E_+ is closed, then E'_\pm separates points in E .*

Proof. (i) The sufficiency of (4.4) is obvious and the necessity follows immediately from proposition 4.3.

(ii) Any $X \notin E_+$, $X \neq 0$ can be separated from 0 by a $Z \in E'_+$. If $X \in E_+$, $X \neq 0$, then by assumption $-X \notin E_+$ and now $-X$ can be separated from 0 by some $Z \in E'_+$. \square

5. Locally Convex Lattice λ -modules

One of our aims is to show that positive λ -linear functions on a Banach lattice λ -module are necessarily continuous. Since this result is even true for λ -convex functions, we shall state it in a more general context in the next section. Here, we will introduce the lattice structure in a locally convex λ -module.

Definition 5.1.

(i) The ordered λ -module E is called a *lattice λ -module* or a *Riesz λ -module* if (E, E_+) has the lattice property, i.e. $\max(X_1, X_2)$ and $\min(X_1, X_2)$ exist for all $X_1, X_2 \in E$.

A module lattice E leads to the definitions:

$$X^+ := \sup(X, 0), \quad X^- := \sup(-X, 0), \quad |X| := X^+ + X^- \quad (5.1)$$

for all $X \in E$.

(ii) The lattice λ -module (E, E_+) is called *Dedekind-complete* (also called *order-complete*) if each net $(X_\iota)_{\iota \in I}$, directed upwards and bounded above by an element of E , has a supremum $\sup_{\iota \in I} X_\iota$ in E .

The following property, also called the Riesz decomposition property, is well known for lattices (see also [1], 8.9).

Proposition 5.2. *A lattice λ -module (E, E_+) has the decomposition property: For $X_1, X_2, Y \in E_+$ with $0 \leq Y \leq X_1 + X_2$, there exists $Y_1, Y_2 \in E_+$ with $Y_1 \leq X_1$, $Y_2 \leq X_2$, and $Y_1 + Y_2 = Y$.*

Proof. We just need to set $Y_1 = Y \wedge X_1 \in E_+$ and $Y_2 = Y - Y_1 \in E_+$ since $Y_2 = Y - Y \wedge X_1 = 0 \vee Y - X_1 \leq 0 \vee X_2 = X_2$. \square

Definitions 5.3.

(i) A λ -seminorm \mathfrak{q} on a lattice λ -module (E, E_+) is called a *lattice λ -seminorm* if it holds that $\mathfrak{q}|X| \leq \mathfrak{q}|X'|$ for all $X, X' \in E$ with $|X| \leq |X'|$. If in addition, $\|\cdot\|$ is a λ -norm we call it a *lattice λ -norm* and $(E, E_+, \|\cdot\|)$ is called a *normed lattice λ -module*.

(ii) If (E, E_+, Ω) is a lattice λ -module whose topology is induced by a family Ω of *lattice λ -seminorms* satisfying assumptions 2.3 and 2.4, then we call (E, E_+, Ω) a *locally convex lattice λ -module*.

- (iii) The normed lattice λ -module $(E, E_+, \|\cdot\|)$ is called a *Banach lattice λ -module* if the normed lattice λ -module $(E, \|\cdot\|)$ is (topologically) complete in the sense of definition 2.5 (ii).

Obviously, a lattice λ -seminorm \mathfrak{q} is symmetric:

$$\mathfrak{q}(X) = \mathfrak{q}(-X) = \mathfrak{q}(|X|) \quad (5.2)$$

for all $X \in E$.

Proposition 5.4. *Let (E, E_+, \mathfrak{Q}) be a locally convex lattice λ -module. Then*

- (i) *The lattice operations $\wedge, \vee : E \times E \rightarrow E, (X_1, X_2) \mapsto X_1 \wedge X_2, X_1 \vee X_2$ are uniformly continuous.*
(ii) *The positive and negative orthants E_+ and E_- are closed.*

Proof. (i) By the translation invariance of \vee (i.e. $X_1 \vee X_2 = ((X_1 - X_2) \vee 0) - X_2$) we may set $X_2 = 0$. Now $|X_1^+ - X_2^+| \leq |X_1 - X_2|$ implies $\mathfrak{q}|X_1^+ - X_2^+| \leq \mathfrak{q}|X_1 - X_2|$ for all $\mathfrak{q} \in \mathfrak{Q}$ which shows the uniform continuity.

(ii) Since $E_+ = [X \rightarrow X^-]^{-1}\{0\}$ and by assumption 2.3 the singleton $\{0\}$ is closed in the Hausdorff space E , the set E_+ is closed. \square

A scalar version of the following result can be found in [1], sections 8.13-15:

Theorem 5.5. *Let (E, E_+, \mathfrak{Q}) be a locally convex lattice λ -module. Then*

- (i) *the positive orthant E_+ is proper and generating, i.e.*

$$E_+ \cap E_- = \{0\} \quad \text{and} \quad E_+ - E_+ = E. \quad (5.3)$$

- (ii) *For a lattice λ -seminorm $\mathfrak{q} \in \mathfrak{Q}$, the conjugate λ -seminorm \mathfrak{q}' is also a lattice λ -seminorm.*
(iii) *The (topological) dual space E' is a lattice λ -module wherein the lattice operation $Z^+ = Z \vee 0$ is given on E_+ by the definition*

$$\langle X, Z^+ \rangle := \text{ess. sup} \{ \langle Y, Z \rangle \mid 0 \leq Y \leq X \} \quad \text{for } X \in E_+. \quad (5.4)$$

Since $Z \leq Z^+$, we have

$$E' = E'_{\pm}. \quad (5.5)$$

- (iv) *The dual E' is topologically complete and Dedekind-complete.*

Proof. (i) holds for any lattice space, since for $X \in E_+ \cap E_-$ we find $X = X^+ = X^- = X^+ \wedge X^- = 0$ and $X = X^+ + X^- \in E_+ + E_- = E_+ - E_+$.

(ii) For $Z, Z' \in E'$ we assume that $|Z| \leq |Z'|$. Then

$$\begin{aligned} \mathfrak{q}'|Z| &= \text{ess. sup} \{ |\langle X, |Z| \rangle| \mid \mathfrak{q}|X| \leq \mathbb{1} \} \leq \text{ess. sup} \{ |\langle X, |Z| \rangle| \mid \mathfrak{q}|X| \leq \mathbb{1} \} \\ &\leq \text{ess. sup} \{ |\langle X, |Z'| \rangle| \mid \mathfrak{q}|X| \leq \mathbb{1} \} = \text{ess. sup} \{ |\langle X, |Z'| \rangle| \mid \mathfrak{q}|X| \leq \mathbb{1} \} \\ &= \mathfrak{q}'|Z'|. \end{aligned}$$

This shows that \mathfrak{q}' is a lattice λ -seminorm.

(iii) Let $Z \in E'$ be a continuous λ -linear function. With the help of proposition 2.6 we find $\eta \in \lambda_{\sharp}$ and $\mathfrak{q} \in \mathfrak{Q}$ such as $|\langle X, Z \rangle| \leq \eta \cdot \mathfrak{q}|X|$ for all $X \in E$. The lattice λ -seminorm character of \mathfrak{q} implies that given $X \in E_+$, we have

$$|\langle Y, Z \rangle| \leq \eta \cdot \mathfrak{q}|X| \quad (5.6)$$

uniformly for all $Y \in E_+$ with $0 \leq Y \leq X$.

For $X \in E_+$ we define Z^+ by (5.4). It follows immediately from (5.6) that

$$|\langle X, Z^+ \rangle| \leq \eta \cdot \mathfrak{q}|X|. \quad (5.7)$$

for all $X \in E_+$. Of course, Z^+ is non-negative on E_+ . For $\zeta \in \lambda_+$ it is evident that $\langle \zeta \cdot X, Z^+ \rangle = \zeta \cdot \langle X, Z^+ \rangle$. Next for $0 \leq Y_i \leq X_i$, $i = 1, 2$ we have $\langle Y_1, Z \rangle + \langle Y_2, Z \rangle = \langle Y_1 + Y_2, Z \rangle \leq \langle X_1 + X_2, Z^+ \rangle$ so that $\langle X_1, Z^+ \rangle + \langle X_2, Z^+ \rangle \leq \langle X_1 + X_2, Z^+ \rangle$. Conversely, for $0 \leq Y \leq X_1 + X_2$ the decomposition property in proposition 5.2 yields elements $0 \leq Y_i \leq X_i$, $i = 1, 2$ with $Y_1 + Y_2 = Y$. Therefore $\langle Y, Z \rangle = \langle Y_1, Z \rangle + \langle Y_2, Z \rangle \leq \langle X_1, Z^+ \rangle + \langle X_2, Z^+ \rangle$ and the inverse inequality $\langle X_1 + X_2, Z^+ \rangle \leq \langle X_1, Z^+ \rangle + \langle X_2, Z^+ \rangle$ follows. Therefore Z^+ is additive and non-negative λ -homogeneous on E_+ . The λ -linearity extends Z^+ uniquely to $E = E_+ + E_-$. Since $|\langle X, Z^+ \rangle| \leq \langle |X|, Z^+ \rangle \leq \eta \cdot \mathfrak{q}||X|| = \eta \cdot \mathfrak{q}|X|$ the continuity condition (5.7) now holds for all $X \in E$: Z^+ is a positive continuous λ -linear function dominating Z . To show that $Z^+ = Z \vee 0$, let \overline{Z} be a positive continuous λ -linear function dominating Z . But the positivity of \overline{Z} shows that for $0 \leq Y \leq X$ one has $\langle Y, Z \rangle \leq \langle Y, \overline{Z} \rangle \leq \langle X, \overline{Z} \rangle$, and therefore $\langle X, Z^+ \rangle \leq \langle X, \overline{Z} \rangle$, hence $Z^+ \leq \overline{Z}$. This shows that $Z^+ = Z \vee 0$.

