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STOCHASTIC DIFFERENTIAL INCLUSIONS WITH MEAN DERIVATIVES HAVING LOWER SEMI-CONTINUOUS RIGHT-HAND SIDES

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Abstract. We prove the existence of solutions of stochastic differential inclusion with mean derivatives relative to the past. The right-hand sides of both the line with forward mean derivative and with quadratic mean derivative are lower semi-continuous. The right-hand of the line with quadratic mean derivative has closed convex images while the right-hand side with forward mean derivative may not hav convex images.).

Introduction

The construction of mean derivatives (forward, backward, etc.) was introduced by E. Nelson (see, e.g., [1, 2, 3]) for the needs of the so called "Stochastic Mechanics" (a version of quantum mechanics). Note that in [4] an additional mean derivative called quadratic, was introduced, that made in principle possible to recover a stochastic process from its various Nelson's mean derivatives and the quadratic one. After that it was found that the equations and inclusions with mean derivatives arise in many problems in mathematical physics, economy, engineering, etc. In particular, inclusions with mean derivatives naturally arise in some problems of optimal control.

In this paper we investigate the inclusions with mean derivatives having lower semi-continuous right-hand sides. A new point here is that both the lines with forward mean derivative and the line with quyadratic mean derivative have set-valued right hand sides and in addition the right-hand side of the line with quadratic mean derivative has closed convex images while the right-hand side of the line with forward mean derivative may not hav convex images. We prove an existence of solution theorem for such inclusion under some natural conditions.

The set of symmetric positive-definite $n \times n$ matrices we denote by $S_+(n)$ and its closure, the set of positive semi-definite matrices, by $\bar{S}_+(n)$.

1. Mean derivatives

Let on certain probability space $(\Omega, \mathcal{F}, \mathsf{P})$ a stochastic process $\xi(t)$ with values in $\mathbb{R}^n$ be given. We suppose that at every t the element $\xi(t)$ belongs to the functional space L_1.

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Denote by $\mathcal{P}_t^\xi$ the σ-subalgebra of $\mathcal{F}$ that is generated by preimages of Borel sets in $\mathbb{R}^n$ under all mappings $\xi(s):\Omega\to\mathbb{R}^n$, $0<s<t$. By $E(\cdot\mid\mathcal{P}_t^\xi)$ we denote the conditional expectation with respect to $\mathcal{P}_t^\xi$. Following Nelson [1, 2, 3] we call $\mathcal{P}_t^\xi$ the past of process $\xi(t)$.

Nelson introduced several constructions of mean derivatives, but in this paper we deal only with the mean derivatives with respect to the "past" according to the following definition

Definition 1.1. ([1, 4]) The forward mean derivative relative to the past ($\mathcal{P}$-mean derivative) $D^{\mathcal{P}}\xi(t)$ of $\xi(t)$ at a time instant t is L_1-random element of the form

$$D^{\mathcal{P}}\xi(t)=\lim_{\triangle t\to+0}E\left(\frac{\xi(t+\triangle t)-\xi(t)}{\triangle t}\,\Big|\,\mathcal{P}_t^\xi\right), \tag{1.1}$$

where the limit is assumed to exist in L_1 and $\triangle t\to+0$ means that $\triangle t$ tends to 0 and $\triangle t>0$.

From the properties of conditional expectation (see [5]) it follows that $D\xi(t)$ can be represented as compositions of $\xi(t)$ and Borel measurable vector fields (regressions)

$$a(t,x)=\lim_{\Delta t\to+0}E(\frac{\xi(t+\Delta t)-\xi(t)}{\Delta t}|\xi(t)=x) \tag{1.2}$$

on $\mathbb{R}^n$. This means that $D\xi(t)=a(t,\xi(t))$.

