SOLUTIONS OF FAR-FIELD AMPLITUDE AND SCATTERING CROSS SECTION FOR A RIGID ELLIPSOID BY INTEGRAL EQUATION TECHNIQUES

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Abstract: I present here the solutions for the boundary value problems of scattering of low-frequency sound waves by an arbitrary rigid ellipsoid by integral equation techniques whose density and compressibility are different from those of the surrounding infinite medium. The analysis is based on a computational scheme in which we first convert the boundary value problems into integral equations. Thereafter, I convert these integral equations to infinite set of algebraic equations. A judicial truncation scheme then helps us in achieving our results. Interesting feature of this computation technique is that the very first truncation of the algebraic system yields the exact solutions for far field amplitude and scattering cross section for a rigid ellipsoid.

1. MATHEMATICAL FORMULATION

I discuss the problem of the irradiation of an obstacle scatterer, occupying a finite region R_2, by an acoustic wave. The boundary of the obstacle is denoted by S while the region exterior is R_1. Let (x_1, x_2, x_3) be a Cartesian coordinate system whose origin O is at the centroid of the scatterer and let $\hat{n}$ denote the unit normal vector to S pointing out of R_2 into the host medium. The incident plane wave propagating in the direction of the unit vector $\hat{\underset{\sim}{b}}$ in the infinite host medium has the potential ϕ^0 (I suppress the time factor $\exp(-i\omega t)$ throughout this analysis) given as

$$\phi^0(\underset{\sim}{x}) = \exp(i\underset{\sim}{k}\cdot\underset{\sim}{x}),\ \underset{\sim}{k} = k\hat{\underset{\sim}{b}} = \frac{2\pi}{\lambda}\hat{\underset{\sim}{b}},\quad \hat{\underset{\sim}{b}} = b_1\hat{i}_1 + b_2\hat{i}_2 + \hat{b}_3 i_3 \tag{1}$$

where $\underset{\sim}{x} = (x_1, x_2, x_3)$, $\hat{i}_1, \hat{i}_2, \hat{i}_3$ are unit vectors along x_1, x_2, x_3 axes respectively while λ is the wave length.

The governing differential equations are

$$(\nabla^2 + k^2)\phi^+(\underset{\sim}{x}) = 0,\quad \underset{\sim}{x} \in R_1,\ k = \frac{2\pi}{\lambda} \tag{2}$$

$$(\nabla^2 + \eta^2 k^2)\phi^-(\underset{\sim}{x}) = 0, \underset{\sim}{x} \in R_2, \tag{3}$$

where the quantity η is the relative index of refraction. If we write

$$\phi^+(\underset{\sim}{x}) = \phi^0(\underset{\sim}{x}) + \phi^s(\underset{\sim}{x}) \tag{4}$$

then $\phi^s(\underset{\sim}{x})$ is the scattered field which satisfies the radiation condition at infinity. The boundary conditions on S are

$$\phi^+(\underset{\sim}{x}_S) = \phi^-(\underset{\sim}{x}_S)$$

$$\left.\frac{\partial\phi^+(\underset{\sim}{x}_S)}{\partial n}\right|_+ = \beta \left.\frac{\partial\phi^-(\underset{\sim}{x}_S)}{\partial n}\right|_-,$$

where β is a non-negative number which gives the ratio of outer to inner compressibilities. The two parameters β and η effectively characterize the composite media. When β =1. η =1, the regions R_1 and R_2 are filled with the same material so that both the potential field and their normal derivatives are continuous across S and, accordingly no scattering occurs. When $\eta \to 0$, I have the scattering problem for a rigid obstacle. However, it is not possible to get the soft body limit from this analysis.