(iv) The first assertion has already been shown in proposition 2.7. For the second we may regard a downward directed net $(Z_\iota)_{\iota \in I}$ bounded below by $0 \in E'$ without loss of generality. We pick one $\iota_0 \in I$ and restrict thereafter the net to the index set $\{\iota \in I \mid \iota \geq \iota_0\}$, again denoted by I . We also find $\eta \in \lambda_{\mathbb{R}}$ and $\mathfrak{q} \in \mathfrak{Q}$ such as

$$0 \leq \langle X, Z_{\iota_0} \rangle \leq \eta \cdot \mathfrak{q}|X|$$

for all $X \in E_+$. Moreover, since $0 \leq \langle X, Z_\iota \rangle \leq \langle X, Z_{\iota_0} \rangle$ for all $\iota \geq \iota_0$ and $X \in E_+$ we even have

$$0 \leq \langle X, Z_\iota \rangle \leq \eta \cdot \mathfrak{q}|X| \quad (5.8)$$

for all $\iota \in I$ and $X \in E_+$.

Next, we define $\inf Z_\iota$ on E_+ by

$$\langle X, \inf Z_\iota \rangle := \text{ess. inf}_{\iota \in I} \langle X, Z_\iota \rangle \quad (5.9)$$

with $X \in E_+$. Obviously, $\langle X, \inf Z_\iota \rangle \in \lambda_+$ and $\inf Z_\iota$ is non-negative homogeneous: $\langle \zeta \cdot X, \inf Z_\iota \rangle = \zeta \cdot \langle X, \inf Z_\iota \rangle$ for $X \in E_+$ and $\zeta \in \lambda_+$. It is also clear to see that $\langle X_1 + X_2, \inf Z_\iota \rangle \geq \langle X_1, \inf Z_\iota \rangle + \langle X_2, \inf Z_\iota \rangle$.

To show the inverse inequality, let's assume that

$$\{\langle X_1 + X_2, \inf Z_\iota \rangle > \langle X_1, \inf Z_\iota \rangle + \langle X_2, \inf Z_\iota \rangle\} \in \mathcal{G}^+.$$

This means that for some $\gamma, \delta \in \mathbb{R}_+$ we have also

$$A := \{\langle X_1 + X_2, \inf Z_\iota \rangle > \gamma > \delta > \langle X_1, \inf Z_\alpha \rangle + \langle X_2, \inf Z_\alpha \rangle\} \in \mathcal{G}^+.$$

Then there exist $\iota_1, \iota_2 \in I$ with

$$A' := A \cap \{ \frac{1}{2}(\gamma + \delta) \geq \langle X_1, Z_{\iota_1} \rangle + \langle X_2, Z_{\iota_2} \rangle \} \in \mathcal{G}^+.$$

Consider $\iota_3 \geq \iota_i$, $i = 1, 2$ such as on A' we have

$$\begin{aligned} \frac{1}{2}(\gamma + \delta) &\geq \langle X_1, Z_{\iota_1} \rangle + \langle X_2, Z_{\iota_2} \rangle \geq \langle X_1, Z_{\iota_3} \rangle + \langle X_2, Z_{\iota_3} \rangle = \langle X_1 + X_2, Z_{\iota_3} \rangle \\ &\geq \text{ess. inf}_{\iota} \langle X_1 + X_2, Z_\iota \rangle \geq \gamma, \end{aligned}$$

which is a contradiction. Therefore the inverse inequality $\langle X_1 + X_2, \inf Z_\iota \rangle \leq \langle X_1, \inf Z_\iota \rangle + \langle X_2, \inf Z_\iota \rangle$ holds too.

Hence we have shown that $\inf Z_\iota$ is a positive, additive, and non-negative λ -homogeneous function on E_+ which has a unique extension to $E = E_+ + E_-$. Since (5.8) holds uniformly in $\iota \in I$ we have also

$$|\langle X, \inf Z_\iota \rangle| \leq \langle |X|, \inf Z_\iota \rangle \leq \eta \cdot \mathfrak{q}|X| \quad (5.10)$$

for all $X \in E$. Therefore $\inf Z_\iota \in E'_+$. As a result of definition in (5.9) it is easy to see that $\inf Z_\iota \geq \overline{Z}$ on E_+ for all \overline{Z} satisfying $\overline{Z} \leq Z_\iota$, $\iota \in I$. This completes the proof. \square

6. λ -convex Functions on Locally Convex Lattice λ -modules

We will start with the following definitions:

Definitions 6.1.

- (i) A λ -valued function $f : E \rightarrow \overline{L}^0(\mathcal{G})$ is λ -convex if for all $X, X' \in E$ and $\zeta \in \lambda$, $0 \leq \zeta \leq \mathbb{1}$ we have $f(\zeta \cdot X + (\mathbb{1} - \zeta) \cdot X') \leq \zeta \cdot f(X) + (\mathbb{1} - \zeta) \cdot f(X')$.
- (ii) The function $f : E \rightarrow \overline{L}^0(\mathcal{G})$ is λ -proper if for all $X \in E$ there exists $\gamma_X \in \mathbb{R}$ such as $f(X) \geq \gamma_X \cdot \mathbb{1}$ and

$$\text{dom} f := \{X \in E \mid f(X) \in \lambda\} \neq \emptyset. \quad (6.1)$$

First we note that a λ -convex function f has the following *locality property*:

Lemma 6.2. *For all $X \in E$ and $A \in \mathcal{G}^+$ we have*

$$\mathbb{1}_A \cdot f(X) = \mathbb{1}_A \cdot f(\mathbb{1}_A \cdot X). \quad (6.2)$$

Proof. Indeed, for $A \in \mathcal{G}$ and $X \in E$ we find $\mathbb{1}_A \cdot f(X) = \mathbb{1}_A \cdot f(\mathbb{1}_A \cdot X + \mathbb{1}_{A^c} \cdot X) \leq \mathbb{1}_A \cdot f(\mathbb{1}_A \cdot X) = \mathbb{1}_A \cdot f(\mathbb{1}_A \cdot X + \mathbb{1}_{A^c} \cdot 0) = \mathbb{1}_A \cdot f(X)$ which implies equalities everywhere and therefore (6.2). \square

With these definitions we can formulate the following result whose proof follows essentially the one of the scalar version (see [1], 9.6):

Theorem 6.3. *Let $(E, E_+, \|\cdot\|)$ be a Banach lattice λ -module. Then any monotone λ -proper λ -convex function $f : E \rightarrow \overline{L}^0(\mathcal{G})$ is continuous on the interior of its domain $\text{dom} f$.*

Proof. Consider $X \in \overset{\circ}{\text{dom}} f$ such as $f(X) \in \lambda$. After the transformation $f(\cdot + X) - f(X)$, we may assume that $X = 0 \in \overset{\circ}{\text{dom}} f$ and $f(0) = 0$. Since λ has a countable neighborhood base of 0, it is enough to show that we get $f(X_\nu) \rightarrow 0 \in \lambda$ for any sequence $(X_\nu)_{\nu \in \mathbb{N}} \subseteq \overset{\circ}{\text{dom}} f$ with $X_\nu \rightarrow 0$.

