Communications on Stochastic Analysis
Vol. 13, No. 1-2, (2019)

INCREASING $\mathcal{C}$-ADDITIVE PROCESSES

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Abstract. It is shown that any infinitely divisible distribution μ on $\mathbb{R}_+$ gives rise to a class of increasing additive processes we call $\mathcal{C}$-additive processes, where $\mathcal{C}$ is a continuous semigroup of cumulant generating functions. The marginal and increment distributions of these pocesses are characterized in terms of their Lévy measure and their drift coefficient. Integral representations of $\mathcal{C}$-additive processes in terms of a Poisson random measure are obtained. The limiting behavior (as $t \to \infty$) of two subclasses of $\mathcal{C}$-additive processes leads to new characterizations of $\mathcal{C}$-selfdecomposable and $\mathcal{C}$-stable distributions on $\mathbb{R}_+$.

1. Introduction

A real (vector)-valued stochastic process $\{X_t\}$ (with $X_0 = 0$) is said to be additive if it has independent increments and it is stochastically continuous with càdlàg paths. The most important subclasses of additive processes are Lévy processes (see, for e.g., Sato [10]), self-similar processes (Sato [9]), semi-selfsimilar processes (Maejima and Sato [6]) and semi-Lévy processes (Sato [11]). Another noteworthy subclass consists of the additive processes that arise as stochastic integrals with a Lévy process integrator and a determinsitic integrand (see Maejima and Ueda [7], Rocha-Arteaga and Sato [8], and references therein). Among the many important properties of an additive process, we cite the infinite divisibilty of its increment and marginal distributions and its Lévy-Itô decomposition.

The purpose of this article is to introduce a class of increasing additive processes taking values on $\mathbb{R}_+$. More specifically, we will show that any infinitely divisible distribution μ on $\mathbb{R}_+$ generates a class of increasing additive processes we call $\mathcal{C}$-additive processes, where $\mathcal{C}$ is a continuous semigroup of cumulant generating functions (see definitions below). These processes will be indexed by a Lebesgue measurable function on $\mathbb{R}_+$ taking values in $(0, 1]$. Their increment and marginal distributions are characterized in terms of their Lévy measures and their drift coefficients. Integral representations of $\mathcal{C}$-additive processes in terms of a Poisson random measure are obtained. Finally, the limiting behavior (as $t \to \infty$) of two subclasses of $\mathcal{C}$-additive processes leads to new characterizations of $\mathcal{C}$-selfdecomposability and $\mathcal{C}$-stability of distributions on $\mathbb{R}_+$ (introduced by van Harn and Steutel [4]).

Received 2019-6-16; Accepted 2019-8-26; Communicated by the editors.
2010 *Mathematics Subject Classification.* Primary 60G51; Secondary 60E07.
Key words and phrases. Composition semigroups, continuous-time branching processes, infinite divisibility, Laplace-Stieltjes transforms, Poisson random measures, weak convergence.
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The paper is organized as follows. In Section 2, we parallel the treatment for the real-valued case in [8] (see their Section 2.1) to introduce the class of increasing $\mathcal{C}$-additive processes. We first establish the existence of a collection of infinitely divisible distributions on $\mathbb{R}_+$ (Theorem 2.2). We proceed to show that these distributions arise as the marignals of increasing $\mathcal{C}$-additive processes, whose existence is proven in Theorem 2.4. We obtain formulas for the drift coefficient and the Lévy measure of the said marginal distributions (Theorem 2.8) and we describe a subclass of increasing $\mathcal{C}$ additive processes generated by a $\mathcal{C}$-stable distribution (Theorem 2.13). We end the section with several several examples. In Section 3, we give an integral representation of an increasing $\mathcal{C}$-additive process in terms of a Poisson random measure under the assumption that the generating measure μ has a bounded Lévy measure ν and no drift coefficient (Theorem 3.2). The case when ν is unbounded is studied in Section 4, where it is shown that an increasing $\mathcal{C}$ additive process arises as a weak limit of increasing $\mathcal{C}$-additive processes generated by driftless measures with finite Lévy measures (Theorem 4.1). Section 5 is devoted to limit theorems. We show that for a subclass of increasing $\mathcal{C}$-additive processes, a weak limit (as $t \to \infty$) exists if and only if the generating measure μ, or its Lévy measure, has a finite log-moment (Theorem 5.2). This result leads to new characterizations of $\mathcal{C}$-selfdecompoable and $\mathcal{C}$-stable distributions (Theorems 5.4 and 5.5 and Corollary 5.6).

We devote the remainder of the section to recalling a few basic facts needed in the sequel.

The collection of infinitely divisible distributions on $\mathbb{R}_+$ will be denoted by $\mathcal{I}(\mathbb{R}_+)$. For $a \geq 0$, δ_a will designate the point mass probability measure at a. We denote the probability law of a random variable X by $\mathcal{L}(X)$ and we use the notation $X \stackrel{d}{=} Y$ to mean $\mathcal{L}(X) = \mathcal{L}(Y)$.

We recall that a distribution μ on $\mathbb{R}_+$ is characterized by its Laplace-Stieltjes transform (LST, hereafter) $\phi(\tau)$ defined by

$$\phi(\tau) = \int_0^\infty e^{-\tau x} \mu(dx).$$

Moreover, $\mu \in \mathcal{I}(\mathbb{R}_+)$ if and only if $\phi(\tau)$ admits the representation

$$\phi(\tau) = e^{-C(\tau)},$$

where $C(\tau)$ has a completely monotone derivative on $(0, \infty)$ (with $C(0) = 0$). The function $C(\tau)$ is referred to as the cumulant generating function (cgf, hereafter) of μ.

Let $\mathcal{C} = (C_t;\ t \geq 0)$ be a continuous composition semigroup of cgf's with the following properties:

$$C_0(\tau) = \tau; \quad C_s \circ C_t(\tau) = C_{s+t}(\tau),\ (s, t \geq 0); \quad \lim_{t \downarrow 0} C_t(\tau) = \tau; \quad \lim_{t \to \infty} C_t(\tau) = 0. \tag{1.1}$$

for every $\tau \geq 0$. We have for every $t \geq 0$, $C_t(\tau) = -\ln \eta_t(\tau)$, where η_t is the LST of a distribution in $\mathcal{I}(\mathbb{R}_+)$. Following Steutel and van Harn [12], Chapter 5, Section 8, we will assume without loss of generality that $C_1'(0) = e^{-1}$ (up to a linear change of the time scale). Several examples of such subgroups are given in Section 2.

The infinitesimal generator U of the semigroup $\mathcal{C}$ is defined by

$$U(\tau) = \lim_{t\downarrow 0}(C_t(\tau) - \tau)/t \quad (\tau \geq 0), \tag{1.2a}$$

and satisfies $U(0) = 0$ and $U(\tau) < 0$ for $\tau > 0$. U admits the representation

$$U(\tau) = a_1\tau + b\tau^2 - \int_0^\infty (e^{-\tau x} - 1 + \tau x/(1+x^2))\, dm(x), \tag{1.2b}$$

where a_1 is a real number, $b \geq 0$, and $m(dx)$ is a Lévy spectral function such that $\int_0^y x^2\, dm(x) < \infty$ for every $y > 0$. The assumption $C_t'(0) = e^{-1}$ forces $a_1 = -1$. Moreover, the following non-explosion condition holds:

$$\left| \int_{0^+}^y U(x)^{-1}\, dx \right| = \infty \text{ for sufficiently small } y > 0.$$

We note that U admits representations that are different from (1.2b), but equivalent to it (see for example Li [5], Chapter 3).

A function related to U, called the A-function, is defined by

$$A(\tau) = \exp\Big\{ \int_\tau^1 (U(x))^{-1}\, dx \Big\}, \qquad (\tau \geq 0;\ A(0) = 0). \tag{1.3}$$

The functions $U(\tau)$ and $A(\tau)$ satisfy the following identities for $t, \tau \geq 0$:

$$\frac{\partial}{\partial t} C_t(\tau) = U(C_t(\tau)) = U(\tau) C_t'(\tau) \quad \text{and} \quad A(C_t(\tau)) = e^{-t} A(\tau)). \tag{1.4}$$

We deduce from the first equation in (1.4) and the continuity of the semigroup $\mathcal{C}$ (see (1.1)) that

$$C_t(\tau) = \tau + \int_0^t U(C_s(\tau))\, ds. \tag{1.5}$$

We note the addtional properties

$$C_t(\tau) < \tau \ (t, \tau > 0) \quad \text{and} \quad \lim_{\tau\downarrow 0} \frac{U(\tau)}{\tau} = -1 \tag{1.6}$$

Let $\mathcal{Z} = (\{Z_x(t)\}, x \geq 0)$ be a collection of independent copies of an $\mathbb{R}_+$-valued subcritical continuous-time branching process driven by the semigroup $\mathcal{C}$ with initial condition $Z_x(0) = x$. The subcriticality of $\{Z_x(t)\}$ follows from the assumption $C_1'(0) = e^{-1}$. We will refer to $\{Z_x(t)\}$ as a $\mathcal{C}$-CB process. Let $\{Q_t(x, dy)\}$ be the infinitely divisible transition semigroup of probability measures associated with $\{Z_x(t)\}$. The LST of $Z_x(t)$ is

$$\int_0^\infty e^{-\tau y} Q_t(x, dy) = e^{-xC_t(\tau)} = \eta_t(\tau)^x. \tag{1.7}$$

Moreover,

$$E(Z_x(t)) = \int_0^\infty y Q_t(x, dy) = xe^{-t} \tag{1.8}$$

We refer the reader to [5], Chapter 3, for more on $\mathcal{C}$-CB processes.

The operator $\alpha \odot_{\mathcal{C}}$ that acts on $\mathbb{R}_+$-valued random variables introduced in [4] (see also [12], Chapter V, Section 8) is defined as follows:

$$\alpha \odot_{\mathcal{C}} X = Z_X(t) \quad (t = -\ln \alpha), \tag{1.9}$$

where $0 < \alpha \leq 1$ and X is an $\mathbb{R}_+$-valued random variable independent of the collection $\mathcal{Z}$ of $\mathcal{C}$-CB processes described above. We note that the process $\{Z_X(t)\}$ of (1.9) is itself a $\mathcal{C}$-CB process starting with X individuals, i.e., $Z_X(0) = X$.

In [4], the operator $\odot_{\mathcal{C}}$ was used in lieu of the standard multiplication to study stability equations for $\mathbb{R}_+$-valued processes with stationary independent increments. Bouzar [1] used $\odot_{\mathcal{C}}$ in similar fashion to introduce a family of discrete time, $\mathbb{R}_+$-valued first order autoregressive processes.

Let $\phi_X(\tau)$ be the LST of an $\mathbb{R}_+$-valued random variable X. Then the LST $\phi_{\alpha \odot_{\mathcal{C}} X}(\tau)$ of $\alpha \odot_{\mathcal{C}} X$ is easily shown to be

$$\phi_{\alpha \odot_{\mathcal{C}} X}(\tau) = \phi_X(C_t(\tau)) \quad (t = -\ln \alpha; \tau \geq 0). \tag{1.10}$$

The following lemma gathers some basic properties of the operator $\odot_{\mathcal{C}}$.

Lemma 1.1. *Let α, $\beta \in (0, 1]$ and X and Y be $\mathbb{R}_+$-valued random variables. Then*

(i) $1 \odot_{\mathcal{C}} X \stackrel{d}{=} X$.
(ii) $\alpha \odot_{\mathcal{C}} (\beta \odot_{\mathcal{C}} X) \stackrel{d}{=} (\alpha\beta) \odot_{\mathcal{C}} X$.
(iii) *If X and Y are independent, then $\alpha \odot_{\mathcal{C}} (X + Y) \stackrel{d}{=} \alpha \odot_{\mathcal{C}} X + \alpha \odot_{\mathcal{C}} Y$.*
(iv) *If X and Y are independent, then so are $\alpha \odot_{\mathcal{C}} X$ and $\beta \odot_{\mathcal{C}} Y$.*
(v) *If $\{X_n\}$ is a sequence of $\mathbb{R}_+$-valued random variables such that $X_n \stackrel{d}{\to} X$, then $\alpha \odot_{\mathcal{C}} X_n \stackrel{d}{\to} \alpha \odot_{\mathcal{C}} X$.*
(vi) $\alpha \odot_{\mathcal{C}} X \stackrel{d}{\to} 0$ *if* $\alpha \downarrow 0$.