Following [4, 6] we introduce the new mean derivative D_2, called quadratic relative to the past, that differentiates an L_1 random process $\xi(t)$, $t\in[0,T]$ according to the rule

$$D_2^{\mathcal{P}}\xi(t)=\lim_{\triangle t\to+0}E\left(\frac{(\xi(t+\triangle t)-\xi(t))(\xi(t+\triangle t)-\xi(t))^*}{\triangle t}\,\Big|\,\mathcal{P}_t^\xi\right), \tag{1.3}$$

where the limit is assumed to exist in L_1, $\triangle t\to+0$ means that $\triangle t$ tends to 0 and $\triangle t>0$. Here $(\xi(t+\triangle t)-\xi(t))$ is considered as a column vector in $\mathbb{R}^n$ while $(\xi(t+\triangle t)-\xi(t))^*$ is a row vector (transposed, or conjugate vector) and the limit is supposed to exists in $L_1(\Omega,\mathcal{F},\mathsf{P})$.

We emphasize that the matrix product of a column on the left and a row on the right is a matrix. It is shown that $D_2\xi(t)$ is a symmetric positive semi-definite matrix function on $[0,T]\times\mathbb{R}^n$.

Consider the Banach space $C^0([0,T],\mathbb{R}^n)$ of continuous curves in $\mathbb{R}^n$ given on $[0,T]$ with usual uniform norm and the σ-algebra $\mathcal{C}$ generated by cylinder sets. By $\mathcal{P}_t$ we denote the σ-subalgebra of $\mathcal{C}$ generated by cylinder sets with bases over $[0,t]\subset[0,T]$. Recall that $\mathcal{C}$ is the Borel σ-algebra on $C^0([0,T],\mathbb{R}^n)$.

Let $a:[0,T]\times\ C^0([0,T],\mathbb{R}^n)\to\mathbb{R}^n$ and $\alpha:[0,T]\times C^0([0,T],\mathbb{R}^n)\to\bar{S}_+(n)$ be measurable mappings.

The equation with $\mathcal{P}$-mean derivatives is a system of the form

$$\begin{cases} D^{\mathcal{P}}\xi(t)=a(t,\xi(\cdot)),\\ D_2^{\mathcal{P}}\xi(t)=\alpha(t,\xi(\cdot)).\end{cases} \tag{1.4}$$

Definition 1.2. ([7]) We say that (1.4) on $\mathbb{R}^n$ has a solution on $[0,T]$ with initial condition $\xi(0)=\xi_0$ if there exists a probability space $(\Omega,\mathcal{F},\mathsf{P})$ and a process $\xi(t)$ given on $(\Omega,\mathcal{F},\mathsf{P})$ and taking values in $\mathbb{R}^n$ such that $\xi(0)=\xi_0$ and for almost all $t\in[0,T]$ equation (1.4) is satisfied P-a.s. by $\xi(t)$.

Let $\Xi : [0,T] \times C^0([0,T],\mathbb{R}^n) \to Z$ be a mapping to some metric space Z. Below we shall often suppose that such mappings with various spaces Z satisfy the following condition:

Condition 1. *For each $t \in [0,T]$ from the fact that the curves $x_1(\cdot)$, $x_2(\cdot) \in C^0([0,T],\mathbb{R}^n)$ coincide for $0 \le s \le t$, it follows that $\Xi(t,x_1(\cdot)) = \Xi(t,x_2(\cdot))$.*

Remark 1.3. Note that the fact that a mapping Ξ satisfies Condition 1, is equivalent to the fact that Ξ at each t is measurable with respect to Borel σ-algebra in Z and $\mathcal{P}_t$ in $\tilde{\Omega}$ (see [8]).

Lemma 1.4. **(**[7]**)** *For a continuous (measurable, smooth) mapping*

$$\alpha : [0,T] \times C^0([0,T],\mathbb{R}^n) \to S_+(n)$$

satisfying Condition 1, there exists a continuous (measurable, smooth, respectively) mapping $A : [0,T] \times C^0([0,T],\mathbb{R}^n) \to L(\mathbb{R}^n,\mathbb{R}^n)$ that satisfies Condition 1 and such that $\alpha(t,x(\cdot)) = A(t,x(\cdot))A^*(t,x(\cdot))$ for each $(t,x(\cdot)) \in \mathbb{R} \times \tilde{\Omega}$.