With the remark following definition 2.1 we take $r > 0$ with $B_{\|\cdot\|, r} \subseteq \overset{\circ}{\text{dom}} f$. For $n \geq 1$, we set $V_n := \{X \in E \mid \|X\| \leq r2^{-n}\}$ so that $V_{n+1} + V_{n+1} \subseteq V_n$. For each $\mu \in \mathbb{N}$ we find $X_{\nu_\mu} \in 1/\mu \cdot V_\mu$. Setting $Y_n := \sum_{\mu=1}^n \mu \cdot |X_{\nu_\mu}|$ we get $Y_n \leq Y_{n+1} \in E_+$ and $0 \leq Y_{n+m} - Y_n = \sum_{\mu=1}^m (n + \mu) \cdot |X_{\nu_{n+\mu}}| \in (V_{n+1} + \dots + V_{n+m}) \subseteq V_n$ for all

$n, m \geq 1$. Thus Y_n is a Cauchy-sequence in E_+ and because of the completeness of E it converges to some $Y \in E$.

Since E_+ is closed by proposition 5.4 (ii), the fact that $E_+ \ni Y_{n+m} - Y_n \rightarrow Y - Y_n$ gives $Y - Y_n \in E_+$, or $Y_n \leq Y$ for all n . By the monotonicity and convexity of f we get the following inequalities for all $n \geq 1$:

$$\begin{aligned} |f(X_n)| \leq |f(|X_n|)| &\leq \left| f\left(\frac{1}{n}(n \cdot |X_n|) + \left(1 - \frac{1}{n}\right) \cdot 0\right) \right| \leq \frac{1}{n} |f(n \cdot |X_n|)| \\ &\leq \frac{1}{n} |f(Y_n)| \leq \frac{1}{n} |f(Y)|. \end{aligned}$$

This shows that $f(X_n) \rightarrow 0$ which is in contradiction to the assumption. \square

We are going to combine the theorem 5.5 and theorem 6.3 in the following generalization of the Namioka-Klee theorem:

Corollary 6.4. *Let $(E, E_+, \|\cdot\|)$ be a Banach lattice λ -module. If E_+^* denotes the set of all (not necessarily continuous) positive λ -linear functions on E then*

$$E_+^* = E'_+ \quad \text{and} \quad E_+^* - E_+^* = E'_\pm. \quad (6.3)$$

Moreover, with the conjugate lattice λ -norm $\|\cdot\|'$ the dual space $(E', E'_+, \|\cdot\|')$ is an order-complete Banach lattice λ -module whose lattice operation $Z \rightarrow Z^+$ is given in (5.4).

Remark 6.5. It is interesting to note that the statement of corollary 6.4 is not correct for general ordered Banach spaces or Banach λ -modules (see example 6.10 in [19]). Some properties linking the order structure with the topology are needed, for example the condition that E_+ is closed and generating (see [18], appendix A).

7. Subdifferentials on Banach Lattice Modules

Now we will focus on the existence of subdifferentials of a λ -convex function f on a locally convex lattice λ -module. We carry over the basic results in [21] and [17] to the case of λ -modules.

Definition 7.1. A function $f : E \rightarrow \overline{L}^0(\mathcal{G})$ on a locally convex λ -module (E, \mathcal{Q}) is *subdifferentiable* at $X_0 \in E$ if there exists a *continuous* λ -linear function $Z \in E'$, called a *subdifferential* of f at X_0 , such as

$$\langle Y - X_0, Z \rangle \leq f(Y) - f(X_0) \quad (7.1)$$

for all $Y \in E$.

A first step towards the existence of a subdifferential is the analysis of the *directional derivative* ∂f of a λ -proper and λ -convex function f on the domain of f . For $X_0 \in \text{dom} f$ and $Y \in E$ we define

$$\partial f(X_0)(Y) := \text{ess. inf}_{\zeta \in \lambda_\#} \frac{f(X_0 + \zeta \cdot Y) - f(X_0)}{\zeta}. \quad (7.2)$$

Proposition 7.2. *Let (E, \mathcal{Q}) be a locally convex λ -module, $f : E \rightarrow \overline{L}^0(\mathcal{G})$ a λ -proper and λ -convex function and $X_0 \in \text{dom} f$. For $Y \in E$ and $\zeta_0, \zeta_1, \zeta_2 \in \lambda_\#$ with $\zeta_0 \leq \mathbb{I}, \zeta_1 \leq \zeta_2$, we get the following results:*

$$(i) \quad \frac{f(X_0 + \zeta_1 \cdot Y) - f(X_0)}{\zeta_1} \leq \frac{f(X_0 + \zeta_2 \cdot Y) - f(X_0)}{\zeta_2}. \quad (7.3)$$

$$(ii) \quad -\partial f(X_0)(-Y) \leq \partial f(X_0)(Y).$$

(iii)

$$\left| \frac{f(X_0 + \zeta_0 Y) - f(X_0)}{\zeta_0} \right| \leq \max [f(X_0 + Y) - f(X_0), f(X_0 - Y) - f(X_0)]. \quad (7.4)$$

(iv) $\partial f(X_0)(Y) \in \lambda$.(v) *The derivative $\partial f(X_0) : E \ni Y \mapsto \partial f(X_0)(Y) \in \lambda$ is λ -convex and has the locality property (6.2).*(vi) *The function $\partial f(X_0) : E \ni Y \mapsto \partial f(X_0)(Y)$ is positive λ -homogeneous and therefore λ -sublinear.*(vii) *The subdifferential inequality holds:*

$$\partial f(X_0)(Y - X_0) \leq f(Y) - f(X_0). \quad (7.5)$$

Proof. (i) The λ -convex representation $X_0 + \zeta_1 \cdot Y = \frac{\zeta_1}{\zeta_2}(X_0 + \zeta_2 \cdot Y) + (1 - \frac{\zeta_1}{\zeta_2}) \cdot X_0$ implies $f(X_0 + \zeta_1 \cdot Y) \leq \frac{\zeta_1}{\zeta_2} \cdot f(X_0 + \zeta_2 \cdot Y) + (1 - \frac{\zeta_1}{\zeta_2}) \cdot f(X_0)$ which is equivalent to (7.3).

(ii) For $\zeta \in \lambda_{\#}$, the convex combination $X_0 = \frac{1}{2}(X_0 + \zeta \cdot Y) + \frac{1}{2}(X_0 - \zeta \cdot Y)$ yields $2f(X_0) \leq f(X_0 + \zeta \cdot Y) + f(X_0 - \zeta \cdot Y)$ or

$$- [f(X_0 - \zeta \cdot Y) - f(X_0)] \leq f(X_0 + \zeta \cdot Y) - f(X_0) \quad (7.6)$$

from which (ii) follows immediately.

(iii) We have $X_0 + \zeta_0 \cdot Y = (1 - \zeta_0) \cdot X_0 + \zeta_0 \cdot (X_0 + Y)$ such as $f(X_0 + \zeta_0 \cdot Y) \leq (1 - \zeta_0) \cdot f(X_0) + \zeta_0 \cdot f(X_0 + Y)$ or equivalently

$$f(X_0 + \zeta_0 \cdot Y) - f(X_0) \leq \zeta_0 \cdot [f(X_0 + Y) - f(X_0)]. \quad (7.7)$$

If we multiply (7.6) by -1 and replace Y by $-Y$ in (7.7) we get

$$- [f(X_0 + \zeta_0 \cdot Y) - f(X_0)] \leq \zeta_0 \cdot [f(X_0 - Y) - f(X_0)]. \quad (7.8)$$

Both (7.7) and (7.8) yield (7.4).

(iv) For $X_0 \in \text{dom} f$ and $Y \in E$, we find $\xi \in \lambda_{\#}$ so that $X_0 + \xi \cdot Y$ and $X_0 - \xi \cdot Y$ lie in $\text{dom} f$ and therefore $\max [f(X_0 + \xi \cdot Y) - f(X_0), f(X_0 - \xi \cdot Y) - f(X_0)] \in \lambda$. By (7.4), we get

$$\begin{aligned} |\partial f(X_0)(Y)| &= \frac{1}{\xi} \left| \text{ess. inf}_{\zeta \in \lambda_{\#}} \frac{f(X_0 + \zeta \cdot \xi \cdot Y) - f(X_0)}{\zeta} \right| \\ &\leq \frac{1}{\xi} \max [f(X_0 + \xi \cdot Y) - f(X_0), f(X_0 - \xi \cdot Y) - f(X_0)] \in \lambda. \end{aligned}$$

(v) For $\xi \in \lambda$, $0 \leq \xi \leq 1$, $\zeta \in \lambda_{\#}$, and $Y, Y' \in E$, we have

$$\frac{f(X_0 + \zeta \cdot (\xi \cdot Y + (1 - \xi) \cdot Y'))}{\zeta} \leq \xi \frac{f(X_0 + \zeta \cdot Y)}{\zeta} + (1 - \xi) \frac{f(X_0 + \zeta \cdot Y')}{\zeta}$$

which implies the λ -convexity of $\partial f(X_0) : E \ni Y \mapsto \partial f(X_0)(Y) \in \lambda$.