Proof. The proof of ((i)-(iii), (v) and (vi) follows starightforwardly from the assumptions on the semigroup $\mathcal{C}$ and equation (1.10). For (iv), we note $\alpha \odot_{\mathcal{C}} X = Z'_X(-\ln \alpha)$ and $\beta \odot_{\mathcal{C}} Y = Z''_Y(-\ln \beta)$, where $\mathcal{Z}' = (\{Z'_x(t)\}, x \geq 0)$ and $\mathcal{Z}'' = (\{Z''_y(t)\}, y \geq 0)$ are independent collections of $\mathcal{C}$-CB processes. It follows that if X and Y are independent, then so are $Z'_X(-\ln \alpha)$ and $Z''_Y(-\ln \beta)$. □

2. A Class of Increasing Additive Processes

We start out by introducing a stochastic integral for step functions (see (2.2) below) taking values in the interval $(0, 1]$.

We recall that a subordinator is an increasing Lévy process that starts at 0.

Let $\mu \in \mathcal{I}(\mathbb{R}_+)$ with LST $K_\mu(\tau)$. We assume the existence of the following processes on some probability space $(\Omega, \mathcal{F}, P)$:

- An $\mathbb{R}_+$-valued subordinator $\{Y_t^{(\mu)}\}$ such that $\mathcal{L}(Y_1^{(\mu)}) = \mu$;
- A collection of independent copies of a $\mathcal{C}$-CB process (see Section 1) $\big(\{Z_x^{(j)}(t)\}, x \geq 0, 1 \leq j \leq n\big)$
- The collections $\big(\{Z_x^{(j)}(t)\}, x \geq 0, 1 \leq j \leq n\big)$ and $\{Y_t^{(\mu)}\})$ are mutually independent.

Let $f(s)$ be a step function defined on an interval $[t_0, t_1] \subset [0, \infty)$, i.e.,

$$f(s) = \sum_{j=1}^{n} a_j I_{[s_{j-1}, s_j)}(s) \tag{2.1}$$

for some subdivision $t_0 = s_o < s_1 < \cdots < s_n = t_1$ of the interval $[t_0, t_1]$, and some $a_j \in (0, 1]$, $j = 1, \cdots, n$.

Let $D_j = Y_{s_j}^{(\mu)} - Y_{s_{j-1}}^{(\mu)}$, $1 \le j \le n$ and define the $\mathbb{R}_+$-valued random variable Y by

$$Y = \sum_{j=1}^{n} Z_{D_j}^{(j)}(-\ln a_j) = \sum_{j=1}^{n} a_j \odot_{\mathcal{C}} (Y_{s_j}^{(\mu)} - Y_{s_{j-1}}^{(\mu)}). \tag{2.2}$$

Importantly, we note that if $f(s)$ of (2.1) admits a representation along a different subdivision of $[t_0, t_1]$, say, $f(s) = \sum_{i=1}^{m} b_i I_{[u_{i-1}, u_i)}(s)$, then

$$\sum_{j=1}^{n} a_j \odot_{\mathcal{C}} (Y_{s_j}^{(\mu)} - Y_{s_{j-1}}^{(\mu)}) \overset{d}{=} \sum_{i=1}^{m} b_i \odot_{\mathcal{C}} (Y_{u_i}^{(\mu)} - Y_{u_{i-1}}^{(\mu)}).$$

This can be seen by noting that Y can be decomposed along the refined subintervals $\big\{[u_{i-1} \vee s_{j-1}, u_i \wedge s_j)\big\}$ (over which $a_j = b_i$). The conclusion follows by using the fact that $\{Y_t^{(\mu)}\}$ has independent stationary increments and Lemma 1.1-(iv).

Proposition 2.1. *The random variable Y of (2.2) has an infintely divisible distribution on $\mathbb{R}_+$ with LST*

$$\phi_{t_0,t_1}(\tau) = \exp\Big\{\int_{t_0}^{t_1} \ln K_\mu\big(C_{-\ln f(s)}(\tau)\big)\, ds\Big\}. \tag{2.3}$$

Proof. Since $\{Y_t^{(\mu)}\}$ is a Lévy process, $[K_\mu(\tau)]^t$ is the LST of $Y_{s+t}^{(\mu)} - Y_s^{(\mu)}$, for $s, t \ge 0$. Let $\Delta_j = s_j - s_{j-1}$, $1 \le j \le n$. By assumption (and also Lemma 1.1-(iv)), the summands in (2.2) are independent. In compatibiltiy with (1.1), we adopt the convention $C_\infty(\tau) = 0$. It follows by (1.10) that the LST of Y is

$$\phi_Y(\tau) = \prod_{j=1}^{n} [K_\mu\big(C_{-\ln a_j}(\tau)\big)]^{\Delta_j} = \exp\Big\{\sum_{j=1}^{n} \Delta_j \ln K_\mu\big(C_{-\ln a_j}(\tau)\big)\Big\}. \tag{2.4}$$

Since

$$\sum_{j=1}^{n} \Delta_j \ln K_\mu\big(C_{-\ln a_j}(\tau) = \int_{t_0}^{t_1} \ln K_\mu\big(C_{-\ln f(s)}(\tau)\big)\, ds,$$

we have shown that $\phi_{t_0,t_1}(\tau)$ of (2.3) satisfies $\phi_{t_0,t_1}(\tau) = \phi_Y(\tau)$, which implies that $\phi_{t_0,t_1}(\tau)$ is an LST. It is clear that the latter is independent of the choice of the subdivision $\{s_0, s_1, \cdots, s_n\}$ of $[t_0, t_1]$. Let now k be a positive integer and μ_k be a probability measure on $\mathbb{R}_+$ such that $\mu = \mu_k^{*k}$. The LST of μ_k is $[K_\mu(\tau)]^{1/k}$. It is easily verified that $\{Y_{t/k}^{(\mu)}\}$ is a subordinator with $\mathcal{L}(Y_{1/k}^{(\mu)}) = \mu_k$. Letting

$$Y_k = \sum_{j=1}^{n} a_j \odot_{\mathcal{C}} (Y_{s_j/k}^{(\mu)} - Y_{s_{j-1}/k}^{(\mu)}).$$

we see (using the argument above) that the LST of Y_k is

$$\phi_{Y_k}(\tau) = \exp\Big\{ \int_{t_0}^{t_1} \ln[K_\mu\big(C_{-\ln f(s)}(\tau)\big)]^{1/k}\, ds \Big\}.$$

Since $\phi_{Y_k}(\tau) = [\phi_{t_0,t_1}(\tau)]^{1/k}$, we conclude that $\phi_{t_0,t_1}i(\tau)$ is the LST of a distribution in $\mathcal{I}(\mathbb{R}_+)$. □

We next extend Proposition 2.1 to $(0,1]$-valued Lebesgue measurable function on $[t_0,t_1]$, The result is to be seen as the analogue of Proposition 29 established by Rocha-Arteaga and Sato (2003) for real valued bounded measurable functions.

Theorem 2.2. *Let $\mu \in \mathcal{I}(\mathbb{R}_+)$ with LST $K_\mu(\tau)$ and let $f(s)$ be a Lebesgue measurable function defined on the interval $[t_0,t_1] \subset [0,\infty)$ and taking values in $(0,1]$. Then the function $\phi_{t_0,t_1}(\tau)$ of (2.3) is the LST of an infinitely divisible distribution μ_{t_0,t_1} on $\mathbb{R}_+$.*

Proof. Let f be as assumed above and let $f_n(s)$ be a sequence of step functions on $[t_0,t_1]$ such that $0 < f_n(s) \le 1$ and $f(s) = \lim_{n\to\infty} f_n(s)$ almost everywhere (a.e.) with respect to the Lebesgue measure. Let

$$\phi_n(\tau) = \exp\Big\{ \int_{t_0}^{t_1} \ln K_\mu\big(C_{-\ln f_n(s)}(\tau)\big)\, ds \Big\}. \qquad (n \ge 1).$$

The continuity of $C_t(\tau)$ as a function of t implies

$$\lim_{n\to\infty} \ln K_\mu\big(C_{-\ln f_n(s)}(\tau)\big) = \ln K_\mu\big(C_{-\ln f(s)}(\tau)\big) \qquad \text{(a.e. [s] ; } \tau \in [0,\infty)),$$

Since $K_\mu(\tau)$ is infinitely divisible, there exists a completely monotone function $\rho(u)$ on $(0,\infty)$ such that $\ln K_\mu(\tau) = -\int_0^\tau \rho(u)\, du$ (Theorem 4.2. p.90, in [12]). We have by (1.6) that $C_{-\ln f_n(s)}(\tau) < \tau$ for every $n \ge 1$, which implies that $|\ln K_\mu(C_{-\ln f_n(s)}(\tau))| \le \int_0^\tau \rho(u)\, du$. It follows by the Lebesgue dominated convergence theorem applied to the sequence $\{\ln K_\mu\big(C_{-\ln f_n(s)}(\tau)\big)\}$ that

$$\lim_{n\to\infty} \ln \phi_n(\tau) = \int_{t_0}^{t_1} \ln K_\mu\big(C_{-\ln f(s)}(\tau)\big)\, ds \qquad (\tau \ge 0).$$

Therefore, $\lim_{n\to\infty} \phi_n(\tau) = \phi_{t_0,t_1}(\tau)$, where $\phi_{t_0,t_1}(\tau)$ is the function in (2.3). Next, we note that $\lim_{\tau\downarrow 0} \ln K_\mu\big(C_{-\ln f(s)}(\tau)\big) = 0$ (as $\lim_{\tau\downarrow 0} C_{-\ln f(s)}(\tau) = 0$). Applying again the Lebesgue dominated convergence theorem as we did above (with the difference that now the index is τ, which we restrict to $[0,1]$ without loss of generality), we have $\lim_{\tau\downarrow 0} \phi_{t_0,t_1}(\tau) = 1$. We conclude by the continuity theorem (Theorem 3.1, Appendix A, in [12]) that $\phi_{t_0,t_1}(\tau)$ is an LST (that is independent of the choice of the limiting sequence of the step functions $\{f_n\}$). As the limit of infinitely divisible LST's, $\phi_{t_0,t_1}(\tau)$ is itself infinitely divisible (Proposition 2.2, p. 79, in [12]). □

In view of the proof of Proposition 2.1, one could adopt the notation

$$Y = \int_{t_0}^{t_1} f(s) \odot_{\mathcal{C}} dY_s^{(\mu)},$$

where Y and f are as in (2.1)-(2.2). This notation could be extended to $(0,1]$-valued lebesgue measurable functions through a weak limit argument (as seen in the proof of Theorem 2.2). Rather, we propose in Sections 3 and 4 some representations in terms of random variables via Poisson random measures.

We denote by $\mathcal{E}(\mathbb{R}_+;(0,1])$ the collection of Lebesgue measurable functions $f(s)$ on $\mathbb{R}_+$ taking values in $(0,1]$.

Let $\mu \in \mathcal{I}(\mathbb{R}_+)$ and let $f \in \mathcal{E}(\mathbb{R}_+;(0,1])$. Applying Theorem 2.2 to the restriction of f to the interval $[s,t]$, $0 \le s < t < \infty$, we can assume the existence of a probability measure $\mu_{s,t} \in$ on $\mathbb{R}_+$ that is infinitely divisible and with LST

$$\phi_{s,t}(\tau) = \int_0^\infty e^{-\tau y}\,\mu_{s,t}(dy) = \exp\Big\{\int_s^t \ln K_\mu\big(C_{-\ln f(y)}(\tau)\big)\,dy\Big\}. \tag{2.5}$$

Proposition 2.3. *The family of probability measures $\mu_{s,t}, 0 \le s \le t < \infty$ of (2.5) satisfies the following properties:*

(i) $\mu_{s,t} * \mu_{t,u} = \mu_{s,u}$ *for* $0 \le s \le t \le u < \infty$.
(ii) $\mu_{s,s} = \delta_0$.
(iii) $\mu_{s,t} \stackrel{d}{\to} \delta_0$ *as* $s \uparrow t$.
(iv) $\mu_{s,t} \stackrel{d}{\to} \delta_0$ *as* $s \downarrow t$.

Proof. (i) follows from the fact that $\phi_{s,u}(\tau) = \phi_{s,t}(\tau)\phi_{t,u}(\tau)$, $0 \le s \le t \le u < \infty$; (ii) from the fact that $\phi_{s,s}(\tau) = 1$; (iii) (resp., (iv)) from the fact that $\lim_{h\uparrow 0}\phi_{s-h,s}(\tau) = 1$ (resp., $\lim_{h\downarrow 0}\phi_{s,s+h}(\tau) = 1$). □

A stochastic process $\{X_t\}$ is said to be additive in law (we refer to [10]) if

(1) it has independent increments;
(2) $X_0 = 0$;
(3) it is stochastically continuous, i.e., for any $\epsilon > 0$, $\lim_{s\to t} P(|X_s - X_t| > \epsilon) = 0$.