The proof of Lemma 1.4 can be found in [7, Lemma 1].

Consider set-valued mappings $\boldsymbol{a}(t,x(\cdot))$ and $\boldsymbol{\alpha}(t,x(\cdot))$ that send $[0,T]\times C^0([0,T],\mathbb{R}^n)$ to $\mathbb{R}^n$ and $\overline{S}_+(n)$, respectively, and in addition satisfy Condition 1. The differential inclusion with forward $\mathcal{P}$-mean derivatives is a system of the form

$$\begin{cases} D^{\mathcal{P}}\xi(t) \in \boldsymbol{a}(t,\xi(\cdot)), \\ D_2^{\mathcal{P}}\xi(t) \in \boldsymbol{\alpha}(t,\xi(\cdot)). \end{cases} \tag{1.5}$$

Definition 1.5. **(**[7]**)** We say that inclusion (1.5) has a solution with initial condition $\xi_0 \in \mathbb{R}^n$ if there exists a probability space and a stochastic process $\xi(t)$ given on it and taking values in $\mathbb{R}^n$, such that $\xi(0) = \xi_0$ and a.s. $\xi(t)$ satisfies inclusion (1.5).

2. Set-valued mappings with lower semi-continuous right-hand sides

A set-valued mapping F from a set X into a set Y is a correspondence that assigns a non-empty subset $F(x) \subset Y$ to every point $x \in X$; $F(x)$ is called the value of x.

In order to distinguish set-valued mappings form single-valued ones we shall denote a set-valued mapping F sending X to Y, by the symbol $F : X \multimap Y$ while for a single-valued mapping we shall keep the notation $f : X \to Y$.

If X and Y are metric spaces, for set-valued mappings there are several different analogues of continuity that in the case of single-valued mappings are transformed into usual continuity (here we do not deal with the description of such notion for set-valued mappings of topological spaces, see, e.g., [9]). In this paper we deal only with lower semi-continuous set-valued mappings, so we give only their definitions.

Definition 2.1. A set-valued mapping F is called lower semicontinuous at the point $x \in X$ if for each $\varepsilon > 0$ there exists a neighbourhood $U(x)$ of x such that from $x' \in U(x)$ it follows that $F(x)$ belongs to the ε-neighbourhood of $F(x')$. F is called lower semicontinuous on X if it is lower semicontinuous at every point of X

An important technical role in investigating set-valued mappings is played by single-valued mappings that approximate the set-valued ones in some sense. We describe two kinds of such single-valued mappings: selectors and ε-approximations.

Definition 2.2. Let $F : X \multimap Y$ be a set-valued mapping. A single-valued mapping $f : X \to Y$ such that for each $x \in X$ the inclusion $f(x) \in F(x)$ holds, is called a selector of F.

Not every set-valued mapping has a continuous selector. For lower semicontinuous set-valued mappings with convex closed values their existence is proved in the classical Michael's Theorem.

Theorem 2.3. (Michael's Theorem) *If X is an arbitrary metric space and Y is a Banach space, a lower semicontinuous mapping such that the value of every point of X is a convex closed set, has a continuous selector.*

If the values of a lower semicontinuous set-valued mapping (generally speaking) are not convex, it may not have continuous selectors. Then the following construction is often very much useful.

Definition 2.4. Let E be a separable Banach space. A non-empty set $Y \subset L^1([0,l],E)$ is called decomposable if $f \cdot \chi(\mathcal{Y}) + g \cdot \chi([0,l]\backslash\mathcal{Y}) \in Y$ for all $f, g \in Y$ and for every measurable subset $\mathcal{Y}$ in $[0,l]$ where χ is the characteristic function of the corresponding set.

The reader can find more details about decomposable sets in [10] and [11].