(vi) Let $0 \leq \xi \in \lambda$. We use the locality of f : On the set $A_0 := \{\xi = 0\}$ the expression $\frac{f(X_0 + \zeta \cdot \xi \cdot Y) - f(X_0)}{\zeta}$ is 0 for all $\zeta \in \lambda_{\sharp}$ such as $0 = \partial f(X_0)(\xi \cdot Y) = \xi \cdot \partial f(X_0)(Y)$ on A_0 . On the other hand, we have for $\varepsilon > 0$ on $A_\varepsilon := \{\xi \geq \varepsilon\}$ that $\frac{f(X_0 + \zeta \cdot \xi \cdot Y) - f(X_0)}{\zeta} = \xi \cdot \frac{f(X_0 + \zeta \cdot \xi \cdot Y) - f(X_0)}{\zeta \cdot \xi}$ which implies $\partial f(X_0)(\xi \cdot Y) = \xi \cdot \partial f(X_0)(Y)$ on A_ε and therefore on $\{\xi \geq 0\}$.

(vii) For $\zeta \in \lambda_{\sharp}$, $\zeta \leq 1$, the convex representation $X_0 + \zeta \cdot (Y - X_0) = \zeta \cdot Y + (1 - \zeta) \cdot X_0$ shows that $\partial f(X_0)(Y - X_0) \leq \frac{f(X_0 + \zeta(Y - X_0)) - f(X_0)}{\zeta} \leq f(Y) - f(X_0)$. \square

Theorem 7.3. *Let $(E, E_+, \|\cdot\|)$ be a Banach lattice λ -module. Then any monotone λ -proper, λ -convex function $f : E \rightarrow \bar{L}^0(\mathcal{G})$ is continuous and subdifferentiable at all $X \in \mathring{\text{dom}}f$.*

Proof. Let $X_0 \in \mathring{\text{dom}}f$ and $0 \neq Y \in E$. For all $\zeta \in \lambda$, we set

$$\langle \zeta \cdot Y, Z \rangle := \zeta \cdot \partial f(X_0)(Y).$$

Z is well-defined on $\lambda \cdot \{Y\}$ since $\zeta \cdot Y = \zeta' \cdot Y$ implies $\zeta = \zeta'$ on $A = \{\|Y\| > 0\}$ and with $\mathbb{1}_{A^c} \cdot Y = 0$ we get by the locality of $\partial f(X_0)$ and the fact that $\partial f(X_0)(0) = 0$:

$$\begin{aligned} \zeta \cdot \partial f(X_0)(Y) &= \mathbb{1}_A \cdot \zeta \cdot \partial f(X_0)(Y) + \mathbb{1}_{A^c} \cdot \zeta \cdot \partial f(X_0)(Y) \\ &= \mathbb{1}_A \cdot \zeta' \cdot \partial f(X_0)(Y) = \zeta' \cdot \partial f(X_0)(Y). \end{aligned}$$

Next, we will show that Z is dominated on $\lambda \cdot \{Y\}$ by the λ -sublinear function $\partial f(X_0)$: For $\zeta \in \lambda$ and $A := \{\zeta \geq 0\}$ the locality and the positive λ -homogeneity of $\partial f(X_0)$ together with (ii) of proposition 7.2 show

$$\begin{aligned} \langle \zeta \cdot Y, Z \rangle &= \zeta \cdot \partial f(X_0)(Y) = \partial f(X_0)(\mathbb{1}_A \cdot \zeta \cdot Y) - \partial f(X_0)(-\mathbb{1}_{A^c} \cdot \zeta \cdot Y) \\ &\leq \partial f(X_0)(\mathbb{1}_A \cdot \zeta \cdot Y) + \partial f(X_0)(\mathbb{1}_{A^c} \cdot \zeta \cdot Y) = \partial f(X_0)(\zeta \cdot Y). \end{aligned}$$

By the Hahn-Banach theorem 3.1, we can extend Z to a λ -linear function on E satisfying

$$\langle Y, Z \rangle \leq \partial f(X_0)(Y) \leq f(X_0 + Y) - f(X_0) \quad (7.9)$$

for all $Y \in E$ where the last inequality comes from (7.5).

Moreover, Z is monotone. Indeed let $Y \in E_+$ and assume that $A := \{\langle Y, Z \rangle < 0\} \in \mathcal{G}^+$. Then (7.9) implies $f(X_0 - Y) \geq f(X_0) - \langle Y, Z \rangle$ so that $\{f(X_0 - Y) > f(X_0)\} \supseteq A \in \mathcal{G}^+$, which is a contradiction to the assumption that f is monotone.

But, since the λ -linear function Z is in particular λ -convex and as shown monotone, theorem 6.3 proves that Z is continuous and consequently a subdifferential to f at X_0 . \square

8. The L_λ^p - and $L_\lambda^{(p_2, p_1)}$ -modules

For $X \in L^0(\mathcal{F})$, we recall the definition of the conditional norm $\|X\|_{p, \mathcal{G}}$ from (2.2). It is easy to check that $\|X\|_{p, \mathcal{G}}$ are λ -norms in the sense of definition 2.1 (iv.c).

Definition 8.1. For $p \in [1, \infty]$ we set

$$L_\lambda^p(\mathcal{F}) := \left\{ X \in L^0(\mathcal{F}) \mid \|X\|_{p, \lambda} := \left\| \|X\|_{p, \mathcal{G}} \right\|_\lambda < \infty \right\}. \quad (8.1)$$

In the first part of this section we keep the σ -algebra \mathcal{F} fixed, so that we can simply write L_λ^p instead of $L_\lambda^p(\mathcal{F})$. The space L_λ^p with the λ -norm $\|\cdot\|_{p,\mathcal{G}}$ is obviously a locally convex normed λ -module.

Theorem 8.2. *For $p \in [1, \infty]$, the normed λ -modules L_λ^p are complete.*

Proof. Let $(X_n)_{n \geq 0}$ be a Cauchy sequence in L_λ^p , i.e.

$$\sup_{n_1, n_2 \geq n_\varepsilon} \|X_{n_1} - X_{n_2}\|_{p,\lambda} < \varepsilon. \quad (8.2)$$

By passing to a subsequence — again denoted by (X_n) — we may assume that for $\nu \geq 1$

$$\|X_\nu - X_{\nu-1}\|_{p,\lambda} < 2^{-\nu}.$$

With $Y_\nu := X_\nu - X_{\nu-1}$ we find $\sum_{\nu \geq n} \|Y_\nu\|_{p,\lambda} \leq 2^{-n+1}$.

We define

$$\bar{Y}_n := \sum_{\nu=1}^n |Y_\nu| \in L_\lambda^p, \quad \bar{Y} := \sum_{\nu \geq 1} |Y_\nu| \geq 0.$$

Now

$$\left\| \|\bar{Y}\|_{p,\mathcal{G}} - \|\bar{Y}_n\|_{p,\mathcal{G}} \right\|_\lambda \leq \left\| \|\bar{Y} - \bar{Y}_n\|_{p,\mathcal{G}} \right\|_\lambda \leq \sum_{\nu \geq n+1} \|Y_\nu\|_{p,\lambda} \leq 2^{-n}.$$

This shows that $\bar{Y} \in L_\lambda^p$ and consequently that $\bar{\zeta} := \|\bar{Y}\|_{p,\mathcal{G}} \in \lambda_+$. For $r > 0$ and $A_r := \{\bar{\zeta} \leq r\} \in \mathcal{G}$ we have according to Beppo Levi's theorem

$$\mathbb{E} \left[\mathbb{1}_{A_r} \cdot \bar{Y}^p \mid \mathcal{G} \right] = \lim_{n \rightarrow \infty} \mathbb{E} \left[\mathbb{1}_{A_r} \cdot \left(\sum_{\nu=1}^n |Y_\nu| \right)^p \mid \mathcal{G} \right] \leq \mathbb{1}_{A_r} r^p.$$

But now $\mathbb{1}_{A_r} \sum_{\nu=1}^\infty Y_\nu =: \mathbb{1}_{A_r} \cdot Y_\infty$ is also in L^p where Y_∞ is defined independently of r . Since for all $r > 0$

$$\mathbb{1}_{A_r} \mathbb{E} [|Y|^p \mid \mathcal{G}]^{1/p} \leq \mathbb{1}_{A_r} \cdot \bar{\zeta}$$

we find that $Y_\infty \in L_\lambda^p$ and $\|Y_\infty - \sum_{\nu=1}^n Y_\nu\|_{p,\lambda} \leq \sum_{\nu > n} \|Y_\nu\|_{p,\lambda} \rightarrow 0$ for $n \rightarrow \infty$. Finally, $\|Y_\infty + X_0 - X_n\|_{p,\lambda} = \|Y - \sum_{\nu=1}^n Y_\nu\|_{p,\lambda} \rightarrow 0$ for $n \rightarrow \infty$. This shows that the subsequence (X_n) has a limit in L_λ^p . It is now easy to see that the original sequence has the same limit. \square

The following proposition is a consequence of the conditional version of Hölder's inequality.