$\{X_t\}$ is an additive process if it is additive in law and is

(4) *càdlàg*, i.e., is almost surely right-continuous with left limits.

Two stochastic processes $\{X_t\}$ and $\{Y_t\}$ are said to be modifications of each other if $P(X_t = Y_t) = 1$ for every $t \ge 0$. They are said to be identical in law if they have the same finite dimensional distributions.

Theorem 2.4. *There is an increasing additive process $\{X_t\}$ on some probability space $(\Omega, \mathcal{F}, P)$ such that for any $0 \le s < t < \infty$, $X_t - X_s$ admits $\mu_{s,t}$ of (2.5) as its distribution. This process is unique in the sense that if $\{X'_t\}$ is an additive process such that $X'_t \stackrel{d}{=} X_t$ for every $t \ge 0$, then $\{X_t\}$ and $\{X'_t\}$ are identical in law.*

Proof. By Proposition 2.3 above and Theorem 9.7- (ii), page 51, in [10], there exits an additive process in law $\{Y_t\}$ such that for any $0 \le s < t < \infty$, $Y_t - Y_s$ has $\mu_{s,t}$ of (2.5) as its distribution. Since $\mu_{s,t}$ has support on $\mathbb{R}_+$, $\{Y_t\}$ is increasing. By Theorem 11.5, p. 63, in [10], $\{Y_t\}$ admits a modification $\{X_t\}$ that is an additive process. It is easily seen that $\mu_{s,t}$ remains the distribution of $X_t - X_s$ and thus $\{X_t\}$ is also increasing. The second statement of the theorem follows from part (iii) of Theorem 9.7-(iii), p. 51, in [10]. □

Definition 2.5. We will refer to the process $\{X_t\}$ of Theorem 2.4 as the *increasing $\mathcal{C}$-additive process* generated by the pair (μ, f) with $\mu \in \mathcal{I}(\mathbb{R}_+)$ and $f \in \mathcal{E}(\mathbb{R}_+;(0,1])$.

We recall that if $\mu \in \mathcal{I}(\mathbb{R}_+)$, then its LST $\phi(\tau)$ admits the canonical representation (Chapter 10 in [10])

$$\phi(\tau) = \exp\Big\{-a\tau - \int_{(0,\infty)} (1-e^{-\tau x})\nu(dx)\Big\} \qquad (\tau \geq 0), \tag{2.6}$$

for some $a \geq 0$ and a measure ν on the Borel sets in $(0,\infty)$ satisfying

$$\int_{(0,\infty)} (1 \wedge x)\nu(dx) < \infty. \tag{2.7}$$

The measure ν is called the Lévy measure of μ and a its drift coefficient. We will refer to the pair (a,ν) in (2.6) as the characteristic couple of $\mu \in \mathcal{I}(\mathbb{R}_+)$.

Next, we characterize the marginal distributions of a $\mathcal{C}$-additive process in terms of their Lévy measures and their drift coefficients.

First, we need to establsih two preliminary results.

Recalling that $\eta_t(\tau) = e^{-C_t(\tau)}$ is the LST of the probability measure of $Q_t(1,dy)$ in $\mathcal{I}(\mathbb{R}_+)$ (see (1.7)) , we will denote the characteristic couple of the latter by (b_t, ν_t), i.e.,

$$C_t(\tau) = b_t\tau + \int_{(0,\infty)} (1-e^{-\tau x})\nu_t(dx). \tag{2.8}$$

We will also denote by $\{Q_t^0(x,\cdot)\}_{t\geq 0}$ the semigroup obtained by restricting the semigroup $\{Q_t(x,\cdot)\}_{t\geq 0}$ to $(0,\infty)$ (see the paragraph preceding equation (1.7)) .

Lemma 2.6. *Let X be an $\mathbb{R}_+$-valued random variable with distribution $\mu \in \mathcal{I}(\mathbb{R}_+)$, generated by the pair (a,ν). Let μ_α be the distribution of $\alpha \odot_{\mathcal{C}} X$ for $\alpha \in (0,1)$. Then $\mu_\alpha \in \mathcal{I}(\mathbb{R}_+)$ and has characteristic couple $(ab_t, \nu_t^{(\mu_\alpha)})$ with $t = -\ln\alpha$ and*

$$\nu_t^{(\mu_\alpha)}(B) = a\nu_t(B) + \int_{(0,\infty)} Q_t^0(x,B)\nu(dx), \quad (B \in \mathcal{B}(0,\infty)). \tag{2.9}$$

Proof. The infinite divisibility of the distribution of $\alpha \odot_{\mathcal{C}} X$ follows from (1.10) and Proposition 3.5, Chapter III, in [12]. We have by (1.10) and (2.6)

$$-\ln\phi_{\alpha\odot_{\mathcal{C}}X}(\tau) = -\ln\phi(C_t(\tau)) = aC_t(\tau) + \int_{(0,\infty)} (1-e^{-xC_t(\tau)})\nu(dx) \quad (t = -\ln\alpha),$$

or, by (1.7) and (2.8),

$$\begin{aligned} -\ln\phi_{\alpha\odot_{\mathcal{C}}X}(\tau) = ab_t\tau &+ a\int_{(0,\infty)} (1-e^{-\tau x})\nu_t(dx) \\ &+ \int_{(0,\infty)} \Big(\int_{(0,\infty)} (1-e^{-\tau y})Q_t^0(x,dy)\Big)\nu(dx). \end{aligned}$$

Therefore,

$$-\ln\phi_{\alpha\odot_{\mathcal{C}}X}(\tau) = ab_t\tau + \int_{(0,\infty)} (1-e^{-x\tau})\nu_t^{(\mu_\alpha)}(dx),$$

with $\nu_t^{(\mu_\alpha)}(dx)$ of (2.9). It remains to show that $\int_{(0,\infty)}(1\wedge x)\nu_t^{(\mu_\alpha)}(dx)<\infty$. Since ν_t is a Lévy measure, it is enough to prove that $\int_{(0,\infty)}(1\wedge y)m_1(dy)<\infty$, where $m_1(dy)=\int_{(0,\infty)}Q_t^0(x,dy)\nu(dx)$. Indeed,

$$\int_{(0,\infty)}(1\wedge y)m_1(dy)=\int_{0,\infty}\Big(\int_{(0,\infty)}(1\wedge y)Q_t^0(x,dy)\Big)\nu(dx).$$

By (1.8), $\int_{(0,\infty)}(1\wedge y)Q_t^0(x,dy)=E(Z_x(t)\wedge 1)\le x\wedge 1$, which implies

$$\int_{(0,\infty)}(1\wedge y)m_1(dy)\le\int_{(0,\infty)}(1\wedge x)\nu(dx)<\infty$$

since ν is a Lévy measure. □

We recall the notion of factoring for an addtive process (we refer to [11]).

Let $\{X_t\}$ be an $\mathbb{R}_+$-valued additive process in law. A pair $(\{\rho_s\}_{s\ge0},\sigma)$ is called a factoring of $\{X_t\}$ if

(1) σ is a measure on $\mathbb{R}_+$ such that $\sigma([0,t])<\infty$ (local boundedness).
(2) σ is diffuse (atomless)
(3) $\rho_s\in\mathcal{I}(\mathbb{R}_+)$ for all $s\ge0$
(4) $\ln\phi_{\rho_s}(\tau)$ is measurable in s for each $t\ge0$ (ϕ_{ρ_s} being the LST of ρ_s).
(5) $\left|\int_0^t\ln\phi_{\rho_s}(\tau)\sigma(ds)\right|<\infty$ for all $t,\ \tau\ge0$.
(6) The LST $\phi_{0,t}$ of X_t admits the following representation for all $t\ge0$

$$\phi_{0,t}(\tau)=\exp\Big\{\int_0^t\ln\phi_{\rho_s}(\tau)\sigma(ds)\Big\}\qquad(t\ge0).\tag{2.10}$$

Proposition 2.7. *Let $\{X_t\}$ be an increasing $\mathcal{C}$-additive process generated by the pair (μ,f) for some $\mu\in\mathcal{I}(\mathbb{R}_+)$ and some $f\in\mathcal{E}(\mathbb{R}_+;(0,1])$. Let ρ_s be the probability distribution with LST $K_\mu(C_{-\ln f(s)}(\tau))$, $s\ge0$, and let $\sigma(dx)=dx$ be the Lebesgue measure on $\mathbb{R}_+$. Then $(\{\rho_s\},\sigma)$ is a factoring of $\{X_t\}$.*

Proof. The proof follows easily from (2.5). We omit the details. □

We denote by $\mathcal{B}_0(\mathbb{R}_+)$ the class of Borel sets B in $\mathbb{R}_+$ such that $\inf_{x\in B}>0$.

Theorem 2.8. *Let $\{X_t\}$ be an increasing $\mathcal{C}$-additive process generated by the pair (μ,f) for some $\mu\in\mathcal{I}(\mathbb{R}_+)$ and some $f\in\mathcal{E}(\mathbb{R}_+;(0,1])$. Let $(\{\rho_s\},\sigma)$ be the factoring of $\{X_t\}$, as described in Proposition 2.7, and let (a,ν) be the characteristic couple of μ. Then, $\mu_{0,t}=\mathcal{L}(X_t)$ has characteristic couple $(c_t^{(X)},\nu_t^{(X)})$:*

$$c_t^{(X)}=a\int_0^t b_{-\ln f(y)}\,dy\quad\text{and}\quad\nu_t^{(X)}(B)=\int_0^t\nu^{(\rho)}_{-\ln f(y)}(B)\,dy,\tag{2.11}$$

for any $B\in\mathcal{B}_0(\mathbb{R}_+)$, where

$$\nu_u^{(\rho)}(B)=a\nu_u(B)+\int_{(0,\infty)}Q_u^0(x,B)\nu(dx)\tag{2.12}$$

and (b_t,ν_t) is as in (2.8).

Proof. First, we note that $K_\mu(C_{-\ln f(y)}(\tau))$ can be seen as the LST of $f(y) \odot_{\mathcal{C}} Y$, where Y is a random variable with μ as its distribution (and $K_\mu(\tau)$ as its LST). By Lemma 2.6 and (2.9), ρ_y has characteristic couple $(ab_{-\ln f(y)}, \nu^{(\rho)}_{-\ln f(y)})$ with $\nu^{(\rho)}_u$ as in (2.12). By Lemma 2.7 (item (7)) in [11], the functions $ab_{-\ln f(y)}$ and $\nu^{(\rho)}_{-\ln f(y)}(B)$ $(B \in \mathcal{B}_0(\mathbb{R}_+))$ are measurable in y. Equation (2.11) then follows from Lemma 2.7 (item (9)) in [11]. □

We note that if μ in Theorem 2.8 has no drift coefficient ($a = 0$), then the increasing $\mathcal{C}$-addtive process generated by (μ, f) is driftless. On the other hand, if the Lévy measure ν of μ is 0, i.e., $\mu = \delta_a$, $a > 0$, then we have the following result.

Corollary 2.9. *Let $a > 0$ and let $\{X^{(a)}_t\}$ be the increasing $\mathcal{C}$-additive process generated by (δ_a, f) for some $f \in \mathcal{E}(\mathbb{R}_+; (0,1])$. Then the probability distribution $\mu^{(a)}_{0,t}$ of $X^{(a)}_t$ has characteristic couple $(c^{(a)}_t, \nu^{(a)}_t)$:*

$$c^{(a)}_t = a \int_0^t b_{-\ln f(y)}\, dy \quad \text{and} \quad \nu^{(a)}_t(B) = a \int_0^t \nu_{-\ln f(y)}(B)\, dy, \tag{2.13}$$

for any $B \in \mathcal{B}_0(\mathbb{R}_+)$.

Formulas for the drift coefficients and the Lévy measures of the distributions of the increments of an increasing $\mathcal{C}$-additive process are easily deduced from (2.11). We omit the details.

$\mathcal{C}$-additive processes satisfy a sort of stability property by scalar multiplication.