To derive the integral equation I use the equation

$$(\nabla^2 + k^2)G(\underset{\sim}{x}, \underset{\sim}{x}') = -\delta(\underset{\sim}{x} - \underset{\sim}{x}'), \qquad \underset{\sim}{x}, \underset{\sim}{x}' \in R = R_1 \cup S \cup R_2, \tag{5}$$

where $\underset{\sim}{x}$ and $\underset{\sim}{x}'$ are, respectively, the field and source points.

$$G(\underset{\sim}{x}, \underset{\sim}{x}') = \frac{\exp(ik\,|\underset{\sim}{x} - \underset{\sim}{x}'|)}{4\pi\,|\underset{\sim}{x} - \underset{\sim}{x}'|}$$

is Green's function when the whole region R is occupied by the host medium and $\delta(\underset{\sim}{x} - \underset{\sim}{x}')$ is the Dirac delta function. When we multiply eqn. (2) by $G(\underset{\sim}{x}, \underset{\sim}{x}')$ and (5) by $\phi^+(\underset{\sim}{x})$ subtract and integrate over the region R_1, and let S_∞ denote the infinite sphere with center at the origin $\underset{\sim}{O}$, we get $\underset{\sim}{x} \in R_1$ or $\underset{\sim}{x} \in R_2$,

$$\left.\begin{matrix} \phi^+(\underset{\sim}{x}'), \; \underset{\sim}{x}' \in R_1 \\ 0, \qquad \underset{\sim}{x}' \in R_2 \end{matrix}\right\} = -\int_S \left[-\phi^+(\underset{\sim}{x}_S)|_+ \frac{\partial G}{\partial n}(\underset{\sim}{x}_S, \underset{\sim}{x}') + G(\underset{\sim}{x}_S, \underset{\sim}{x}') \left.\frac{\partial\phi^+}{\partial n}(\underset{\sim}{x}_S)\right|_+ \right] dS$$

$$+\int_{S_\infty} \left[G(\underset{\sim}{x}_S, \underset{\sim}{x}') \frac{\partial\phi^+}{\partial n}(\underset{\sim}{x}_S) - \phi^+(\underset{\sim}{x}_S) \frac{\partial G}{\partial n}(\underset{\sim}{x}_S, \underset{\sim}{x}')) \right] dS_\infty$$

$$= \int_S \left[\phi^+(\underset{\sim}{x}_S)|_- \frac{\partial G}{\partial n}(\underset{\sim}{x}_S, \underset{\sim}{x}') - \beta G(\underset{\sim}{x}_S, \underset{\sim}{x}') \frac{\partial\phi^-}{\partial n}(\underset{\sim}{x}_S) \right]_- dS + \Phi^0(\underset{\sim}{x}')$$

$$= \phi^0(\underset{\sim}{x}') + \int_{R_2} \{div[\phi^-(\underset{\sim}{x})\nabla G(\underset{\sim}{x}, \underset{\sim}{x}') - \beta G(\underset{\sim}{x}, \underset{\sim}{x}')\nabla\phi^-(\underset{\sim}{x})]\} dR_2$$

$$= \phi^0(\underset{\sim}{x}') + \int_{R_2} [\phi^-(\underset{\sim}{x})\nabla^2 G(\underset{\sim}{x}, \underset{\sim}{x}') - \beta G(\underset{\sim}{x}, \underset{\sim}{x}')\nabla^2\phi^-(\underset{\sim}{x})$$

$$+(1-\beta)\nabla\phi^-(\underset{\sim}{x})\cdot\nabla G(\underset{\sim}{x}, \underset{\sim}{x}')]dR_2, \quad \underset{\sim}{x}' \in R_1 \text{ or } R_2 \tag{6}$$

Now we use eqn. (5) and set $B = \beta - 1$ and $C = (\beta\eta^2 - 1)$ so that eqn. (6) becomes

$$\phi^{\pm}(\underset{\sim}{x}) = \phi^0(\underset{\sim}{x}) - \int_{R_2} [B\nabla'\phi^-(\underset{\sim}{x})\cdot\nabla' G(\underset{\sim}{x}, \underset{\sim}{x}') - Ck^2 G(\underset{\sim}{x}, \underset{\sim}{x}')\phi^-(\underset{\sim}{x}')]dR_2', \; \underset{\sim}{x} \in R \tag{7}$$

This formula is valid everywhere and constitutes the governing integral equation to determine $\phi^-(\underset{\sim}{x})$ in R_2. After $\phi^-(\underset{\sim}{x})$ has been thus determined, I substitute it in this very equation and find $\phi^+(\underset{\sim}{x})$.