Theorem 2.5. *Let (Ξ, d) be a separable metric space, X be a Banach space. Consider the space $Y = L^1(([0,T],\mathcal{B},\lambda),X))$ of integrable maps from $[0,T]$ into X (here $\mathcal{B}$ is the Borel σ-algebra and λ is the normalised Lebesgue measure). If a set-valued map $G : \Xi \to Y$ is lower semicontinuous and has closed decomposable images, it has a continuous selector.*

Theorem 2.5 is a particular case of the Fryszkowski-Bressan-Colombo Theorem, see, e.g., [10, Lemma 9.2].

Below we are using only deterministic initial values, i.e., ξ_0 being a point in $\mathbb{R}^n$.

3. The main result

Theorem 3.1. *Let the set-valued vector field*

$$\mathbf{a} : [0,T] \times C^0([0,T],\mathbb{R}^n) \to \mathbb{R}^n$$

be lower semi-continuous, have closed images, uniformly bounded, i.e.,

$$\|a(t,x(\cdot))\| < C \tag{3.1}$$

for some $C > 0$ and for all $a(t,x(\cdot)) \in \mathbf{a}(t,x(\cdot))$, and satisfy Condition 1.

Let $\boldsymbol{\alpha} : [0,T] \times C^0([0,T],\mathbb{R}^n) \to S_+(n)$ be set-valued, lower semi-continuous, have closed convex values, be uniformly bounded and satisfy Condition 1. We also suppose that there exist constants $C_0 > 0$ and $C_1 > C_0$ such that the following inequality

$$C_0 < tr\,\alpha(t,x(\cdot)) < C_1, \tag{3.2}$$

for every $\alpha(t,x(\cdot)) \in \boldsymbol{\alpha}(t,x(\cdot))$ holds uniformly.

Then under the above hypotheses the inclusion

$$\begin{cases} D^{\mathcal{P}}\xi(t) \in \mathbf{a}(t,\xi(\cdot)), \\ D_2^{\mathcal{P}}\xi(t) \in \boldsymbol{\alpha}(t,\xi(\cdot)). \end{cases} \tag{3.3}$$

for the initial condition $\xi(0) = \xi_0$ has a solution well defined on the entire interval $t \in [0,T]$.

Proof. First of all, by Theorem 2.3 (Michael's Theorem) there exists a continuous selector $\alpha(t,\xi(\cdot))$ of $\boldsymbol{\alpha}(t,\xi(\cdot))$ that evidently satisfies Condition 1 and inequality (3.2).

Let $x(\cdot)$ be a continuous curve. Consider the set-valued vector field $\mathbf{a}(t,x(\cdot))$ along $x(\cdot)$ and denote by $\mathbf{Pa}(\cdot,x(\cdot))$ the set of all measurable selectors of $\mathbf{a}(t,x(\cdot))$, i.e., the set of measurable maps $\{f:\mathbb{R}\to\mathbb{R}^n : f(x(t))\in\mathbf{a}(t,x(\cdot))\}$. It is obvious that since estimate (3.1) is satisfied, all those selectors are integrable on any finite interval in $\mathbb{R}$ with respect to Lebesgue measure.

In $C^0([0,T],\mathbb{R}^n)$ introduce the σ-algebra $\tilde{\mathcal{F}}$ generated by cylindrical sets. By $\tilde{\mathcal{P}}_t$ denote the σ-algebra generated by cylindrical sets over $[0,t]\subset[0,T]$.