Proposition 2.10. *Let $\{X_t\}$ be an increasing $\mathcal{C}$-additive process generated by the pair (μ, f) for some $\mu \in \mathcal{I}(\mathbb{R}_+)$ and some $f \in \mathcal{E}(\mathbb{R}_+; (0,1])$. Let $c \in (0,1)$. Then the increasing $\mathcal{C}$-additive process $\{X_t(c)\}$ generated by the pair (μ, cf) satisfies $X_t(c) \stackrel{d}{=} c \odot_{\mathcal{C}} X_t$ for every $t > 0$. The drift coefficient of $X_t(c)$ is $c^{X(c)}_t = b_{-\ln c} c^{(X)}_t$ and its Lévy measure is*

$$\nu^{X(c)}_t(B) = c^{(X)}_t \nu_{-\ln c}(B) + \int_{(0,\infty)} Q^0_{-\ln c}(x, B) \nu^{(X)}_t(dx) \quad (B \in \mathcal{B}_0(\mathbb{R}_+)).$$

Proof. Let $0 < s < t$. We have by Lemma 1.1, $c \odot_{\mathcal{C}} X_t = c \odot_{\mathcal{C}} (X_s + X_t - X_s) \stackrel{d}{=} c \odot_{\mathcal{C}} X_s + c \odot_{\mathcal{C}} (X_t - X_s)$. Independence and a simple LST argument based on Theorem 2.4 and (1.10) establishes that $X_t(c) \stackrel{d}{=} c \odot_{\mathcal{C}} X_t$. The formulas for the drift and the Lévy measure of $X_t(c)$ follow from Lemma 2.6 and Theorem 2.8. □

The Lévy-Itô decomposition (Chapter 4, Section 19, in [10]) applies to $\mathcal{C}$-additive processes as follows.

Theorem 2.11. *Let $\mu \in \mathcal{I}(\mathbb{R}_+)$ with characteristic couple (a, ν) and $f \in \mathcal{E}(\mathbb{R}_+; (0,1])$. Let $\{X_t\}$ be an increasing $\mathcal{C}$-additive process generated by the pair (μ, f) and with characteristic couple $(c^{(X)}_t, \nu^{(X)}_t)$ given by (2.11). Then there exists a Poisson random measure $M^{(\mu,f)}$ on $(0,\infty) \times (0,\infty)$ with mean measure $\widetilde{\nu}((0,t] \times B) = \nu^{(X)}_t(B)$, $B \in \mathcal{B}(\mathbb{R}_+)$, such that*

$$X_t = c^{(X)}_t + \int\int_{(0,t] \times \mathbb{R}_+} x M^{(\mu,f)}(ds,\, dx). \tag{2.14}$$

Proof. Since X_t is infinitely divisible, its Lévy measure $\nu_t^{(X)}$ satisfies the property $\int_{(0.\infty)}(x \wedge 1)\nu_t^{(X)}(dx) < \infty$ (see 2.7), which implies $\int_{(0,1]} x\nu_t^{(X)}(dx) < \infty$. Then both the existence $M^{(\mu,f)}$ and equation (4.3) follow from Theorem 19.3, p. 121, in [10]. □

The Poisson random measure $M^{(\mu,f)}$ in (2.14) regulates the number of jumps and their sizes of the process $\{X_t\}$ (see [10], Theorem 19.2, p. 120). We note that for $c \in (0,1)$ and $f \in \mathcal{E}(\mathbb{R}_+;(0,1])$, the Poisson random measure $M^{(\mu,cf)}$ has mean measure

$$\widetilde{\nu}^{(c)}((0,t] \times B) = c_t^{(X)}\nu_{-\ln c}(B) + \int_{(0,\infty)} Q^0_{-\ln c}(x,B)\,\nu_t^{(X)}(dx) \quad (B \in \mathcal{B}(\mathbb{R}_+)).$$

Remarks 2.12. *(i) The probability measure μ_{t_0,t_1} of Theorem 2.2 and its LST of (2.3), with $f(s) = e^{-s}$, arose in the context of $\mathcal{C}$-CB processes with immigration (or $\mathcal{C}$-CBI processes). In this case, the function $-\ln K_\mu(\tau)$ is the immigration mechanism of the $\mathcal{C}$-CBI process and the infinitesimal generator $U(\tau)$ (see (1.2a-b) is the branching mechanism of the process. The immigration process is the incresing $\mathcal{C}$-additive process generated by the pair (μ, f). We refer the reader to [5], Chapter 3, Section 3.3 for more details.*

(ii) The following condition on the infinitesimal generator U of the semigroup $\mathcal{C} = (C_t : t \geq 0)$ was introduced in [5] (Condition 3.6, p. 60):

$$U(\tau) < 0 \text{ for } \tau \geq \theta \text{ and } \int_\theta^\infty |U(y)|^{-1}\,dy < \infty, \tag{2.15}$$

for some constant $\theta > 0$. We note that in our case the first part of (2.15) is true (see Section 1). By Theorem 3.10, p. 61, in [5], the condition (2.15) holds if and only if the drift coefficient b_t in (2.8) satisfies $b_t = 0$ for every $t > 0$. Therefore, if one assumes (2.15), then by Theorem 2.8, any $\mathcal{C}$-additive process is a pure jump process as in this case $c_t^{(X)} = 0$ (by 2.11).

The next result identifies a class of increasing $\mathcal{C}$-additive processes with $\mathcal{C}$-stable marginal distributions. We recall a few basic facts about these distributions (see [4]).

An $\mathbb{R}_+$-valued random variable X (or its distribution), with LST $\phi(\tau)$, is said to have a $\mathcal{C}$-stable distribution if for every $t > 0$, there exists $b > 0$ such that

$$\phi(\tau) = \phi(C_t(\tau))^b \qquad (\tau \geq 0). \tag{2.16}$$

If μ is a $\mathcal{C}$-stable distribution on $\mathbb{R}_+$, then $\mu \in \mathcal{I}(\mathbb{R}_+)$ and its LST $\phi(\tau)$ admits the canonical representation

$$\phi(\tau) = \exp\{-\lambda A(\tau)^\gamma\} \qquad (\tau \geq 0), \tag{2.17}$$

where $\gamma \in (0,1]$ and $\lambda > 0$ (with $A(\tau)$ of (1.3)). The constant γ is called the exponent of the $\mathcal{C}$-stable distribution.

Theorem 2.13. *Assume that μ is a $\mathcal{C}$-stable distribution with exponent $\gamma \in (0,1]$ and LST (2.17) for some $\lambda > 0$. Let $\{X_t\}$ be the increasing $\mathcal{C}$-additive process* *generated by the pair (μ, f) for some $f \in \mathcal{E}(\mathbb{R}_+; (0,1])$. Then the distributions of X_t $(t > 0)$ and $X_t - X_s$ $(0 \le s < t < \infty)$ are $\mathcal{C}$-stable with LST*

$$\phi_{s,t}(\tau) = \exp\Big\{-\lambda\Big(\int_s^t f(y)^\gamma\, dy\Big) A(\tau)^\gamma\Big\} \qquad (\tau \ge 0). \tag{2.18}$$

Proof. By assumption, Theorem 2.4, (2.17) and (1.4), we have

$$\ln K_\mu(C_{-\ln f(s)}(\tau)) = -\lambda\big[A(C_{-\ln f(s)}(\tau))\big]^\gamma = -\lambda f(s)^\gamma A(\tau)^\gamma \qquad (\tau \ge 0),$$

which, along with (2.5), implies (2.18). □

We conclude the section by applying the main results above to a family of semigroups of cgf's which we denote by $\mathcal{C}^{(\beta,d)} = \big\{(C_t^{(\beta,d)}(\tau), t \ge 0) : \beta \in (0,1], d \ge 0\big\}$. It is a modified version of an example in [5] (Chapter 3 , page 62).

For $\beta \in (0,1]$ and $d \ge 0$, let

$$C_t^{(\beta,d)}(\tau) = e^{-t}\tau\big[1 + d(1 - e^{-\beta t})\tau^\beta\big]^{-1/\beta} \quad (t, \tau \ge 0). \tag{2.19}$$

If $d = 0$, then $C_t^{(\beta,0)}(\tau) = e^{-t}\tau$ which implies $\mathcal{C}^{(\beta,0)} = \mathcal{C}^{(1,0)}$ for any $\beta \in (0,1]$. In this case, we have $\alpha \odot_{\mathcal{C}^{(1,0)}} X \stackrel{d}{=} \alpha X$ (see (1.9)) and thus $\alpha \odot_{\mathcal{C}^{(1,0)}} X$ corresponds to the ordinary multiplication (see also [4]).

Assuming $d > 0$ and letting $\lambda_t = [d(1 - e^{-\beta t})]^{-1}$ $(t > 0)$, we have

$$C_t^{(\beta,d)}(\tau) = e^{-t}\tau\Big(\frac{\lambda_t}{\lambda_t + \tau^\beta}\Big)^{1/\beta} \quad \text{and} \quad \frac{\partial}{\partial\tau}C_t^{(\beta,d)}(\tau) = e^{-t}\Big(\frac{\lambda_t}{\lambda_t + \tau^\beta}\Big)^{1+1/\beta}. \tag{2.20}$$

Noting that the function $\phi_1(\tau) = \Big(\frac{\lambda_t}{\lambda_t + \tau^\beta}\Big)^{1+1/\beta}$ is the LST of a compound-gamma distribution, where the primary distribution is gamma with parameters λ_t and $1 + 1/\beta$ and the secondary distribution is the standard stable distribution with exponent β and LST $e^{-\tau^\beta}$, it follows that $C_t^{(\beta,d)}(\tau)$ has a completely monotone derivative and hence is a cgf. It is easily verified that $\mathcal{C}^{(\beta,d)}$ forms a continuous semigroup of cgf's.

We denote by $\{Q_t^{(\beta,d)}(x, dy)\}$ the transition semigroup associated with $\mathcal{C}^{(\beta,d)}$ (see (1.7)). Straightforward calculations show that for any $\beta \in (0,1]$ and $d \ge 0$, $\frac{\partial}{\partial\tau}C_t^{(\beta,d)}(\tau)\Big|_{\tau=0} = e^{-t}$ and

$$U^{(\beta,d)}(\tau) = -\tau(1 + d\tau^\beta), \quad A^{(\beta,d)}(\tau) = \Big[\frac{(1+d)\tau^\beta}{1 + d\tau^\beta}\Big]^{1/\beta}. \tag{2.21}$$

The characteristic couple of the probability measure $Q_t^{(1,0)}(1, dy)$ is $(e^{-t}, 0)$. The transition semigroup associated with $\mathcal{C}^{(1,0)}$ is $Q_t^{(1,0)}(x, dy) = \delta_{xe^{-t}}(dy)$. If $\mu \in \mathcal{I}(\mathbb{R}_+)$ has characteristic couple (a, ν) and $f \in \mathcal{E}(\mathbb{R}_+; (0,1])$, then the increasing $\mathcal{C}^{(1,0)}$-additive process $\{X_t\}$ generated by (μ, f) has characeristics (see (2.11))

$$c_t^{(X)} = a\int_0^t f(y)\, dy \quad \text{and} \quad \nu_t^{(X)}(B) = \int_0^t \nu\big([f(y)]^{-1}B\big)\, dy, \tag{2.22}$$

for any $B \in \mathcal{B}_0(\mathbb{R}_+)$. Moreover, if μ is $\mathcal{C}^{(1,0)}$-stable (or stable in the standard sense, since $A^{(1,0)}(\tau)=\tau$) with LST $\phi(\tau)=e^{-\lambda\tau^\gamma}$, $\lambda>0$ (see (2.17)), then $\{X_t\}$ has a stable marginal distribution with LST $\phi_{0,t}(\tau)=\exp\Big\{-\lambda\Big(\int_0^t f(y)^\gamma\,dy\Big)\tau^\gamma\Big\}$.