To solve the integral equation (7) I use the expansions

$$\phi^0(\underset{\sim}{x}) = \exp(i\underset{\sim}{k}.\, \underset{\sim}{x}) = \sum_{n=0}^{\infty} \frac{(ik)^n}{n!}(\hat{\underset{\sim}{b}}\cdot\underset{\sim}{x})^n = \sum_{n=0}^{\infty} \frac{(ik)^n}{n!}\phi_n^0(\underset{\sim}{x}) \tag{8a}$$

where

$$\phi_n^{(0)}(\underset{\sim}{x}) = (\hat{\underset{\sim}{b}}\cdot\underset{\sim}{x})^n.$$

$$G(\underset{\sim}{x}, \underset{\sim}{x}') = \frac{e^{ik|\underset{\sim}{x}-\underset{\sim}{x}'|}}{4\pi|\underset{\sim}{x}-\underset{\sim}{x}'|} = \frac{1}{4\pi R} + \frac{ik}{4\pi} - \frac{k^2\tilde{R}}{8\pi} + \ldots \qquad \tilde{R} = |\underset{\sim}{x} - \underset{\sim}{x}| \tag{8b}$$

$$\phi^{\pm}(\underset{\sim}{x}) = \sum_{n=0}^{\infty} \frac{(ik)^n}{n!}\phi_n^{\pm}(\underset{\sim}{x}),$$

where the functions $\phi_n^{\pm}(\underset{\sim}{x})$ are independent of the parameter *k*. Substituting these expansions in integral equation (7) I get the following integral equations for the function $\phi_n^{\pm}(\underset{\sim}{x}),\; n = 0,1,2,....,$

$$\phi_0^{\pm}(\underset{\sim}{x}) = \phi_0^0(\underset{\sim}{x}) - B\int_{R_2} \nabla'\phi_0^-(\underset{\sim}{x})\cdot\nabla'\left(\frac{1}{4\pi\tilde{R}}\right)dR_2', \tag{9}$$

$$\phi_1^{\pm}(\underset{\sim}{x}) = \phi_1^0(\underset{\sim}{x}) - B\int_{R_2} \nabla'\phi_1^-(\underset{\sim}{x})\cdot\nabla'\left(\frac{1}{4\pi\tilde{R}}\right)dR_2' \tag{10}$$

$$\phi_2^{\pm}(\underset{\sim}{x}) = \phi_2^0(\underset{\sim}{x}) - \int_{R_2} \left\{B\left[\nabla'\phi_0^-(\underset{\sim}{x})\cdot\nabla'\frac{\tilde{R}}{4\pi} + \nabla'\phi_2^-(\underset{\sim}{x}').\nabla'\left(\frac{1}{4\pi\tilde{R}}\right)\right] + C\left(\frac{1}{2\pi\tilde{R}}\right)\phi_0^-(\underset{\sim}{x})\right\}dR_2' \tag{11}$$

and so on and where $\underset{\sim}{x} \in R$. Also

$$\psi_0^0(\underset{\sim}{x}) = 1, \quad \phi^0(\underset{\sim}{x}) = \hat{\underset{\sim}{b}}\cdot\underset{\sim}{x}, \qquad \phi_2^0(\underset{\sim}{x}) = (\hat{\underset{\sim}{b}}\cdot\underset{\sim}{x})^2$$

are the terms in the series (8a).

Equations (9)-(11) are the integral equations of the potential theory and can be processed by the truncation scheme as explained in references [12-15]. As mentioned before, I first derive the solutions in the interior region and then use them to obtain the exterior soutions. I illustrate the method for a triaxial ellipsoid.