Proposition 8.3. *Consider $p \in [1, \infty)$ and q its conjugate exponent according to (2.3). Then we have*

$$\|X \cdot Z\|_{1,\lambda} \leq \|X\|_{p,\lambda} \cdot \|Z\|_{q,\lambda} \quad (8.3)$$

for all $X \in L_\lambda^p$ and $Z \in L_\lambda^q$.

Proof. For $X \in L_\lambda^p$, $Z \in L_\lambda^q$, and $n \in \mathbb{N}$, Hölder's inequality for conditional expectations yields

$$\begin{aligned} \|(|X| \wedge n) \cdot (|Z| \wedge n)\|_{1,\lambda} &= \left\| \mathbb{E} [(|X| \wedge n) \cdot (|Z| \wedge n) \mid \mathcal{G}] \right\|_\lambda \\ &\leq \left\| \mathbb{E} [(|X| \wedge n)^p \mid \mathcal{G}]^{1/p} \cdot \mathbb{E} [(|Z| \wedge n)^q \mid \mathcal{G}]^{1/q} \right\|_\lambda \leq \|X\|_{p,\lambda} \cdot \|Z\|_{q,\lambda}. \end{aligned}$$

Taking the limit $n \rightarrow \infty$, we get (8.3) by the monotone convergence theorem. \square

Theorem 8.4. *Consider $p \in [1, \infty)$ and q its conjugate exponent as given in (2.3).*

(i) *For $Z \in L_\lambda^q$, the mapping $\langle \cdot, Z \rangle : L_\lambda^p \rightarrow \lambda$ given by*

$$\langle X, Z \rangle := \mathbb{E}[X \cdot Z | \mathcal{G}], \quad \text{for } X \in L_\lambda^p \quad (8.4)$$

defines a continuous λ -linear function on L_λ^p .

(ii) *Conversely, if $\langle \cdot, Z \rangle : L_\lambda^p \rightarrow \lambda$ is a continuous λ -linear function, then there exists a unique $Z \in L_\lambda^q$ such as (8.4) holds for all $X \in L_\lambda^p$.*

Proof. (i) Obviously, (8.4) defines a λ -linear function which is continuous by (8.3) and proposition 2.6.

(ii) Let $\langle \cdot, Z \rangle$ be a continuous λ -linear function on L_λ^p . Because of proposition 2.6, there exists $\eta \in \lambda_\#$ with $|\langle X, Z \rangle| \leq \eta \|X\|_{p, \mathcal{G}}$.

We define the λ -linear function $\langle \cdot, \tilde{Z} \rangle : L_\lambda^p \rightarrow \lambda$ by $\langle X, \tilde{Z} \rangle := \langle X, Z \rangle / \eta$ such as $|\langle X, \tilde{Z} \rangle| \leq \|X\|_{p, \mathcal{G}}$. We find that $\mathbb{E}[|\langle X, \tilde{Z} \rangle|^p] \leq \mathbb{E}[\mathbb{E}[|X|^p | \mathcal{G}]] = \mathbb{E}[|X|^p] = \|X\|_p^p$ on L_λ^p and Jensen's inequality implies that for all $X \in L_\lambda^p$, we have

$$\mathbb{E}[\langle X, \tilde{Z} \rangle] \leq \mathbb{E}[|\langle X, \tilde{Z} \rangle|] \leq \mathbb{E}[|\langle X, \tilde{Z} \rangle|^p]^{1/p} \leq \|X\|_p.$$

Now, $\langle \cdot, \hat{Z} \rangle : L_\lambda^p \rightarrow \mathbb{R}$, defined by $\langle X, \hat{Z} \rangle := \mathbb{E}[\langle X, \tilde{Z} \rangle]$ is a real linear function on L_λ^p satisfying $\langle X, \hat{Z} \rangle \leq \|X\|_p$. With the help of the classical Hahn-Banach theorem, it can be extended to a linear function, again denoted by \hat{Z} on L^p with $\langle X, \hat{Z} \rangle \leq \|X\|_p$ for all $X \in L^p$. The topological dual of L^p being L^q , we find a unique $Z' \in L^q$ such as $\langle X, \hat{Z} \rangle = \mathbb{E}[X \cdot Z']$ for all $X \in L^p$. In particular, $\langle \mathbb{1}_A \cdot X, \hat{Z} \rangle = \mathbb{E}[\mathbb{1}_A \cdot X \cdot Z']$ for all $A \in \mathcal{G}^+$ and $X \in L_\lambda^p$, whence we have

$$\mathbb{E}[X \cdot Z' | \mathcal{G}] = \langle X, \tilde{Z} \rangle \leq \|X\|_{p, \mathcal{G}} \quad (8.5)$$

for all $X \in L_\lambda^p$. Since $\|Z'\|_{q, \mathcal{G}} = \text{ess. sup} \{ \mathbb{E}[X \cdot Z' | \mathcal{G}] \mid \|X\|_{p, \mathcal{G}} \leq 1 \}$, we get $\|Z'\|_{q, \mathcal{G}} \leq 1$ and a fortiori $\|Z'\|_{q, \lambda} \leq 1$. Finally, setting $Z'' = Z' \cdot \eta \in L_\lambda^q$ we see that

$$\langle X, Z \rangle = \eta \cdot \langle X, \tilde{Z} \rangle = \eta \cdot \mathbb{E}[X \cdot Z' | \mathcal{G}] = \mathbb{E}[X \cdot Z'' | \mathcal{G}].$$

The uniqueness of Z' implies that of Z'' . This shows part (ii) of the theorem. \square

Now, we will fix the nested σ -algebras $\mathcal{G} \subseteq \mathcal{G}_1 \subseteq \mathcal{F}$ and set $\lambda_1 := L^\infty(\mathcal{G}_1)$. Thereby we will define the following spaces.

Definition 8.5. For $\overline{\mathcal{G}} = (\mathcal{G}, \mathcal{G}_1)$ and $\overline{p} = (p_1, p_2) \in [1, \infty]^2$, we define

(i) for $X \in L^0(\mathcal{F})$ the λ -norm $\|\cdot\|_{\overline{p}, \overline{\mathcal{G}}}$ by

$$\|X\|_{\overline{p}, \overline{\mathcal{G}}} := \left\| \|X\|_{p_2, \mathcal{G}_1} \right\|_{p_1, \mathcal{G}}, \quad (8.6)$$

(ii) the space $L_\lambda^{\bar{p}}$ by

$$L_\lambda^{\bar{p}} := L_\lambda^{\bar{p}}(\bar{\mathcal{G}}) := \left\{ X \in L^0(\mathcal{F}) \mid \|X\|_{\bar{p},\lambda} := \left\| \|X\|_{\bar{p},\mathcal{G}} \right\|_\lambda < \infty \right\}. \quad (8.7)$$

Similar to theorems 8.2 and 8.4 we get the following results.

Theorem 8.6.

- (i) For $\bar{\mathcal{G}} = (\mathcal{G}, \mathcal{G}_1)$ and $\bar{p} = (p_1, p_2) \in [1, \infty]^2$, the normed λ -module $L_\lambda^{\bar{p}}$ is complete.
- (ii) For $\bar{p} = (p_1, p_2) \in [1, \infty]^2$ we set $\bar{q} = (q_1, q_2) \in (1, \infty]^2$ where q_i is the conjugate exponent to p_i , $i = 1, 2$.

Then $L_\lambda^{\bar{q}} = L_{\lambda_1}^{q_2} \times L_\lambda^{q_1}$ is the dual space of $L_\lambda^{\bar{p}}$. The duality and the norm inequality are given by

$$\langle X, Z \rangle := \mathbb{E}[X \cdot Z \mid \mathcal{G}], \quad (8.8)$$

$$|\langle X, Z \rangle| \leq \|X\|_{\bar{p},\bar{\mathcal{G}}} \|Z\|_{\bar{q},\bar{\mathcal{G}}}, \quad (8.9)$$

for $X \in L_\lambda^{\bar{p}}$ and $Z \in L_\lambda^{\bar{q}}$.

The proof of these statements follows the same lines as those for theorem 8.2 and 8.4.

Remark 8.7. Of course, the last theorem can be generalized to any finite number of nested σ -algebras $\mathcal{G} \subseteq \mathcal{G}_1 \subseteq \dots \subseteq \mathcal{G}_T$. This will be important when dealing with multi-period time-consistent risk assessments.