For $d>0$, the compound-gamma distribution with LST $\phi_1(\tau)$ above has probability density function (pdf)

$$h(x)=\frac{\beta\lambda_t^{1+1/\beta}}{\Gamma(1/\beta)}\int_0^\infty g_y(x)y^{1/\beta}e^{-\lambda_t y}\,dy, \tag{2.23}$$

where $g_y(x)$ is the pdf of the stable distribution on $\mathbb{R}_+$ with LST $e^{-y\tau^\beta}$. It follows that the characteristic couple $\big(b_t^{(\beta,d)},\nu_t^{(\beta,d)}\big)$ of $Q_t^{(\beta,d)}(1,dy)$ is given by

$$b_t^{(\beta,d)}=\exp\Big\{-e^{-t}\int_{(0,\infty)}\frac{h(x)}{x}\,dx\Big\}\quad\text{and}\quad \nu_t^{(\beta,d)}(B)=e^{-t}\int_B\frac{h(x)}{x}\,dx. \tag{2.24}$$

When $\beta=1$ and $d>0$, additional formulas can be stated more explicitly. In this case, $\lambda_t=d(1-e^{-t})^{-1}$, $h(x)$ is the pdf of a gamma distribution with parameters $(\lambda_t,2)$ and

$$b_t^{(1,d)}=e^{-\lambda_t e^{-t}}\quad\text{and}\quad \nu_t^{(1,d)}(B)=\lambda_t^2e^{-t}\int_B e^{-\lambda_t y}\,dy. \tag{2.25}$$

The transition semigroup $Q_t^{(1,d)}(x,dy)$ is the Poisson $(\lambda_t x e^{-t})$ compounding of the exponential distribution with parameter λ_t. Let $\mu\in\mathcal{I}(\mathbb{R}_+)$ with characteristic couple (a,ν) $f\in\mathcal{E}(\mathbb{R}_+;(0,1])$ $(f(y)<1)$, and $\{X_t\}$ the $\mathcal{C}^{(1,d)}$-additive process $\{X_t\}$ generated by (μ,f). Define $k(y)=\frac{f(y)}{1-f(y)}$. The characteristics of $\{X_t\}$ (see (2.11)) are shown to be

$$c_t^{(X)}=a\int_0^t e^{-k(y)/d}\,dy\quad\text{and}\quad \nu_t^{(X)}(dy)=\Big(\int_0^t H(y,z)\,dz\Big)\,dy, \tag{2.26}$$

where

$$H(y,z)=k(z)(1+k(z))e^{-\frac{1+k(z)}{d}y}\Big(\frac{a}{d^2}+\sum_{n=1}^\infty\frac{ny^{n-1}}{(n!)^2d^{2n}}\int_{(0,\infty)}x^ne^{-\frac{k(z)}{d}x}\,\nu(dx)\Big). \tag{2.27}$$

By (2.21), we can write $A^{(1,d)}(\tau)=-\ln\phi_1(\tau)$, where $\phi_1(\tau)=\exp\{-\frac{1+d}{d}(1-G(\tau)\}$ and $G(\tau)$ is the LST of an exponential distribution with parameter $1/d$. Therefore, any $\mathcal{C}^{(1,d)}$-stable distribution with exponent $\gamma\in(0,1]$ is a compound distribution, where the primary distribution is a $\mathcal{C}^{(1,0)}$-stable (or standard stable) distribution with exponent γ and the secondary distribution is the compound Poisson distribution with LST $\phi_1(\tau)$.

Assume μ is $\mathcal{C}^{(1,d)}$-stable and $f\in\mathcal{E}(\mathbb{R}_+;(0,1])$. Then the increasing $\mathcal{C}^{(1,d)}$-additive process $\{X_t\}$ generated by the pair (μ,f) has a $\mathcal{C}^{(1,d)}$-stable marginal distribution with LST $\phi_{0,t}(\tau)=\exp\Big\{-\lambda\Big(\int_0^t f(y)^\gamma\,dy\Big)(-\ln\phi_1(\tau))^\gamma\Big\}$, $\lambda>0$.

3. $\mathcal{C}$-additive Processes and Poisson Random Measures (ν Finite)

We assume first that $\mu \in \mathcal{I}(\mathbb{R}_+)$ has characteristic couple $(0,\nu)$ with a finite Lévy measure ν and no drift coefficient ($a = 0$). Letting $c = \nu((0,\infty))$, we rewrite $\nu = c\sigma$, where σ a probability distribution on $\mathbb{R}_+$ such that $\sigma(\{0\}) = 0$. We note that the LST of μ is $K_\mu(\tau) = \exp\{-c(1-\phi_\sigma(\tau))\}$, where ϕ_σ is the LST of σ.

We assume the existence of the following processes on some probability space $(\Omega, \mathcal{F}, P)$:

- An $\mathbb{R}_+$-valued subordinator $\{Y_t^{(\mu)}\}$ such that $\mathcal{L}(Y_1^{(\mu)}) = \mu$. $\{Y_t^{(\mu)}\}$ is necessarily a compound Poisson process with intensity c, jump times $\{T_i\}_{i\geq 1}$, and (iid) jump sizes $\{D_i\}$ with common distribution σ (see Çinlar [2], Chapter VII, Section 7).
- A collection of independent copies of a $\mathcal{C}$-CB process $\big(\{Z_x^{(i)}(t)\}, x \geq 0, i \geq 1\big)$.
- The collections $\big(\{Z_x^{(i)}(t)\}, x \geq 0, i \geq 1\big)$ and $\{Y_t^{(\mu)}\})$ are mutually independent.

For $i \geq 1$ and $t \geq 0$, we define $Y_i(t) = Z_{D_i}^{(i)}(t)$, with $Y_i(0) = D_i$. By definition and the above assumptions, $(\{Y_i(t\}, i \geq 1)$ constitutes a sequence of iid $\mathcal{C}$-CB processes (see Section 1).

Since the processes $\{Z_x^{(i)}(t)\}$, $i \geq 1$, are Markovian, their common transition semigroup of probabilities $\{Q_t(x,dy)\}$ (see (1.7)) extends uniquely to a probability measure $\mathcal{Q}(x,\mathbf{B})$ on $(\mathbb{R}_+^{[0,\infty)}, \mathcal{B}(\mathbb{R}_+)^{[0,\infty)})$ for every $x \geq 0$. Moreover, for every $\mathbf{B} \in \mathcal{B}(\mathbb{R}_+)^{[0,\infty)}$, $\mathcal{Q}(x,\mathbf{B})$ is measurable as a function of x on $\mathbb{R}_+$ (see for example [10], Chapter 2, Section 10).

If $\mathbf{B}$ is a rectangle in $\mathcal{B}(\mathbb{R}_+)^{[0,\infty)}$ of the form $\mathbf{B} = \{\mathbf{y} \in \mathbb{R}_+^{[0,\infty)} : \mathbf{y}(t_i) \in B_i, i = 0, 1, \cdots, n\}$, where the B_i's are Borel sets in $\mathbb{R}_+$ and $0 = t_0 < t_1 < \cdots < t_n$, then

$$\begin{aligned}\mathcal{Q}(x,\mathbf{B}) = \int_{B_n} Q_{t_n - t_{n-1}}(x_{n-1}, dx_n) \int_{B_{n-1}} Q_{t_{n-1}-t_{n-2}}(x_{n-2}, dx_{n-1}) \cdots \\ \cdots \int_{B_1} Q_{t_1 - t_0}(x_0, dx_1) \int_{B_0} \delta_x(dx_0).\end{aligned} \tag{3.1}$$

The processes $\{Y_i(t)\}$, $i \geq 1$, have a random initial position (namely D_i). Their common probability law on $\mathcal{B}(\mathbb{R}_+)^{[0,\infty)}$, denoted by π, is given by (cf. Remark 10.8, p. 58, in [10])

$$\pi(\mathbf{B}) = \int_0^\infty \mathcal{Q}(x,\mathbf{B})\sigma(dx) \qquad (\mathbf{B} \in \mathcal{B}(\mathbb{R}_+)^{[0,\infty)}). \tag{3.2}$$

The sequence $\{T_i\}$ forms a Poisson random measure on $\mathbb{R}_+$ with mean measure $c\,\text{Leb}$. Since the sequences $\{Y_i(\cdot)\}$ and $\{T_i\}$ are independent, it follows by Corollary 3.5, p. 265, in Çinlar (2011) that the sequence $\{(T_i, Y_i(\cdot))\}$ is a Poisson random measure M on $\mathbb{R}_+ \times \mathbb{R}_+^{[0,\infty)}$ with mean measure $c\,\text{Leb} \otimes \pi$ (Leb is for Lebesgue measure), with π of (3.2). For a measurable function h on $\mathbb{R}_+ \times \mathbb{R}_+^{[0,\infty)}$, the random variable $Mh = \int_{\mathbb{R}_+ \times \mathbb{R}_+^{[0,\infty)}} h(s, \mathbf{y})\, M(ds, d\mathbf{y})$ on $\Omega, \mathcal{F}, P)$ takes the form

$$Mh = \sum_{k=1}^{\infty} h(T_k, Y_k(\cdot)). \tag{3.3}$$

Theorem 3.1. *Let $f(s)$ be a Lebesgue measurable function defined on the interval $[t_0, t_1] \subset [0, \infty)$ and taking values in $(0, 1]$. Define the function g on $\mathbb{R}_+ \times \mathbb{R}_+^{[0,\infty)}$ by $g(s, \mathbf{y}) = \mathbf{y}(-\ln f(s)) I_{[t_0,t_1]}(s)$. Then the random variable Mg is given by*

$$Mg = \sum_{k=1}^{\infty} Y_k(-\ln f(T_k)) I_{[t_0,t_1]}(T_k), \tag{3.4}$$

and has LST

$$\phi(\tau) = \exp\Big\{-c \int_{t_0}^{t_1} (1 - \phi_\sigma\big(C_{-\ln f(s)}(\tau)\big)\, ds\Big\}. \tag{3.5}$$

Proof. First we note that g is measurable on $\mathbb{R}_+ \times \mathbb{R}_+^{[0,\infty)}$ as the projection operator $\mathrm{pr}_t(\mathbf{y}) = \mathbf{y}(t)$, $t \geq 0$, from $\mathbb{R}_+^{[0,\infty)}$ to $\mathbb{R}_+$ is $\mathcal{B}(\mathbb{R}_+)^{[0,\infty)}$-measurable. Equation (3.4) follows straightforwardly from the definitions of the random measure M and (3.3), with $h(s, \mathbf{y}) = g(s, \mathbf{y})$. The LST $\phi(\tau)$ of Mg satisfies

$$\begin{aligned}\phi(\tau) = E(e^{-M(\tau g)}) &= \exp\Big\{-c \int_{\mathbb{R}_+ \times \mathbb{R}_+^{[0,\infty)}} \big(1 - e^{-\tau g(s,\mathbf{y})}\big)\, \mathrm{Leb} \otimes \pi(ds, d\mathbf{y})\Big\} \\ &= \exp\Big\{-c \int_{t_0}^{t_1} \Big(\int_{\mathbb{R}_+^{[0,\infty)}} \big(1 - e^{-\tau \mathbf{y}(-\ln f(s))}\big)\, \pi(d\mathbf{y})\Big)\Big\}.\end{aligned}$$

We have $\int_{\mathbb{R}_+^{[0,\infty)}} \big(1 - e^{-\tau \mathbf{y}(-\ln f(s))}\big)\, \pi(d\mathbf{y}) = E\big(1 - e^{-\tau Y_1(-\ln f(s))}\big)$, as π is the probability law of $\{Y_1(t)\}$ (cf. Proposition 10.6, p. 57, in [10]). Since the latter is a $\mathcal{C}$-CB process with $Y_1(0) = D_1$, it follows (see (1.9)) that $Y_1(-\ln f(s))) \stackrel{d}{=} f(s) \odot_{\mathcal{C}} D_1$ and thus, by (1.10), $E\big(1 - e^{-\tau Y_1(-\ln f(s))}\big) = 1 - \phi_\sigma(C_{-\ln f(s)}(\tau))$ (recall σ is the common distribution of the D_i's). Equation (3.5) ensues. $\square$

Noting that by definition $Y_i(t) \stackrel{d}{=} e^{-t} \odot_{\mathcal{C}} D_i$ for every $i \geq 1$ and $t \geq 0$, we have the following representation of Mg of (3.4):

$$Mg \stackrel{d}{=} \sum_{k=1}^{\infty} \big(f(T_k)) \odot_{\mathcal{C}} D_k\big) I_{[t_0,t_1]}(T_k), \tag{3.6}$$

where the operation $A \odot_{\mathcal{C}} X$ is extended to a random element A taking values in $[0, 1]$ (and cumulative distribution function $F_A(a)$) via its LST:

$$\phi_{A \odot_{\mathcal{C}} X}(\tau) = \int_0^1 \phi_{a \odot_{\mathcal{C}} X}(\tau) F_A(da). \tag{3.7}$$