INTERIOR SOLUTION

For a triaxial ellipsoid whose semiaxes have the lengths a_1, a_2 and a_3, the interior region R_2 is

$$R_2 : \frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} + \frac{x_3^2}{a_3^2} < 1, \quad a_1 > a_2 > a_3 > 0 .$$

Since, in the integral eqns. (9)-(11), the terms $\phi_n^0 = (\hat{\underset{\sim}{b}} \cdot \underset{\sim}{x})^n$ are the known n^{th} degree homogeneous functions in x_1, x_2 and x_3, it follows from the analysis in [12-15] that the

(i) zeroth order approximation of the integral eqns. (9) yields the exact interior solution $\phi_0^-(\underset{\sim}{x})$.

(ii) first order approximation of eqn. (10) yields the exact solution for $\phi_1^-(\underset{\sim}{x})$;

(iii) second order approximation of eqn. (11) yields the exact solution for $\phi_2^-(\underset{\sim}{x})$, and so on.

To find the exact solution of equation (9) we write it as

$$\phi_0^-(\underset{\sim}{x}) = \phi_0^0(\underset{\sim}{x}) - \frac{B}{4\pi} \int_{R_2} \left(\frac{1}{|\underset{\sim}{x} - \underset{\sim}{x}'|} \right)_{,i'} \phi_{0,i'}^-(\underset{\sim}{x}') dR_2', \qquad \underset{\sim}{x} \in R_2 \tag{12}$$

When we differentiate both sides m times with respect to $\underset{\sim}{x}$, we have

$$\phi_{0,p_1,\dots,p_m}^-(\underset{\sim}{x}) = \phi_{0,p_1,p_2,\dots,p_m}^0(\underset{\sim}{x}) - \frac{B}{4\pi} \int_{R_2} \left(\frac{1}{|\underset{\sim}{x} - \underset{\sim}{x}'|} \right)_{i'p_1p_2,\dots p_m} \phi_{0,i'}^-(\underset{\sim}{x}') dR_2', \tag{13}$$

where p's have the values 1, 2, 3, . Next, I use Taylor's expansion

$$\phi_{0,i}^-(\underset{\sim}{x}) = \sum_{s=0}^{\infty} \frac{1}{s!} \phi_{0,q_1q_2\dots q_s}^-(\underset{\sim}{0}) x_{q_1} x_{q_2} \dots x_{q_s} ,$$

where q's have the values 1, 2, 3, in (13) and get

$$\phi_{0,p_1p_2\dots p_m}^-(\underset{\sim}{0}) = \psi_{0,p_1p_2,\dots p_m}^0(\underset{\sim}{0}) + \frac{(-1)^{m+1}}{4\pi} B \sum_{s=0}^{\infty} \frac{1}{s!} T_{ip_1p_2\dots p_m,q_1q_2\dots q_s} \times \phi_{0,iq_1q_2,\dots q_s}^-(\underset{\sim}{0}) \tag{14}$$

while the quantities $T_{ip_1p_2,\dots p_m,q_1q_2,\dots q_s}$ are the shape factors defined as

$$T_{ip_1p_2,\dots p_m,q_1q_2,\dots,q_s} = \int_{R_2} \left(\frac{1}{|\underset{\sim}{x}|}\right)_{,ip_1,p_2,\dots p_m} x_{q_1}x_{q_2}\dots x_{q_s}\,dR_2 \tag{15}$$

Since $\phi_0^0(\underset{\sim}{x}) = 1$, only the term $\phi_0^-(0)$ is nonzero while $\phi^-_{0,p_1p_2,\dots p_m}(\underset{\sim}{0}) = 0$, and I find from (14) that $\phi_0^-(\underset{\sim}{0}) = \phi_0^0(\underset{\sim}{0}) = 1,$ so that

$$\phi_0^-(\underset{\sim}{x}) = 1,\ \ x \in R_2 \tag{16}$$

Processing integral eqn. (10) in the same fashion, I have

$$\phi^-_{1,p_1p_2,\dots p_m}(\underset{\sim}{0}) = \phi^0_{1,p_1p_2,\dots p_m}(\underset{\sim}{0}) + \frac{(-1)^{m+1}}{4\pi}B\sum_{s=0}^{\infty}\frac{1}{s!}T_{i,p_1p_2,\dots p_m,q_1,q_2,\dots q_s} \times \phi_{1,iq_1q_2,\dots q_s}(\underset{\sim}{0}) \tag{17}$$