Consider the set-valued mapping B that sends $x(\cdot)\in C^0([0,T],\mathbb{R}^n)$ into $\mathbf{Pa}(\cdot,x(\cdot))$. Because of estimate (3.1) all selectors from $\mathbf{Pa}(\cdot,x(\cdot))$ are integrable (see above), hence B takes values in the space $L^1(([0,T],\mathcal{B},\lambda),\mathbb{R}^n)$. It is known (see, e.g., [11, Section 5.5]) that under the above-mentioned conditions

$$B: C^0([0,T],\mathbb{R}^n)\to L^1(([0,T],\mathcal{B},\lambda),\mathbb{R}^n)$$

is lower semicontinuous and for any $x(\cdot)\in C^0([0,T],\mathbb{R}^n)$ the set $\mathbf{Pa}(\cdot,x(\cdot))$ (the image $B(x(\cdot))$) is decomposable and closed. Thus, by Lemma 2.5 B has a continuous selector $b: C^0([0,T],\mathbb{R}^n)\to L^1(([0,T],\mathcal{B},\lambda),\mathbb{R}^n)$.

For any $t\in[0,T]$ let us introduce the map $f_t: C^0([0,T],\mathbb{R}^n)\to C^0([0,T],\mathbb{R}^n)$ sending a curve $x(\cdot)\in C^0([0,T],\mathbb{R}^n)$ into the curve

$$f_t(\tau,x(\cdot))=\begin{cases} x(\tau) & \text{for } \tau\in[0,t]\\ x(t) & \text{for } \tau\in[t,l]\end{cases}.$$

Obviously the map f_t is continuous. Since $f_t(\tau,x(\cdot))$ belongs to $C^0([0,T],\mathbb{R}^n)$, the curve $b(f_t(\tau,x(\cdot)))\in L^1(([0,T],\mathcal{B},\lambda),\mathbb{R}^n)$ is well defined. By construction $b(f_t(\tau,x(\cdot)))\in\mathbf{a}(\tau,x(\tau))$ for almost all $\tau\in[0,t]$ and so this selector continuously depends on t in $L^1(([0,T],\mathcal{B},\lambda),\mathbb{R}^n)$.

Introduce the map $a:[0,T]\times C^0([0,T],\mathbb{R}^n)\to\mathbb{R}^n$ by the formula

$$a(t,x(\cdot))=b(f_t(\tau,x(\cdot))). \tag{3.4}$$

By the construction this map is continuous jointly in $t\in[0,T]$ and $x(\cdot)\in C^0([0,T],\mathbb{R}^n)$. It is obvious that if $x_1(\cdot)$ and $x_2(\cdot)$ coincide on $[0,t]$ then $a(t,x_1(\cdot))=a(t,x_2(\cdot))$. This means that $a(t,x(\cdot))$ is measurable with respect to $\tilde{\mathcal{P}}_t$. (see Remark 1.3).

Taking into account (3.1) one can easily derive that $\|a(t,x(\cdot))\|$ is uniformly bounded.

In follows from Lemma 1.4 that there exists continuous mapping $A:[0,T]\times C^0([0,T],\mathbb{R}^n)\to L(\mathbb{R}^n,\mathbb{R}^n)$ that satisfies Condition 1 and such that $\alpha(t,x(\cdot))=A(t,x(\cdot))A^*(t,x(\cdot))$ for each $(t,x(\cdot))\in\mathbb{R}\times C^0([0,T],\mathbb{R}^n)$.

Consider the diffusion type equation

$$d\xi(t)=A(t,\xi(\cdot))dw(t). \tag{3.5}$$

By [8, Sec. III.2, Theorem 4] equation (3.5) has a solution. Note that it follows from (3.2) that $A(t,x(\cdot))$ is uniformly bounded and has uniformly bounded converse operator. Then by Corollary 1 to [8, Sec. III.2, Theorem 2] the equation

$$d\xi(t)=a(t,\xi(\cdot))dt+A(t,\xi(\cdot))dw(t) \tag{3.6}$$

has a solution as well. One can easily see that the solution $\xi(t)$ of (3.6) is a solution of (3.3). From the general theory of equations with mean derivatives it follows that $D^{\mathcal{P}}\xi(t) = a(t, \xi(\cdot) \in \mathbf{a}(t, \xi(\cdot))$ and $D_2^{\mathcal{P}}\xi(t) = \alpha(t, \xi(\cdot))$. $\square$

Note that the solution of (3.6) is not unique.

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