Theorem 3.2. *Let $\mu \in \mathcal{I}(\mathbb{R}_+)$ with characteristic couple $(0, c\sigma)$, where c and σ are as defined above. Let the sequences $\{D_i\}$, $\{T_i\}$, and $\{Y_i(\cdot)\}$, are as defined above. Let $f \in \mathcal{E}(\mathbb{R}_+; (0, 1])$ and $g_t(s, \mathbf{y}) = \mathbf{y}(-\ln f(s)) I_{[0,t]}(s)$. Then*

$$X_t = Mg_t = \sum_{k=1}^{\infty} Y_k(-\ln f(T_k)) I_{[0,t]}(T_k), \tag{3.8}$$

is an increasing $\mathcal{C}$*-additive process generated by* (μ, f)*. Moreover, the characteristic couple of* $\mathcal{L}(X_t)$ *is* $(0, \nu_t^{(X)})$*, with*

$$\nu_t^{(X)}(B) = c\int_0^t \int_{(0,\infty)} Q^0_{-\ln f(s)}(x, B)\sigma(dx)\, ds, \qquad B \in \mathcal{B}_0(\mathbb{R}_+).$$

Proof. Clearly, $X_0 = 0$. Since $Mg_t = \int_{\mathbb{R}_+\times\mathbb{R}_+^{[0,\infty)}} \mathbf{y}(-\ln f(s))I_{[0,t]}(s)\, M(ds, d\mathbf{y})$, the càdlàg property of $\{X_t\}$ follows. By (2.5), the LST of $\mu_{s,t} = \mathcal{L}(X_t - X_s)$, $0 \le s < t < \infty$, is

$$\phi_{s,t}(\tau) = \exp\Big\{-c\int_s^t (1 - \phi_\sigma\big(C_{-\ln f(s)}(\tau)\big)\, ds\Big\}.$$

Therefore, $\mu_{s,t}$ satisfies the properties (i)-(iv) in Proposition 2.3. It follows by (i) and an induction argument that for $0 \le t_0 < t_1 < \cdots < t_n$, the probability law $m_{t_0,\cdots,t_n} = \mathcal{L}(X_{t_0}, \cdots, X_{t_n})$ satisfies

$$\begin{aligned} m_{t_0,\cdots,t_n}(B_0 \times B_1 \times B_n) = \int_0^\infty \cdots \int_0^\infty &\mu_{0,t_0}(dy_0)I_{B_0}(y_0)\mu_{t_0,t_1}(dy_1)I_{B_1}(y_0 + y_1) \\ &\times \cdots \mu_{t_{n-1},t_n}(dy_n)I_{B_n}(y_0 + \cdots + y_n), \end{aligned} \tag{3.9}$$

where $B_0, \cdots, B_n$ are Borel sets in $\mathbb{R}_+$. A standard argument (see, for e.g., the proof of (ii) $\Rightarrow$ (i) of Theorem 9.7, p. 51, in [10]) implies

$$\begin{aligned} &E(\Big[\exp\Big\{-\sum_{i=1}^n \tau_i(X_{t_i} - X_{t_{i-1}})\Big\}\Big] \\ &\qquad = \int_0^\infty \cdots \int_0^\infty \exp\Big\{-\sum_{i=1}^n \tau_i y_i\Big\}\mu_{t_0,t_1}(dy_1)\cdots\mu_{t_{n-1},t_n}(dy_n) \quad (3.10) \\ &\qquad = \prod_{i=1}^n \int_0^\infty e^{-\tau_i y_i}\mu_{t_{i-1},t_i}(dy_i), \end{aligned}$$

which implies that the increments $X_{t_1} - X_{t_0}, \cdots, X_{t_n} - X_{t_{n-1}}$ are independent. Stochastic continuity of $\{X_t\}$ is insured by (iii) and (iv) of Proposition 2.3. The formula for $\nu_t^{(X)}$ follows from (2.11)-(2.12). □

The following corollary is a direct consequence of Theorem 3.2 and (3.6).

Corollary 3.3. *Under the assumptions of Theorem 3.2, the increasing* $\mathcal{C}$*-additive process* $\{X_t\}$ *of (3.8) admits the representation*

$$X_t = Mg_t \stackrel{d}{=} \sum_{k=1}^\infty \big(f(T_k)) \odot_{\mathcal{C}} D_k\big)I_{[0,t]}(T_k), \tag{3.11}$$

Next, we give a decomposition theorem for increasing $\mathcal{C}$-addiive processes. We state first a useful result whose proof is straightforward.

Lemma 3.4. *Let* $W_0, W_1, \cdots, W_n$ $(n \ge 1)$ *be a sequence of independent* $\mathbb{R}_+$*-valued random variables and* S_j*,* $0 \le j \le n$*, be the sequence of its partial sums* *(with $S_0 = W_0$). Then the joint LST, $h(\tau_0, \tau_1, \cdots, \tau_n)$, of $(S_0, S_1, \cdots, S_n)$ is given by*

$$h(\tau_0, \tau_1, \cdots, \tau_n) = \prod_{l=0}^{n} h_l(r_l), \tag{3.12}$$

where h_l is the LST of W_l, $\tau_l \geq 0$, and $r_l = \sum_{k=l}^{n} \tau_k$, $0 \leq l \leq n$.

Theorem 3.5. *Let $\mu \in \mathcal{I}(\mathbb{R}_+)$ with characteristic couple (a, ν), where $a \geq 0$, and ν is bounded (thus $\nu = c\sigma$, $c \geq 0$, where σ is a probability measure on $\mathbb{R}_+$). Let $f \in \mathcal{E}(\mathbb{R}_+; (0,1])$ and let $\{X_t\}$ be an increasing $\mathcal{C}$-additive process generated by (μ, f) on some probability space $(\Omega, \mathcal{F}, P)$. There exists an increasing $\mathcal{C}$-additive process $\{\widetilde{X}_t\}$ on some probability space $(\Omega_0, \mathcal{F}_0, P_0)$ that is identical in law to $\{X_t\}$ and such that*

$$\widetilde{X}_t = \widetilde{X}_t^{(a)} + Mg_t, \tag{3.13}$$

where

(i) *$\{Mg_t\}$ is an increasing $\mathcal{C}$-additive process generated by (μ_1, f), where $\mu_1 \in \mathcal{I}(\mathbb{R}_+)$ has characteristic couple $(0, c\sigma)$, M is the Poisson random measure of (3.3) on $(\Omega_0, \mathcal{F}_0, P_0)$ with mean measure $c\,\mathrm{Leb} \otimes \pi$, with π of (3.2), and $g_t(s, \mathbf{y}) = \mathbf{y}(-\ln f(s)) I_{[0,t]}(s)$ (cf. Theorem 3.2);*

(ii) *$\{\widetilde{X}_t^{(a)}\}$ is an increasing $\mathcal{C}$-additve process $(\Omega_0, \mathcal{F}_0, P_0)$, generated by the pair (δ_a, f) (cf. Corollary 2.9);*

(iii) *$\{\widetilde{X}_t^{(a)}\}$ and $\{Mg_t\}$ are independent.*

Moreover, the characteristic couple $(c_t^{(X)}, \nu_t^{(X)})$ of $\mathcal{L}(X_t)$ is $c_t^{(X)} = a \int_0^t b_{-\ln f(s)}\, ds$ and

$$\nu_t^{(X)}(B) = a \int_0^t \nu_{-\ln f(s)}(B)\, ds + c \int_0^t \int_{(0,\infty)} Q^0_{-\ln f(s)}(x, B)\sigma(dx)\, ds.$$

Proof. Let $\mu_1 \in \mathcal{I}(\mathbb{R}_+)$ with characteristic couple $(0, c\sigma)$ and consider a probability space $(\Omega_1, \mathcal{F}_1, P_1)$, where the following random elements are defined:

- $\{Y_t^{(\mu_1)}\}$ is a $\mathbb{R}_+$-valued compound Poisson process with intensity c, jump times $\{T_i\}_{i \geq 1}$, and (iid) jump sizes $\{D_i\}$ with common distribution σ;
- $\big(\{Z_x^{(i)}(t)\}, x \geq 0, i \geq 1\big)$ is a collection of independent copies of a $\mathcal{C}$-CB process;
- $\{Y_t^{(\mu_1)}\}$, $\big(\{Z_x^{(i)}(t)\}, x \geq 0, i \geq 1\big)$ and $\{\widetilde{X}_t^{(a)}\}$ are mutually independent.

Let M be the Poisson random measure defined by (3.3) with mean measure $c\,\mathrm{Leb} \otimes \pi$ and with π of (3.2). By Theorem 3.2, $\{Mg_t\}$ of (3.8) is an increasing $\mathcal{C}$-additive process generated by the pair (μ_1, f). Moreover, extending $(\Omega_1, \mathcal{F}_1, P_1)$ to $(\Omega_0, \mathcal{F}_0, P_0)$, we construct an increasing $\mathcal{C}$-additve process $\{\widetilde{X}_t^{(a)}\}$, generated by the pair (δ_a, f) that is independent of $\{Mg_t\}$. The process $\{\widetilde{X}_t\}$ defined by (3.13) is clearly an increasing $\mathcal{C}$-additive process. For $0 \leq t_0 < t_1 < \cdots < t_n < \infty$, let $W_l = Mg_{t_l} - Mg_{t_{l-1}}$ and $W_l' = \widetilde{X}_{t_l}^{(a)} - \widetilde{X}_{t_{l-1}}^{(a)}$, $1 \leq l \leq n$ (with $W_0 = Mg_{t_0}$ and $W_0' = \widetilde{X}_{t_0}^{(a)}$). By (2.5) and Lemma 3.4, the joint LST of $Mg_{t_0}, Mg_{t_1}, \cdots, Mg_{t_n}$

(as partial sums of the W_l's) is

$$h^{(1)}(\tau_0,\tau_1,\cdots,\tau_n)=\prod_{l=0}^{n}\exp\Big\{\int_{t_{l-1}}^{t_l}\ln K_{\mu_1}\big(C_{-\ln f(s)}(r_l)\big)\,ds\Big\}\quad(t_{-1}=0),\tag{3.14}$$

and the joint LST of $\widetilde{X}_{t_0}^{(a)},\widetilde{X}_{t_1}^{(a)},\cdots,\widetilde{X}_{t_n}^{(a)}$ (as partial sums of the W_l''s) is

$$h^{(2)}(\tau_0,\tau_1,\cdots,\tau_n)=\prod_{l=0}^{n}\exp\Big\{-a\int_{t_{l-1}}^{t_l}C_{-\ln f(s)}(r_l)\,ds\Big\}\quad(t_{-1}=0),\tag{3.15}$$

where $r_l=\sum_{k=l}^{n}\tau_k$ and $\tau_i\geq 0$ $(0\leq i\leq n)$. It follows by independence of $\{\widetilde{X}_t^{(a)}\}$ and $\{Mg_t\}$, (3.14) and (3.15), that the joint LST of $\widetilde{X}_{t_0},\widetilde{X}_{t_1},\cdots,\widetilde{X}_{t_n}$ is

$$\begin{aligned}h(\tau_0,\tau_1,\cdots,\tau_n)=&h^{(1)}(\tau_0,\tau_1,\cdots,\tau_n)\ h^{(2)}(\tau_0,\tau_1,\cdots,\tau_n)\\ =&\prod_{l=0}^{n}\exp\Big\{\int_{t_{l-1}}^{t_l}\big(-aC_{-\ln f(s)}(r_l)+\ln K_{\mu_1}\big(C_{-\ln f(s)}(r_l)\big)\,ds\Big\}.\end{aligned}\tag{3.16}$$

The joint LST of $X_{t_0},X_{t_1},\cdots,X_{t_n}$ is easily seen that the right-hand side of (3.16), leading to the conclusion that $\{\widetilde{X}_t\}$ is an increasing $\mathcal{C}$-additive process that is identical in law to $\{X_t\}$. □

We conclude the section by briefly discussing a different representation for an increasing $\mathcal{C}$-additve process without drift in terms of a Poisson random measure. Let $\mu\in\mathcal{I}(\mathbb{R}_+)$ with characteristic couple $(0,\nu)$, with ν bounded. Consider the measure Leb $\otimes\,\nu\mathcal{Q}$ on $\big(\mathbb{R}_+\times\mathbb{R}_+^{[0,\infty)},\mathcal{B}(\mathbb{R}_+)\otimes\mathcal{B}(\mathbb{R}_+)^{[0,\infty)}\big)$, where

$$\nu\mathcal{Q}(\mathbf{B})=\int_{(0,\infty)}\mathcal{Q}(x,\mathbf{B})\nu(dx)\qquad \mathbf{B}\in\mathcal{B}(\mathbb{R}_+)^{[0,\infty)}.$$

It is clear that $\nu\mathcal{Q}$ is finite, which implies that Leb $\otimes\,\nu\mathcal{Q}$ is σ-finite, since the Lebesgue measure Leb is σ-finite on $\mathbb{R}_+$. By Proposition 19.4, p. 122, in [10], there exists a Poisson random measure M on $\mathbb{R}_+\times\mathbb{R}_+^{[0,\infty)}$ with mean measure Leb $\otimes\,\nu\mathcal{Q}$. Letting $f\in\mathcal{E}(\mathbb{R}_+;(0,1])$ and $g_t(s,\mathbf{y})=\mathbf{y}(-\ln f(s))I_{[0,t]}(s)$, we have

$$-\ln E\big(e^{-\tau Mg_t}\big)=\int_0^t\Big(\int_{\mathbb{R}_+}\int_{\mathbb{R}_+^{[0,\infty)}}(1-e^{-\tau g_t(s,\mathbf{y})})\mathcal{Q}(x,d\mathbf{y})\nu(dx)\Big)\,ds\Big).$$

Now,

$$\int_{\mathbb{R}_+^{[0,\infty)}}(1-e^{-\tau g_t(s,\mathbf{y})})\mathcal{Q}(x,d\mathbf{y})=1-E\big(e^{-\tau Z_x(-\ln f(s))}\big)=1-e^{-xC_{-\ln f(s)}(\tau)},$$

(the second equation above following from (1.7)), which implies that the marginal LST $\phi_{0,t}(\tau)$ of the process $\{Mg_t\}$ takes the form (2.5). Using the same argument as in the proof of Theorem 3.2, we conclude that $\{Mg_t\}$ is an increasing $\mathcal{C}$-additive process. We also note that the representation (3.13) obtained for an increasing $\mathcal{C}$-additive process with drift and bounded ν remains valid in this context.