Now $\phi_1^0(\underset{\sim}{x}) = (\hat{\underset{\sim}{b}}.\ \underset{\sim}{x}) = b_1x_1 + b_2x_2 + b_3x_3$, I find that $\phi^0_{1,p_1p_2,\dots p_m}(\underset{\sim}{0}) = 0,\ m \geq 2$ and consequently

$$\phi^-_{1,p_1p_2,\dots p_m}(\underset{\sim}{0}) = 0,\ m \geq 2$$

and the only non-zero coefficients in Taylor's expansion of $\phi_1^-(\underset{\sim}{x})$ are $\phi^-_{1,p}(\underset{\sim}{0}),\ p = 1,2,3$ Indeed for $m = 0,\ \phi_1^-(\underset{\sim}{0}) = \phi_1^0(\underset{\sim}{0}) = 0$ and for $m = 1$, $s = 0$, relation (17) yields

$$\phi^-_{1,p}(\underset{\sim}{0}) = \phi^0_{1,p}(\underset{\sim}{0}) + \frac{B}{4\pi}T_{ip}\phi^-_{1,i}(\underset{\sim}{0}) = b_p + \frac{B}{4\pi}T_{pp}\phi^-_{1,p}(\underset{\sim}{0}),\ \text{p not summed.}$$

where I have used the fact that $T_{ip} = 0,\ i \neq p$ in view of the symmetry of the ellipsoid. Thus, the foregoing relation yields

$$\phi^-_{1,p}(\underset{\sim}{0}) = \frac{b_p}{1 - \dfrac{B}{4\pi}T_{pp}},\ \text{p not summed.} \tag{18}$$

To find the factors T_{pp} explicitly, I appeal to relation (15) and set $|\underset{\sim}{x}| = r,$ so that

$$T_{11} = \int_{R_2}\frac{\partial^2}{\partial x_1^2}\left(\frac{1}{r}\right)dR_2 = \frac{a_2a_3}{a_1}\int_{\sum_{s=1}^{3}x_i^2<1}\frac{\partial^2}{\partial x_1^2}\left(\frac{1}{(a_1^2x_1^2 + a_2^2x_2^2 + a_3^2x_3^2)^{1/2}}\right)dx_1dx_2dx_3$$

$$= -a_1a_2a_3\int_S \frac{x_1^2}{r_1^3}dS,\ \ r_1 = \sqrt{(a_1^2x_1^2 + a_2^2x_2^2 + a_3^2x_3^2)},$$

where $S : \sum_{i=1}^{3} x_i^2 = 1,$ is the sphere of unit radius. Thus

$$T_{pp} = -2\pi V I_1(a_p)\,,\ \text{p not summed} \tag{19}$$

where $V = a_1 a_2 a_3$ and

$$I_1(a_p) = \int_0^\infty \frac{du}{(u+a_p^2)R_u},\ R_u = \sqrt{(u+a_1^2)(u+a_2^2)(u+a_3^2)} \tag{20}$$

Substituting the value (19) in (18) we have

$$\phi_{1,p}^-(0) = \frac{b_p}{1+\dfrac{BV}{2}I_1(a_p)},\ \ p = 1,2,3\ \ (21)$$

Accordingly, the exact solution of the integral equation (10) is

$$\phi_1^-(\underset{\sim}{x}) = \phi_{1,p}^-(\underset{\sim}{0})x_p = \sum_{p=1}^{3} \frac{b_p x_p}{1+\dfrac{BV}{2}I_1(a_p)},\ x \in R_2 \tag{22}$$