9. Orlicz and Morse λ -modules

Let $\theta : [0, \infty) \rightarrow [0, \infty)$ be a convex, increasing function with $\theta(0) = 0$ and $\lim_{r \rightarrow \infty} \theta(r)/r = \infty$. This last condition restricts the kind of Orlicz λ -modules we are regarding, but leaves enough interesting examples. Notice that above $r_\theta^* := \inf\{r > 0 \mid \theta(r) > 0\} \in \mathbb{R}_+$, i.e. in the interval $[r_\theta^*, \infty)$ the function θ is strictly increasing such that its inverse θ^{-1} is well-defined there. From $\theta(r) \leq 1/2 \theta(2r) > r_\theta^*$ for r large, one derives that $r \leq \theta^{-1}(1/2 \theta(2r))$ such as $\theta^{-1}(r) \rightarrow \infty$ for $r \rightarrow \infty$.

As θ is a convex increasing function on $[0, \infty)$, the set of its subdifferentials at $r \in [0, \infty)$ is not empty: By $\partial\theta(r) := \lim_{\varepsilon \searrow 0} (\theta(r + \varepsilon) - \theta(r))/\varepsilon$, $\partial\theta(0) := 0$, we denote right-subdifferential such as $r \mapsto \partial\theta(r)$ is an increasing right-continuous function on $[0, \infty)$ with $\theta(r) = \int_0^r \partial\theta(u) du$ for all $r \in [0, \infty)$.

We still need the following inequality as a consequence of the convexity of θ :

$$\theta\left(\frac{r_1 + r_2}{c_1 + c_2}\right) \leq \frac{c_1}{c_1 + c_2} \theta\left(\frac{r_1}{c_1}\right) + \frac{c_2}{c_1 + c_2} \theta\left(\frac{r_2}{c_2}\right) \quad (9.1)$$

for all $r_1, r_2 \in \mathbb{R}_+$ and $c_1, c_2 > 0$.

We define the Orlicz λ -module L_λ^θ by

$$L_\lambda^\theta := L_\lambda^\theta(\mathcal{F}) = \left\{ X \in L^0(\mathcal{F}) \mid \left\| \mathbb{E}[\theta(|X/\zeta|) \mid \mathcal{G}] \right\|_\lambda < \infty \text{ for some } \zeta \in \lambda_\# \right\}. \quad (9.2)$$

With the help of (9.1), one checks that L_λ^θ is indeed a λ -module. On L_λ^θ one introduces the *Luxemburg λ -norm* $\|\cdot\|_{\theta,\mathcal{G}} : L_\lambda^\theta \rightarrow \lambda_+$:

$$\|X\|_{\theta,\mathcal{G}} := \text{ess. inf} \left\{ \zeta \in \lambda_\# \mid \mathbb{E}[\theta(|X/\zeta|) \mid \mathcal{G}] \leq 1 \right\}. \quad (9.3)$$

We justify calling $\|\cdot\|_{\theta, \mathcal{G}}$ a λ -norm because of the following result:

Proposition 9.1. $(L_\lambda^\theta, L_{\lambda,+}^\theta := \{X \in L_\lambda^\theta \mid X \geq 0\}, \|\cdot\|_{\theta, \mathcal{G}})$ is a normed lattice λ -module with lattice λ -norm $\|\cdot\|_{\theta, \mathcal{G}}$.

Proof. First we will show that $\|\cdot\|_{\theta, \mathcal{G}}$ is a λ -norm. For $X \in L_\lambda^\theta$ and $A := \{\|X\|_{\theta, \mathcal{G}} = 0\}$ we get $\mathbb{1}_A \cdot \mathbb{E}[\theta(|X/\varepsilon|) | \mathcal{G}] \leq 1$ for all $\varepsilon > 0$. Since $\theta(r) \rightarrow \infty$ for $r \rightarrow \infty$, we must have $\{X = 0\} \supseteq A$. The λ -homogeneity is trivial and the subadditivity follows again from (9.1). Since θ is increasing, $|X_1| \leq |X_2|$ implies that $\|X_1\|_{\theta, \mathcal{G}} \leq \|X_2\|_{\theta, \mathcal{G}}$ such as $\|\cdot\|_{\theta, \mathcal{G}}$ is indeed a lattice λ -norm. \square

The following inequalities are simple generalizations from the classical theory of Orlicz spaces (We recall the convention $0/0 = 0$):

Proposition 9.2. If $X \in L_\lambda^\theta$, then

$$\mathbb{E} \left[\theta \left(\left| X / \|X\|_{\theta, \mathcal{G}} \right| \right) | \mathcal{G} \right] \leq 1 \quad (9.4)$$

and

$$\left\{ \|X\|_{\theta, \mathcal{G}} \leq 1 \right\} = \left\{ \mathbb{E} \left[\theta(|X|) | \mathcal{G} \right] \leq 1 \right\}. \quad (9.5)$$

Proof. The preceding proof showed that $\{\|X\|_{\theta, \mathcal{G}} = 0\} \subseteq \{X = 0\}$, hence we get $\mathbb{E}[\theta(|X|) | \mathcal{G}] = 0$ on $\{\|X\|_{\theta, \mathcal{G}} = 0\}$. By definition the inclusion $\{\zeta > \|X\|_{\theta, \mathcal{G}} \in \lambda\} \subseteq \{\mathbb{E}[\theta(|X/\zeta|) | \mathcal{G}] \leq 1\}$ holds for all $\zeta \in \lambda_\#$. Therefore $\mathbb{E} \left[\theta \left(\left| X / \|X\|_{\theta, \mathcal{G}} \right| \right) | \mathcal{G} \right] \leq 1$ by a conditional version of Beppo Levi's theorem. This implies the relation \subseteq in (9.5). The inverse relation \supseteq is obvious. \square

Theorem 9.3. $(L_\lambda^\theta, L_{\lambda,+}^\theta, \|\cdot\|_{\theta, \mathcal{G}})$ is complete, i.e. it is a Banach lattice λ -module.

Proof. For a $\|\cdot\|_{\theta, \mathcal{G}}$ -Cauchy sequence $(X_n)_{n \geq 0}$ in L_λ^θ , let us assume that we have $\sup_{n_1, n_2 \geq n_\varepsilon} \|X_{n_1} - X_{n_2}\|_{\theta, \mathcal{G}} < \varepsilon$. Then by Jensen's inequality we get

$$\begin{aligned} 1 &\geq \mathbb{E}[\theta(|X_{n_1} - X_{n_2}|/\varepsilon) | \mathcal{G}] \geq \theta(\mathbb{E}[|X_{n_1} - X_{n_2}|/\varepsilon | \mathcal{G}]) \quad \text{or} \\ \varepsilon \cdot \theta^{-1}(1) &\geq \mathbb{E}[|X_{n_1} - X_{n_2}| | \mathcal{G}]. \end{aligned}$$

This shows that (X_n) is a Cauchy-sequence in L^1 which has a limit, say X . We conclude that $1 \geq \mathbb{E}[\theta(|X_{n_1} - X_{n_2}|/\varepsilon) | \mathcal{G}] \rightarrow_{n_2 \rightarrow \infty} \mathbb{E}[\theta(|X_{n_1} - X|/\varepsilon) | \mathcal{G}]$ a.s. such as $\|X\|_{\theta, \mathcal{G}} \leq \|X_{n_1}\|_{\theta, \mathcal{G}} + \varepsilon$. This proves that $X \in L_\lambda^\theta$ and $X_n \rightarrow X$ in L_λ^θ . \square

For $s \geq 0$, consider

$$\theta^*(s) := \sup_{r \geq 0} (r \cdot s - \theta(r)), \quad (9.6)$$

the Legendre-Fenchel transform of θ . It is again convex and increasing with $\theta^*(0) = 0$. Since the sup in (9.6) can be restricted to the finite interval $[0, \tilde{r}(s) := \sup\{r | \theta(r)/r \leq s\}]$ the transform θ^* is finite on $[0, \infty)$. Moreover, it satisfies again the condition

$$\lim_{s \rightarrow \infty} \theta^*(s)/s = \infty \quad (9.7)$$

since for $s > 0$ we get $\theta^*(s)/s \geq (\theta^{-1}(s) \cdot s - \theta(\theta^{-1}(s)))/s = \theta^{-1}(s) - 1 \rightarrow \infty$ for $s \rightarrow \infty$.