4. $\mathcal{C}$-additive Processes and Poisson Random Measures (ν Unbounded)

In this section, we discuss the case where $\mu \in \mathcal{I}(\mathbb{R}_+)$ has an unbounded Lévy measure ν. As in the preceding section, we assume first that μ has no drift coefficient ($a = 0$).

We assume the existence of the following processes on some probability space $(\Omega, \mathcal{F}, P)$:

- An $\mathbb{R}_+$-valued subordinator $\{Y_t^{(\mu)}\}$ such that $\mathcal{L}(Y_1^{(\mu)}) = \mu$. $\{Y_t^{(\mu)}\}$ is necessarily a pure-jump Lévy process (see [2], Chapter 7, Section 7).
- A collection of independent copies of a $\mathcal{C}$-CB process $\big(\{Z_x^{(i)}(t)\}, x \geq 0, i \geq 1\big)$.
- The collections $\big(\{Z_x^{(i)}(t)\}, x \geq 0, i \geq 1\big)$ and $\{Y_t^{(\mu)}\})$ are mutually independent.

For every $\epsilon > 0$, let $\{Y_t^{(\mu),\epsilon}\}$ be the pure-jump process where the jumps are those of $\{Y_t^{(\mu)}\}$ with sizes greater than ϵ. Let ν_ϵ be the trace of ν in (ϵ, ∞), i.e., for any Borel set B in $\mathbb{R}_+$, $\nu_\epsilon(B) = \nu(B \cap (\epsilon, \infty))$. Condition (2.7) implies that $c_\epsilon = \nu_\epsilon((\epsilon, \infty)) < \infty$. We define the probability law $\sigma_\epsilon(A) = c_\epsilon^{-1} \nu_\epsilon(A)$. The process $\{Y_t^{(\mu),\epsilon}\}$ is compound Poisson whose jump times $\{T_n^\epsilon\}$ form a Poisson process with rate c_ϵ and whose jump sizes $\{D_n^\epsilon\}$ (independent of $\{T_n^\epsilon\}$) have common distribution σ_ϵ (see, for e.g, Çinlar (2011), p. 365). Moreover, independence of $\big(\{Z_x^{(i)}(t)\}, x \geq 0, i \geq 1\big)$ and $\{Y_t^{(\mu)}\})$ implies that for any $\epsilon > 0$, $\big(\{Z_x^{(i)}(t)\}, x \geq 0, i \geq 1\big)$ and $\{Y_t^{(\mu),\epsilon}\})$ are independent. For $i \geq 1$ and $t \geq 0$, we define $Y_i^\epsilon(t) = Z_{D_i^\epsilon}^{(i)}(t)$, with $Y_i^\epsilon(0) = D_i^\epsilon$. By definition and the above assumptions, $(\{Y_i^\epsilon(t\}, i \geq 1)$ constitutes a sequence of iid $\mathcal{C}$-CB processes. Therefore (see Section 3 and (3.3)), $M^\epsilon = \{(T_i^\epsilon, Y_i^\epsilon(\cdot))\}$ is a Poisson random measure on $\mathbb{R}_+ \times \mathbb{R}_+^{[0,\infty)}$ with mean measure $c_\epsilon \,\mathrm{Leb} \otimes \pi^\epsilon$, where

$$\pi^\epsilon(\mathbf{B}) = \int_0^\infty \mathcal{Q}(x, \mathbf{B}) \sigma_\epsilon(dx) = \frac{1}{c_\epsilon} \int_{(\epsilon,\infty)} \mathcal{Q}(x, \mathbf{B}) \nu(dx) \qquad (\mathbf{B} \in \mathcal{B}(\mathbb{R}_+)^{[0,\infty)}).$$

We will make use repeatedly of the following easily established fact without further reference. For any nonneagitve measurable function $k(x)$ over $\mathbb{R}_+$

$$\int_0^\infty k(x) \sigma_\epsilon(dx) = c_\epsilon^{-1} \int_{(\epsilon,\infty)} k(x) \nu(dx).$$

Theorem 4.1. *Let $\mu \in \mathcal{I}(\mathbb{R}_+)$ with characteristic couple $(0, \nu)$, ν unbounded. Let $f \in \mathcal{E}(\mathbb{R}_+; (0, 1])$ and $g_t(s, \mathbf{y}) = \mathbf{y}(-\ln f(s)) I_{[0,t]}(s)$. Then the finite dimensional distributions of any increasing $\mathcal{C}$-additve process $\{X_t\}$ generated by the pair (μ, f) (on some probability space) arise as the limit of the finite dimensional distributions of $\{M^\epsilon g_t\}$ as $\epsilon \downarrow 0$, where (cf. Theorem 3.2)*

$$M^\epsilon g_t = \sum_{k=1}^\infty Y_k^\epsilon(-\ln f(T_k^\epsilon)) I_{[0,t]}(T_k^\epsilon). \tag{4.1}$$

Proof. We proceed as in the proof of Theorem 3.5. For $0 \le t_0 < t_1 < \cdots < t_n < \infty$, let $W_l = M^\epsilon g_{t_l} - M^\epsilon g_{t_{l-1}}$ and $W'_l = X_{t_l} - X_{t_{l-1}}$, $1 \le l \le n$ (with $W_0 = M^\epsilon g_{t_0}$ and $W'_0 = X_{t_0}$). By Theorem 3.1 (equation (3.5)), Lemma 3.3, and (2.5), the joint LST of $M^\epsilon g_{t_0}, M^\epsilon g_{t_1}, \cdots, M^\epsilon g_{t_n}$ (as partial sums of the W_l's) is

$$E\Big(\exp\Big\{-\sum_{l=0}^{n} \tau_l M^\epsilon g_{t_l}\Big\}\Big) = \prod_{l=0}^{n} \exp\Big\{-c_\epsilon \int_{t_{l-1}}^{t_1} \big(1 - \phi_{\sigma_\epsilon}\big(C_{-\ln f(s)}(r_l)\big)\big)\, ds\Big\},$$

where $r_l = \sum_{k=l}^{n} \tau_k$, $t_{-1} = 0$, and $\tau_l \ge 0$ $(0 \le l \le n)$. Since

$$1 - \phi_{\sigma_\epsilon}(\tau) = c_\epsilon^{-1} \int_{(\epsilon,\infty)} (1 - e^{-\tau x})\nu(dx),$$

it follows that

$$E\Big(\exp\Big\{-\sum_{l=0}^{n} \tau_l M^\epsilon g_{t_l}\Big\}\Big) = \prod_{l=0}^{n} \exp\Big\{-\int_{t_{l-1}}^{t_1} \int_{(\epsilon,\infty)} \big(1 - e^{-xC_{-\ln f(s)}(r_l)}\big)\nu(dx)\, ds\Big\}.$$

Therefore,

$$\begin{aligned}
&\lim_{\epsilon\downarrow 0} E\Big(\exp\Big\{-\sum_{l=0}^{n} \tau_l M^\epsilon g_{t_l}\Big\}\Big) \\
&\quad = \prod_{l=0}^{n} \exp\Big\{-\int_{t_{l-1}}^{t_1} \int_{(0,\infty)} \big(1 - e^{-xC_{-\ln f(s)}(r_l)}\big)\nu(dx)\, ds\Big\} \\
&\quad = \prod_{l=0}^{n} \exp\Big\{\int_{t_{l-1}}^{t_1} \ln K_\mu(C_{-\ln f(s)}(r_l))\, ds\Big\}.
\end{aligned}$$

It is clear (again by (2.5) and Lemma 3.4) that the right-hand side of the second equation above is the joint LST of $X_{t_0}, X_{t_1}, \cdots, X_{t_n}$. □

The following result is a direct consequence of Theorem 4.1 and (3.6), where the notation w- lim is taken to mean weak limit.

Corollary 4.2. *Under the assumptions of Theorem 4.1, the $\mathcal{C}$-additive process $\{X_t\}$ admits the representation*

$$X_t \stackrel{d}{=} \text{w-}\lim_{\epsilon\downarrow 0} \sum_{k=1}^{\infty} \big(f(T_k^\epsilon)) \odot_{\mathcal{C}} D_k^\epsilon\big) I_{[0,t]}(T_k^\epsilon), \tag{4.2}$$

We conclude with an extension of Theorem 3.5. The proof is omitted.

Corollary 4.3. *Let $\mu \in \mathcal{I}(\mathbb{R}_+)$ with characteristic couple (a,ν), where $a \ge 0$ and ν is unbounded. Denote by μ_1 the component of μ generated by $(0,\nu)$. Let $f \in \mathcal{E}(\mathbb{R}_+;(0,1])$ and let $\{X_t\}$ be an increasing $\mathcal{C}$-additive process generated by (μ, f) on some probability space $(\Omega, \mathcal{F}, P)$. There exists an increasing $\mathcal{C}$-additive process $\{\widetilde{X}_t\}$ on some probability space $(\Omega_0, \mathcal{F}_0, P_0)$ that is identical in law to $\{X_t\}$ and such that*

$$\widetilde{X}_t = \widetilde{X}_t^{(a)} + \text{w-}\lim_{\epsilon\downarrow 0} M^\epsilon g_t, \tag{4.3}$$

where

(i) $(\{M^\epsilon g_t\}, \epsilon > 0)$ *(as defiend by (4.1)) is the family of increasing* $\mathcal{C}$*-additive processes on* $(\Omega_0, \mathcal{F}_0, P_0)$*, generated by* (μ_1, f)*;*

(ii) $\{\widetilde{X}_t^{(a)}\}$ *is an increasing* $\mathcal{C}$*-additve process on* $(\Omega_0, \mathcal{F}_0, P_0)$*, generated by the pair* (δ_a, f)*;*

(iii) $\{\widetilde{X}_t^{(a)}\}$ *and the familiy* $(\{M^\epsilon g_t\}, \epsilon > 0)$ *are independent.*

5. Convergence Results

In this section, we present some weak convergence results for increasing $\mathcal{C}$-additive processes generated by the pair (μ, f), when f is restricted to a subclass of $\mathcal{E}(\mathbb{R}_+; (0,1])$.

We denote by $\mathcal{E}_1(\mathbb{R}_+; (0,1])$ the subclass of $\mathcal{E}(\mathbb{R}_+; (0,1])$ that consist of all the functions $f(s) = \exp\{-\int_0^s k(y)\,dy\}$, where $k(y)$ is positive and continuous over $[0,\infty)$, with $\lim_{y\to\infty} k(y) = l_f$ for some $l_f \in (0,\infty)$.