FAR-FIELD AMPLITUDE AND SCATTERING CROSS SECTION

In relation (4) I have defined the scattered field $\phi^s(\underset{\sim}{x})$ in the region R_1. With the information gathered in the previous sections I can evaluate the field $\phi^s(\underset{\sim}{x})$. Indeed, from relations (4) and (7) I have

$$\phi^s(\underset{\sim}{x}) = -\int_{R_2} [B\nabla'\phi^-(\underset{\sim}{x}')\cdot\nabla' G(\underset{\sim}{x},\underset{\sim}{x}') - Ck^2 G(\underset{\sim}{x},\underset{\sim}{x}')\phi^-(\underset{\sim}{x})]dR_2' \simeq g(\hat{\underset{\sim}{x}},\hat{\underset{\sim}{k}})\frac{e^{ikr}}{ikr}\cdot$$

$$\text{as } r = |\underset{\sim}{x}| \to \infty \tag{23}$$

where

$$g(\hat{\underset{\sim}{x}},\hat{\underset{\sim}{k}}) = \frac{ik^2}{4\pi}\int_{R_2} [iB\hat{\underset{\sim}{x}}\cdot\nabla\phi^-(\underset{\sim}{x}') + Ck\phi^-(\underset{\sim}{x}')]\exp(-ik\hat{\underset{\sim}{x}}\cdot\underset{\sim}{x}')dR_2' \tag{24}$$

is the far-field amplitude and $\hat{\underset{\sim}{x}}$ and $\hat{\underset{\sim}{k}}$ are unit vectors.

Next, I substitute the expansions

$$\phi^-(\underset{\sim}{x}) = \phi_0^-(\underset{\sim}{x}) + ik\phi_1^-(\underset{\sim}{x}) + \frac{(ik)^2}{2!}\phi_2^-(\underset{\sim}{x}) + \ldots,$$

and

$$\exp(-ik(\hat{\underset{\sim}{x}}\cdot\underset{\sim}{x}') = \sum_{n=0}^{\infty}\frac{(-ik)^n}{n!}(\hat{\underset{\sim}{x}}\cdot\underset{\sim}{x}'),$$

in (24) and obtain the first approximation,

$$g(\hat{\underset{\sim}{x}},\, \hat{k}) = \frac{ik^3}{4\pi}\left[\frac{4\pi VC}{3} - B\hat{\underset{\sim}{x}}\cdot\int_{R_2} \nabla\phi_1^-(\underset{\sim}{x}')dR_2'\right] + O(k^4)\,. \tag{25}$$

This is the approximation first given by Rayleigh [18]. Now, from relation (22) I find that

$$\nabla\phi_1^-(\underset{\sim}{x}) = \sum_{p=1}^{3} \frac{b_p \hat{\underset{\sim}{i}}_p}{1+\dfrac{BV}{2}I_1(a_p)}$$

is a constant vector, so that (25) yields

$$g(\hat{\underset{\sim}{x}},\, \hat{\underset{\sim}{k}}) = \frac{ik^3V}{3}\left[C - B\sum_{P=1}^{3} \frac{b_p\hat{x}_p}{1+\dfrac{BV}{3}I_1(a_p)}\right] + O(k^4). \tag{26}$$

By letting B → −1 and C → −1 in equation (26), I get the formula of Far-Field amplitude for a rigid ellipsoid

$$g(\hat{\underset{\sim}{x}},\hat{\underset{\sim}{k}}) = \frac{ik^3V}{3}\left[-1+\sum_{p=1}^{3}\left(\frac{\underset{\sim}{b}_p\cdot\hat{\underset{\sim}{x}}_p}{1-\dfrac{V}{2}I_1(a_p)}\right)\right] + O(k^4) \tag{27}$$

where the integral $I_1(a_p)$ is defined by relation (20).