Now Young's inequality (see [8] theorem 2.1.4) tells us that

(i) the right-subdifferentials $\partial\theta$ and $\partial\theta^*$ on $(0, \infty)$ are reciprocally right-continuous inverses:

$$\partial\theta^*(s) = \sup\{r \mid \partial\theta(r) \leq s\} \quad \text{and} \quad \partial\theta(r) = \sup\{s \mid \partial\theta^*(s) \leq r\}, \quad (9.8)$$

(ii) for all $r, s \in [0, \infty)$

$$r \cdot s \leq \theta(r) + \theta^*(s), \quad (9.9)$$

(iii) and

$$r \cdot s = \theta(r) + \theta^*(s) \quad \iff \quad r = \partial\theta^*(s) \quad \text{or} \quad s = \partial\theta(r). \quad (9.10)$$

Now, we will prove the natural generalization of Hölder's inequality on Orlicz spaces for λ -modules:

Proposition 9.4. *For $X \in L_\lambda^\theta$ and $Z \in L_\lambda^{\theta^*}$ the ‘‘Hölder's inequality’’ holds:*

$$\mathbb{E}[ZX \mid \mathcal{G}] \leq 2 \cdot \|X\|_{\theta, \mathcal{G}} \|Z\|_{\theta^*, \mathcal{G}}. \quad (9.11)$$

Because of (9.11) any $Z \in L_\lambda^{\theta^*}$ defines a continuous λ -linear function on L_λ^θ .

Proof. For $X \in L_\lambda^\theta$ and $Z \in L_\lambda^{\theta^*}$ set $A_\varepsilon := \{\|X\|_{\theta, \mathcal{G}} \geq \varepsilon \text{ and } \|Z\|_{\theta^*, \mathcal{G}} \geq \varepsilon\}$. Then

$$\mathbb{1}_{A_\varepsilon} \cdot \frac{X}{\|X\|_{\theta, \mathcal{G}}} \cdot \frac{Z}{\|Z\|_{\theta^*, \mathcal{G}}} \leq \mathbb{1}_{A_\varepsilon} \cdot \theta\left(\frac{X}{\|X\|_{\theta, \mathcal{G}}}\right) \cdot \theta^*\left(\frac{Z}{\|Z\|_{\theta^*, \mathcal{G}}}\right) \leq 2 \cdot \mathbb{1}_{A_\varepsilon}.$$

It follows that $\mathbb{1}_{A_\varepsilon} \cdot \mathbb{E}[X \cdot Z \mid \mathcal{G}] \leq 2 \cdot \mathbb{1}_{A_\varepsilon} \cdot \|X\|_{\theta, \mathcal{G}} \cdot \|Z\|_{\theta^*, \mathcal{G}}$ which implies (9.11) for $\varepsilon \rightarrow 0$. \square

For further investigations on the dual λ -module of an Orlicz λ -module, we will first introduce the operator λ -norm $\|\cdot\|_{\theta, \mathcal{G}}^*$, also called Orlicz λ -norm,

$$\|Z\|_{\theta, \mathcal{G}}^* := \text{ess. sup} \left\{ |\langle X, Z \rangle| \mid \|X\|_{\theta, \mathcal{G}} \leq 1 \right\}. \quad (9.12)$$

where $\langle \cdot, Z \rangle$ is a λ -linear function on L_λ^θ or a subspace of it.

The following proposition shows that $\|\cdot\|_{\theta, \mathcal{G}}^*$ is indeed a λ -norm on $L_\lambda^{\theta^*}$. Moreover, the two λ -norms $\|\cdot\|_{\theta^*, \mathcal{G}}$ and $\|\cdot\|_{\theta, \mathcal{G}}^*$ on $L_\lambda^{\theta^*}$ are equivalent.

Proposition 9.5.

- (i) *The functional $\|\cdot\|_{\theta, \mathcal{G}}^*$ is a λ -norm on $L_\lambda^{\theta^*}$.*
- (ii) *For $Z \in L_\lambda^{\theta^*}$ one has*

$$\|Z\|_{\theta^*, \mathcal{G}} \leq \|Z\|_{\theta, \mathcal{G}}^* \leq 2 \|Z\|_{\theta^*, \mathcal{G}}. \quad (9.13)$$

- (iii) *For σ -subalgebras $\mathcal{G}_1 \subseteq \mathcal{G}_2 \subseteq \mathcal{F}$ and $Z \in L_{\lambda(\mathcal{G}_2)}^{\theta^*}$ one has*

$$\mathbb{E} \left[\|Z\|_{\theta, \mathcal{G}_2}^* \mid \mathcal{G}_1 \right] \leq \|Z\|_{\theta, \mathcal{G}_1}^*. \quad (9.14)$$

Proof. (i) Since the properties (iv, a - b) of definition 2.1 are evident for $\|\cdot\|_{\theta, \mathcal{G}}^*$, it remains to be shown that (iv, c) holds or more precisely that $\|Z\|_{\theta, \mathcal{G}}^* = 0$ implies $Z = 0$, the converse being obvious.

Assume w.l.g. that $A := \{Z > 0\} \in \mathcal{F}^+$ and take $r > 0$ with $\theta(r) \leq 1/\mathbb{P}[A]$. Then $\|r \cdot \mathbb{1}_A\|_{\theta, \mathcal{G}} \leq 1$ and $\{\mathbb{E}[r \cdot \mathbb{1}_A \cdot Z | \mathcal{G}] > 0\} \in \mathcal{G}^+$, hence $\|Z\|_{\theta, \mathcal{G}}^* \neq 0$.

(ii) We claim that for all $X \in L_{\lambda}^{\theta^*}$, we have

$$\mathbb{E} \left[\theta^* \left(\left| Z / \|Z\|_{\theta, \mathcal{G}}^* \right| \right) \middle| \mathcal{G} \right] \leq 1. \quad (9.15)$$

Now the first inequality of (9.13) follows from (9.15), while the second is an immediate consequence of proposition 9.4.

To show (9.15) for $Z = |Z| \geq 0$ in $L_{\lambda}^{\theta^*}$, we set $B_{\gamma} := \{\theta(\partial\theta^*(Z/\|Z\|_{\theta, \mathcal{G}}^*)) \leq \gamma\}$ and $X := \mathbb{1}_{B_{\gamma}} \cdot \partial\theta^*(Z/\|Z\|_{\theta, \mathcal{G}}^*)$ for some $B \in \mathcal{F}^+$. From (9.9), we conclude that

$$\mathbb{E} \left[\mathbb{1}_{B_{\gamma}} \cdot \theta^* \left(Z / \|Z\|_{\theta, \mathcal{G}}^* \right) \middle| \mathcal{G} \right] + \mathbb{E} \left[\theta(X) \middle| \mathcal{G} \right] = \mathbb{E} \left[X \cdot Z / \|Z\|_{\theta, \mathcal{G}}^* \middle| \mathcal{G} \right]. \quad (9.16)$$

From (9.5), we know that $A_1 := \{\mathbb{E}[\theta(X) | \mathcal{G}] \leq 1\} = \{\|X\|_{\theta, \mathcal{G}} \leq 1\}$ and by definition of $\|\cdot\|_{\theta, \mathcal{G}}^*$ that $\mathbb{1}_{A_1} \cdot \mathbb{E}[X \cdot Z | \mathcal{G}] \leq \mathbb{1}_{A_1} \cdot \|Z\|_{\theta, \mathcal{G}}^*$. Therefore one gets

$$\mathbb{1}_{A_1} \cdot \mathbb{E}[\mathbb{1}_{B_{\gamma}} \cdot \theta^*(Z/\|Z\|_{\theta, \mathcal{G}}^*) | \mathcal{G}] \leq \mathbb{1}_{A_1} \cdot \mathbb{E}[X \cdot Z / \|Z\|_{\theta, \mathcal{G}}^* | \mathcal{G}] \leq \mathbb{1}_{A_1}.$$

On $A_2 := \{1 < \mathbb{E}[\theta(X) | \mathcal{G}] \leq \gamma\} = A_1^c$ we use the convexity inequality $\theta(r/\alpha) \leq \theta(r)/\alpha$ for all $1 \leq \alpha$ and $r \in [0, \infty)$ to get $\mathbb{1}_{A_2} \cdot \mathbb{E}[\theta(X/\mathbb{E}[\theta(X) | \mathcal{G}]) | \mathcal{G}] \leq \mathbb{1}_{A_2} \cdot \mathbb{E}[\theta(X) | \mathcal{G}] / \mathbb{E}[\theta(X) | \mathcal{G}] \leq \mathbb{1}_{A_2}$. Again the definition of $\|\cdot\|_{\theta, \mathcal{G}}^*$ implies that

$$\mathbb{1}_{A_2} \cdot \mathbb{E}[X \cdot Z / \mathbb{E}[\theta(X) | \mathcal{G}] | \mathcal{G}] \leq \mathbb{1}_{A_2} \cdot \|Z\|_{\theta, \mathcal{G}}^*$$

or equivalently $\mathbb{1}_{A_2} \cdot \mathbb{E}[X \cdot Z / \|Z\|_{\theta, \mathcal{G}}^* | \mathcal{G}] \leq \mathbb{1}_{A_2} \cdot \mathbb{E}[\theta(X) | \mathcal{G}]$. With regard to (9.16), this means that $\mathbb{1}_{A_2} \cdot \mathbb{E}[\mathbb{1}_{B_{\gamma}} \cdot \theta^*(Z/\|Z\|_{\theta, \mathcal{G}}^*) | \mathcal{G}] = 0$.