Lemma 5.1. *Let* $\mu \in \mathcal{I}(\mathbb{R}_+)$ *and* $f \in \mathcal{E}_1(\mathbb{R}_+; (0,1])$*. The following assertions are equivalent for any* $\tau > 0$*:*

(i) $\lim_{t\to\infty} \int_0^t \big|\ln K_\mu(\tau)\big(C_{-\ln f(s)}(\tau)\big)\big|\,ds < \infty$*;*

(ii) $\int_0^\tau \frac{(1/U(v))\ln K_\mu(v)}{k\circ f^{-1}(A(v)/A(\tau))}\,dv < \infty$*;*

where $A(v)$ *is the A-function in (1.3) and* f^{-1} *is the inverse function of* f*.*

Proof. We have by assumption $\int_0^\infty k(y)\,dy = \infty$, which implies that the function f is stricltly decreasing, which implies that $-\ln f(s) = \int_0^s k(y)\,dy$ is strictly increasing with $\lim_{s\to 0^+} -\ln f(s) = 0$ and $\lim_{s\to\infty} -\ln f(s) = \infty$. Let $v = C_{-\ln f(s)}(\tau)$ for $0 \le s \le \tau$. By the first equation of (1.4), v is a strictly decreasing function of s. The second part of (1.4) implies $A(v) = f(s)A(\tau)$ or $s = f^{-1}(A(v)/A(\tau))$ (as f is invertible). Since $\frac{A'(v)}{A(v)} = -\frac{1}{U(v)}$, it follows that

$$ds = -\frac{1/U(v)}{k\circ f^{-1}(A(v)/A(\tau))}dv.$$

If $s = 0$, then $v = C_0(\tau) = \tau$ and if $s = t$, then $v = C_{-\ln f(t)}(\tau)$. Therefore, the change of variable $v = C_{-\ln f(s)}(\tau)$, along with the first part of (1.6), implies

$$\int_0^t \big|\ln K_\mu(\tau)\big(C_{-\ln f(s)}(\tau)\big)\big|\,ds = \int_{C_{-\ln f(t)}(\tau)}^\tau \frac{(1/U(v))\ln K_\mu(v)}{k\circ f^{-1}(A(v)/A(\tau))}\,dv. \tag{5.1}$$

This concludes the proof, since by (1.1) $\lim_{t\to\infty} C_{-\ln f(t)}(\tau) = 0$. □

Define $\ln^+ x = \max(0, \ln x)$. We denote by $\mathcal{I}_{\log}(\mathbb{R}_+)$ the subset of distributions $\mu \in \mathcal{I}(\mathbb{R}_+)$ that satisfy the condition $\int_0^\infty \ln^+ x\mu(dx) < \infty$.

Theorem 5.2. *Let* $\mu \in \mathcal{I}(\mathbb{R}_+)$*, with characteristic couple* (a,ν) *and* $f \in \mathcal{E}_1(\mathbb{R}_+; (0,1])$*. Let* $\{X_t\}$ *be an increasing* $\mathcal{C}$*-additive process generated by the pair* (μ, f)*. The following assertions are equivalent.*

(i) *The probability distribution* $\mu_{0,t}$ *of* X_t *converges to a probability measure* $\mu_{0,\infty} \in \mathcal{I}(\mathbb{R}_+)$*.*

(ii) $\int_0^\infty \ln^+ x\, \nu(dx) < \infty$.

(iii) $\mu \in \mathcal{I}_{log}(\mathbb{R}_+)$.

Moreover, the LST $\phi_{0,\infty}(\tau)$ of $\mu_{0,\infty}$ admits the representations

$$\ln \phi_{0,\infty}(\tau) = \int_0^\tau \frac{(1/U(v)) \ln K_\mu(v)}{k \circ f^{-1}(A(v)/A(\tau))}\, dv = \int_0^\infty \ln K_\mu\big(C_{-\ln f(s)}(\tau)\big)\, ds. \tag{5.2}$$

Proof. Since $\lim_{v\to 0} f^{-1}(A(v)/A(\tau)) = +\infty$, we have by (1.6) that

$$\frac{(1/U(v)) \ln K_\mu(v)}{k \circ f^{-1}(A(v)/A(\tau))} \sim -\frac{1}{l_f} \frac{\ln K_\mu(v)}{U(v)}, \quad \text{as } v \to 0^+. \tag{5.3}$$

Therefore, by Lemma 5.1 and the continuity theorem (Theorem 3.1, Appendix A, in [12]), X_t converges in distribution as $t \to \infty$ if and only if

$$\int_0^\tau -\frac{\ln K_\mu(v)}{U(v)}\, dv < \infty. \tag{5.4}$$

Using the exact same proof as that of Corollary 3.21, p. 67, in [5], one can show that (5.4) is equivalent to (ii), thus proving (i) $\Leftrightarrow$ (ii). By Proposition 3.2, Appendix A, in [12], (5.4) holds if and only if (iii) is true. This shows (i) $\Leftrightarrow$ (iii). Since $\mu_{0,t} \in \mathcal{I}(\mathbb{R}_+)$ for every $t > 0$, its limit $\mu_{0,\infty}$ must be infinitely divisible. The representation (5.2) follows from (5.1) and the continuity theorem (cited above) as the function $\exp\Big\{\int_0^\tau \frac{(1/U(v)) \ln K_\mu(v)}{k \circ f^{-1}(A(v)/A(\tau))}\, dv\Big\}$ is clearly right-continuous at 0. □

Corollary 5.3. *Let μ be a $\mathcal{C}$-stable distribution with exponent $\gamma \in (0, 1]$ and let $f \in \mathcal{E}_1(\mathbb{R}_+; (0, 1])$. Let $\{X_t\}$ be an increasing $\mathcal{C}$-additive process generated by the pair (μ, f). Then $\mu \in \mathcal{I}_{log}(\mathbb{R}_+)$ and the limiting probability distribution $\mu_{0,\infty}$ of $\{X_t\}$ is $\mathcal{C}$-stable with exponent γ and its LST admits the representation*

$$\ln \phi_{0,\infty}(\tau) = -\lambda\Big(\int_0^\infty f(y)^\gamma\, dy\Big) A(\tau)^\gamma \qquad (\lambda > 0). \tag{5.5}$$

Proof. By (2.17), $K_\mu(\tau) = -\lambda A(\tau)^\gamma$ for some $\lambda > 0$. A simple integration exercise shows that (5.4). and thus (5.2), hold, which implies that $\mu \in \mathcal{I}_{\log}(\mathbb{R}_+)$. Equation (5.5) follows from (2.18) and the contnuity theorem. The representation (5.5) also implies that $\mu_{0,\infty}$ is $\mathcal{C}$-stable with exponent γ (again by (2.17)). □

Next, we discuss the case where f is in the subest of $\mathcal{E}_1(\mathbb{R}_+; (0, 1])$ consisting of the functions $f_\alpha(s) = \alpha^s$, $\alpha \in (0, 1)$. In this case, $k_\alpha(x) = l_{f_\alpha} = -\ln \alpha$.

We first recall a notion of self-decomposability for distributions on $\mathbb{R}_+$ introduced in [4] (see also Hansen [3]).

A probability distribution κ on $\mathbb{R}_+$, with LST $\phi(\tau)$, is said to be $\mathcal{C}$-self-decomposable if for every $t > 0$, there exists a probability distribution κ_t, with LST $\phi_t(\tau)$, such that

$$\phi(\tau) = \phi(C_t(\tau))\phi_t(\tau) \quad (\tau \geq 0). \tag{5.6}$$

Both κ and κ_t are necessarily infinitely divisible.

A distribution on $\mathbb{R}_+$ is $\mathcal{C}$-selfdecomposable if and only if its LST $\phi(\tau)$ admits the representation

$$\ln\phi(\tau) = \int_0^\tau \frac{\ln\phi_0(v)}{U(v)}\,dv = \int_0^\infty \ln\phi_0(C_t(\tau))\,dt \quad (\tau \ge 0), \tag{5.7}$$

where $\phi_0(\tau)$ is the LST of a distribution in $\mathcal{I}(\mathbb{R}_+)$ such that

$$\int_0^1 -\frac{\ln\phi_0(v)}{U(v)}\,dv < \infty. \tag{5.8}$$

Theorem 5.2 restricted to the f_α functions states as follows.

Theorem 5.4. *Let $\mu \in \mathcal{I}(\mathbb{R}_+)$, with characteristic couple (a,ν) and $\alpha \in (0,1)$. Let $\{X_t^{[\alpha]}\}$ be an increasing $\mathcal{C}$-additive process generated by the pair (μ, f_α). The following assertions are equivalent.*

(i) *The probability distribution $\mu_{0,t}^{[\alpha]}$ of $X_t^{[\alpha]}$ converges to a probability measure $\mu_{0,\infty}^{[\alpha]} \in \mathcal{I}(\mathbb{R}_+)$.*

[ii] $\int_0^\infty \ln^+ x\,\nu(dx) < \infty$.

(iii) $\mu \in \mathcal{I}_{log}(\mathbb{R}_+)$.

Moreover, the limiting distribution $\mu_{0,\infty}^{[\alpha]}$ is $\mathcal{C}$-selfdecomposable with an LST of the form (5.6) with $\phi_0(\tau) = K_\mu^{-1/\ln\alpha}(\tau)$.

Proof. Only the last statement requires a proof. But then (5.2) can be rewritten as (5.7) with $\phi_0(\tau) = K_\mu^{-1/\ln\alpha}(\tau)$. Since $\mu \in \mathcal{I}(\mathbb{R}_+)$, ϕ_0 is the LST of a distribution in $\mathcal{I}(\mathbb{R}_+)$ that satisfies (5.8). □

The converse holds.

Theorem 5.5. *Any $\mathcal{C}$-selfdecomposable distribution arises as a the limiting distrubution of an increasing C-additive process defined over some probability space.*

Proof. Let μ be a $\mathcal{C}$-selfdecomposable distribution and $f(s) = e^{-s}$. By Theorem 2.3, there exists an increasing $\mathcal{C}$-additive process $\{X_t\}$ defined on some probability space $(\Omega, \mathcal{F}, P)$ and generated by the pair (μ, f). Since $\ln\phi_{s,t}(\tau) = \int_s^t \ln K_\mu(C_y(\tau))\,dy$, $(0 \le s < t < \infty)$, it easily follows by Theorem 5.2 that equations (5.2) and (5.7) are identical, which in turn implies $\phi_{0,\infty}(\tau)$ is the LST of μ. □

We conclude with a new characterization of $\mathcal{C}$-stable distributions.

Corollary 5.6. *Let $\mu \in \mathcal{I}(\mathbb{R}_+)$ and $\alpha \in (0,1)$. The following statements are equivalent.*

(i) *(i)μ is $\mathcal{C}$-stable with exponent $\gamma \in (0,1]$.*

(ii) *(ii) $\mu \in \mathcal{I}_{log}(\mathbb{R}_+)$ and the increasing $\mathcal{C}$-additive process $\{X_t^{[\alpha]}\}$ generated by the pair (μ, f_α) has a limiting distribution $\mu_{0,\infty}^{[\alpha]}$ with an LST satisfying*

$$\ln\phi_{0,\infty}^{[\alpha]}(\tau) = c\ln K_\mu(\tau) \qquad (c = -1/(\gamma\ln\alpha)). \tag{5.9}$$

Proof. (i) $\Rightarrow$ (ii) is a straightforward application of Corolary 5.3 with $f = f_\alpha$. Assuming (ii), $\mu_{0,\infty}^{[\alpha]}$ is $\mathcal{C}$-selfdecomposable by Theorem 5.4. By (5.6), with $\phi_0(\tau) = K_\mu^{-1/\ln\alpha}(\tau)$, we have,

$$\ln \phi_{0,\infty}^{[\alpha]}(\tau) = \frac{-1}{\ln\alpha}\int_0^\infty \ln K_\mu(C_t(\tau))\,dt = \frac{1}{\ln\alpha}\int_0^\tau \frac{\ln K_\mu(v)}{U(v)}\,dv, \tag{5.10}$$

where the second equation follows the change of variable $v = C_t(\tau)$, (1.1) and (1.4). Combining (5.9) and (5.10) leads to the differential equation

$$c\frac{d}{dv}\ln K_\mu(v) = \frac{1}{\ln\alpha}\frac{\ln K_\mu(v}{U(v)} = \frac{-1}{\ln\alpha}\frac{A'(v)}{A(v)}\ln K_\mu(v),$$

whose solution is $\ln K_\mu(\tau) = -\lambda A(\tau)^\gamma$ with $\lambda = -\ln K_\mu(1)$ and $\gamma = -1/(c\ln\alpha)$. It is easily seen that the LST $K_\mu(\tau)$ satisfies (2.16), which implies that μ is $\mathcal{C}$-stable with exponent γ (with the latter being forcibly in $(0,1]$). □

Acknowledgment. The author is grateful to a referee for very valuable comments and suggestions.

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