At this stage I make a few observations about the scattering amplitude g: (i) it satisfies the reciprocity relation; (ii) it satisfies the scattering theorem; (iii) if the scatterer has inversion symmetry $(i.e.\underset{\sim}{x}\in S$ implies $-\underset{\sim}{x}\in S)$, then I can assume that

$$g(\hat{\underset{\sim}{x}},\hat{\underset{\sim}{k}}) = ik^3A_3(\hat{\underset{\sim}{x}},\, \hat{\underset{\sim}{k}}) + k^4A_4(\hat{\underset{\sim}{x}},\, \hat{\underset{\sim}{k}}) + ik^5A_5(\hat{\underset{\sim}{x}},\, \hat{\underset{\sim}{k}}) + k^6A_6(\hat{\underset{\sim}{x}},\, \hat{\underset{\sim}{k}}) + ... \tag{28}$$

Where $A_n(\hat{\underset{\sim}{x}},\hat{\underset{\sim}{k}})$ are real functions and in view of the above mentioned observations

$$A_4(\hat{\underset{\sim}{x}},\hat{\underset{\sim}{k}}) = 0 \tag{29}$$

$$A_6(\hat{\underset{\sim}{x}},\hat{\underset{\sim}{k}}) = -\frac{1}{4\pi}\int A_3(\hat{\underset{\sim}{p}}\cdot\hat{\underset{\sim}{x}})A_3(\hat{\underset{\sim}{p}}\cdot\hat{\underset{\sim}{x}})d\Omega(\hat{p}) \tag{30}$$

and so on. Accordingly, the first approximation to Re$g(\hat{\underset{\sim}{x}},\hat{\underset{\sim}{k}})$ is of order k^6.

By comparing relations (26) and (28) it follows that

$$A_3 = \frac{V}{3}\left[C - B\sum_{P=1}^{3} \frac{b_p \hat{x}_p}{\left[1+\frac{BV}{3}I_1(a_p)\right]}\right]. \tag{31}$$

Then from (29) I find that

$$A_6(x\hat{k}, \hat{\underset{\sim}{k}}) = -\frac{1}{4\pi}\int (A_3(\hat{\underset{\sim}{p}}\cdot\hat{\underset{\sim}{x}}))^2 d\Omega(\hat{p})$$

$$= -\frac{V^2}{36\pi}\int_0^{\pi}\int_0^{2\pi}\left\{C - B\left[\frac{b_1 \sin\theta\cos\phi}{A_1} + \frac{b_2\sin\theta\sin\varphi}{A_2} + \frac{b_3\cos\theta}{A_3}\right]\right\}^2 \sin\theta d\theta d\phi$$

$$= -\frac{V^2}{9}\left\{C + \frac{B^2}{3}\left[\frac{b_1^2}{A_1^2} + \frac{b_2^2}{A_2^2} + \frac{b_3^2}{A_3^2}\right]\right\} \tag{32}$$

where

$$A_p = 1 + \frac{BV}{2}I_1(a_p),\ \ p = 1, 2, 3.$$

The scattering cross section σ_S is given by the formula

$$\sigma_S = \frac{1}{k^2}\int |g(\hat{\underset{\sim}{p}}, \hat{\underset{\sim}{k}})|^2 \, d\Omega(\hat{p}) = -\frac{4\pi}{k^2}\mathrm{Reg}(\hat{\underset{\sim}{k}}, \hat{\underset{\sim}{k}})$$

Substituting in it the value of $g(\hat{\underset{\sim}{p}}, \hat{\underset{\sim}{x}})$ from (28) to (32) we have

$$\sigma_S = -4\pi k^2 A_6(\hat{\underset{\sim}{k}}, \hat{\underset{\sim}{k}}) + \mathrm{O}(\mathrm{k}^6) = \frac{4\pi V^2}{9}k^4\left\{C^2 + \frac{B^2}{3}\sum_{p=1}^{3}\frac{b_p^2}{A_p^2}\right\} + O(k^6). \tag{33}$$

By letting B $\to -1$ and C $\to -1$ in equation (33) I obtain the corresponding formula of scattering cross section σ_S for a rigid ellipsoid

$$\sigma_S = \frac{4\pi V^2}{9}k^4\left\{1 + \frac{1}{3}\sum_{p=1}^{3}\frac{b_p^2}{A_p^2}\right\} + O(k^6). \tag{34}$$

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