We have shown that $\mathbb{E}[\mathbb{1}_{B_{\gamma}} \cdot \theta^*(Z/\|Z\|_{\theta, \mathcal{G}}^*) | \mathcal{G}] \leq 1$ for all $\gamma > 0$ such as we get the desired inequality (9.15) by the conditional version of Beppo Levi's theorem since $\bigcup_{\gamma > 0} B_{\gamma} = \Omega$.

(iii) Consider $Z \in L_{\lambda(\mathcal{G}_2)}^{\theta^*} \supseteq L_{\lambda(\mathcal{G}_1)}^{\theta^*}$. For $X \in L_{\lambda(\mathcal{G}_2)}^{\theta}$ with $\|X\|_{\theta, \mathcal{G}_2} \leq 1$, we apply (9.5) twice to conclude $\mathbb{E}[\theta(|X|) | \mathcal{G}_2] \leq 1$, hence $\mathbb{E}[\theta(|X|) | \mathcal{G}_1] \leq 1$ and therefore $\|X\|_{\theta, \mathcal{G}_1} \leq 1$. This yields

$$\mathbb{E}[\mathbb{E}[X \cdot Z | \mathcal{G}_2] | \mathcal{G}_1] = \mathbb{E}[X \cdot Z | \mathcal{G}_1] \leq \|Z\|_{\theta, \mathcal{G}_1}^*.$$

The set of \mathcal{G}_2 -measurable random variables $\{\mathbb{E}[X \cdot Z | \mathcal{G}_2] \mid \|X\|_{\theta, \mathcal{G}_2} \leq 1\}$ is upward directed so that Beppo Levi's theorem, again in its conditional form, implies (9.14) because of

$$\mathbb{E}[\|Z\|_{\theta, \mathcal{G}_2}^* | \mathcal{G}_1] = \mathbb{E}[\text{ess. sup} \left\{ \mathbb{E}[X \cdot Z | \mathcal{G}_2] \mid \|X\|_{\theta, \mathcal{G}_2} \leq 1 \right\} | \mathcal{G}_1] \leq \|Z\|_{\theta, \mathcal{G}_1}^*.$$

□

It is well-known that generally the dual space of an Orlicz space with growth function θ is bigger than L^{θ^*} (see e.g. [20]). On the other hand, the latter space is dual to the Morse space (also called Orlicz heart) with growth function θ . We therefore introduce the Morse λ -module:

$$M_{\lambda}^{\theta} := M_{\lambda}^{\theta}(\mathcal{F}) := \{X \in L^0(\mathcal{F}) \mid \|\mathbb{E}[\theta(|X/\xi|) | \mathcal{G}]\|_{\lambda} < \infty \text{ for all } \xi \in \lambda_{\#}\}. \quad (9.17)$$

Since we assumed that θ is bounded on bounded sets of $[0, \infty)$ we get immediately that $L^\infty \subseteq M_\lambda^\theta$. Moreover, a close inspection of the proof of (9.13) shows that only bounded test random variables X are needed there. This allows for the following generalization:

$$\|Z\|_{\theta^*, \mathcal{G}} \leq \text{ess. sup} \left\{ |\langle X, Z \rangle| \mid X \in M_\lambda^\theta \text{ and } \|X\|_{\theta, \mathcal{G}} \leq 1 \right\} \leq \|Z\|_{\theta, \mathcal{G}}^* \leq 2 \|Z\|_{\theta^*, \mathcal{G}}. \quad (9.18)$$

The Morse λ -module M_λ^θ has the following properties:

Theorem 9.6.

- (i) M_λ^θ is a closed subspace of L_λ^θ .
- (ii) The λ -module $L_\lambda^{\theta^*}$ is the dual λ -module of M_λ^θ : $(M_\lambda^\theta)' = L_\lambda^{\theta^*}$.

Proof. (i) Let $\zeta \in \lambda_{\sharp}$ and $(X_\iota)_{\iota \in I}$ be a net in M_λ^θ such as X_ι converge to X in L_λ^θ , i.e. $\|X_\iota - X\|_{\theta, \mathcal{G}} \rightarrow 0$. By proposition 9.1 we also have $\|\zeta \cdot X - \zeta \cdot X_\iota\|_{\theta, \mathcal{G}} \rightarrow 0$ so that $\|\zeta \cdot X\|_{\theta, \mathcal{G}} \leq \|\zeta \cdot X - \zeta \cdot X_\iota\|_{\theta, \mathcal{G}} + \|\zeta \cdot X_\iota\|_{\theta, \mathcal{G}} \in \lambda$ since by assumption $X_\iota \in M_\lambda^\theta$.

(ii) By proposition 9.4 we already know that $L_\lambda^{\theta^*} \subseteq (M_\lambda^\theta)'$.

Conversely, let $\langle \cdot, Z \rangle$ be a continuous λ -linear function on M_λ^θ . This means that there exists $\eta \in \lambda_{\sharp}$ with $|\langle X, Z \rangle| \leq \eta \cdot \|X\|_{\theta, \mathcal{G}}$ for all $X \in M_\lambda^\theta$. We define $\langle \cdot, \widehat{Z} \rangle : M_\lambda^\theta \rightarrow \mathbb{R}$ by $\langle X, \widehat{Z} \rangle := \mathbb{E}[\langle X, Z \rangle / \eta]$ such as $|\langle X, \widehat{Z} \rangle| \leq \mathbb{E}[\|X\|_{\theta, \mathcal{G}}]$ for all $X \in M_\lambda^\theta$. Applying (9.13) and (9.14) (with $\mathcal{G}_2 = \mathcal{G}$, $\mathcal{G}_1 = \{\emptyset, \Omega\}$, and θ^* replaced by θ) we get

$$\left| \langle X, \widehat{Z} \rangle \right| \leq \mathbb{E}[\|X\|_{\theta^*, \mathcal{G}}^*] \leq \|X\|_{\theta^*}^* \leq 2 \cdot \|X\|_{\theta}.$$

where $\|\cdot\|_{\theta} = \|X\|_{\theta, \{\emptyset, \Omega\}}$ and similar for $\|\cdot\|_{\theta^*}^*$. Because of the Hahn-Banach theorem, the linear function $\langle \cdot, \widehat{Z} \rangle$ can be extended to a linear function on $M^\theta := M_{\lambda(\{\emptyset, \Omega\})}^\theta$, again denoted by $\langle \cdot, \widehat{Z} \rangle$ and still satisfying $|\langle X, \widehat{Z} \rangle| \leq 2 \cdot \|X\|_{\theta}$. This means that $\langle \cdot, \widehat{Z} \rangle$ is continuous on M^θ . By theorem 2.2.11 in [8], we get an element $Z' \in L^{\theta^*}$ such as $\langle X, \widehat{Z} \rangle = \mathbb{E}[X \cdot Z']$ for all $X \in M^\theta$. In particular for $X \in M_\lambda^\theta$ and $A \in \mathcal{G}^+$ it follows from $\mathbb{E}[\mathbb{1}_A \cdot X \cdot Z'] = \mathbb{E}[\mathbb{1}_A \cdot \langle X, Z \rangle / \eta]$ that with $Z'' = \eta \cdot z'$

$$\langle X, Z \rangle = \mathbb{E}[X \cdot Z'' | \mathcal{G}] \leq \eta \cdot \|X\|_{\theta, \mathcal{G}}$$

for all $X \in M_\lambda^\theta$. Therefore by (9.18) we find $\|Z''\|_{\theta^*, \mathcal{G}} \leq \eta$. This completes the proof. \square

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KARL-THEODOR EISELE: LABORATOIRE DE RECHERCHE EN GESTION ET ÉCONOMIE, INSTITUT DE RECHERCHE MATHÉMATIQUE AVANCÉE, UNIVERSITÉ DE STRASBOURG, PEGE, 61 AVENUE DE LA FORÊT-NOIRE, F-67085 STRASBOURG CEDEX, FRANCE

E-mail address: eisele@unistra.fr

SONIA TAIEB: UNIVERSITÉ EL-MANAR, FACULTÉ DES SCIENCES, CAMPUS UNIVERSITAIRE, 2092 EL MANAR, TUNIS

E-mail address: ettaeib@yahoo.fr