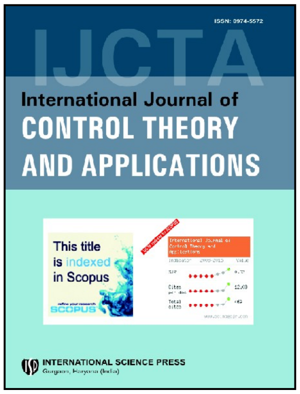

International Journal of Control Theory and Applications

ISSN : 0974-5572

Volume 10 • Number 11 • 2017

Best Septic Approximation of Circular Arcs with Fifteen Equioscillations

Abedallah Rababah

Department of Mathematics, Jordan University of Science and Technology, 22110 Irbid, Jordan
E-mail: rababah@just.edu.jo

Abstract: The issue of polynomial approximation of degree seven for circular arcs is considered. This septic approximation is found in such away that the Chebyshev error function is of degree fourteen with the least deviation from the x-axis. The Chebyshev error function equioscillates fifteen times rather than nine times equioscillations that are mathematically guaranteed by the Borel and Chebyshev theorems. The error function is characterized by having the same extremas and roots of the Chebyshev polynomial of degree fourteen.

keywords: Bézier curves; septic approximation; circular arc; equioscillation; CAD.

1. INTRODUCTION

Let $C[a,b]$ be the set of continuous functions on the closed interval $[a,b]$. The uniform (Chebyshev) norm on the linear space $C[a,b]$ is defined by

$$\| f \|_\infty = \max_{a\le x\le b} | f(x) |, \quad \forall f \in C[a,b].$$

Let P_n be the set of all polynomials of degree n, i.e. polynomials of the form $P_n(x) = \sum_{i=0}^{n} a_i x^i$. The space P_n is of finite dimension, closed, and convex because it is a subspace of $C[a,b]$. For a function $f \in C[a,b]$, define the deviation $\Delta(P_n)$ of $P_n \in P_n$ from f by

$$\Delta(P_n) = \| f - P_n \|_\infty .$$

Also define

$$E_n = E_n(f) = \inf\{\Delta(P_n), \forall P_n \in P_n\}.$$

It is clear that

$$E_n \geq 0, \quad E_0 \geq E_1 \geq E_2 \geq E_3 \geq \cdots.$$

E_n is closed and, therefore, is compact.

For a function $f \in C[a,b]$, a polynomial P_n^* is called the polynomial of best uniform approximation to *f* if $\Delta(P_n^*) = E_n$.

E. Borel proved that there exists in P_n a polynomial P_n^* for which $\Delta(P_n^*) = E_n$, see [9]. A function $E(x)$ is said to equioscillate $n+2$ times on $[a,b]$ if there are $n+2$ points, $a \leq x_1 < x_2 < \cdots < x_{n+1} < x_{n+2} \leq b$ with

$$|E(x_i)| = \max_{a \leq x \leq b} |E(x)|, 1 \leq i \leq n+2$$

and $E(x_{i+1})$ has opposite sign of $E(x_i), 1 \leq i \leq n+1$. Chebyshev proved that this polynomial P_n^* is unique and is characterized by the existence of $n+2$ points (Chebyshev alternant, $x_1 < x_2 < \cdots < x_{n+2}$) which alternately satisfy

$$P_n^*(x_i) - f(x_i) = \mp E_n, i = 1,2,\ldots,n+2.$$

However, for computational purposes, only polynomials of degrees 0 and 1 are characterized, see [9]. There is no method or characterization to find polynomials of best approximation of degrees $n \geq 2$. The improvement in this field is very slow and it is a challenging issue to tackle this problem; "As a matter of fact, the latter problem involves such formidable difficulties that a general solution has not been found to this day [9]".

Let $\tilde{P}_n$ be the set of all monic polynomials of degree n on $I = [0,1]$, i.e. polynomials of the form $P_n(x) = x^n + \sum_{i=0}^{n-1} a_i x^i$. It is well-known that the shifted monic Chebyshev polynomial $\tilde{T}_n(2t-1)$ has the smallest maximum value on *I*, i.e.

$$\| \tilde{T}_n \|_\infty \leq \| \tilde{P}_n \|_\infty, \quad \forall \tilde{P}_n \in \tilde{P}_n,$$

where equality holds only if $\tilde{P}_n = \tilde{T}_n$. The Chebyshev polynomial $\tilde{T}_n(2t-1)$ equioscillates $n+2$ times in $[0,1]$.

The polynomial approximation of degree seven for the circle is treated; the polynomial of best approximation that equioscillates fifteen times rather than nine times is constructed. This is a substantial improvement in the field of approximation theory. A process to locate the best approximation of degree seven that equioscillates nine times is not possible so far.

Circular arcs are widely used to represent motions and model many objects. The implicit form of the circle is impractical for computer applications, and the trigonometric equations are also impractical and there is a need for a convenient and practical form to generate points on the circle. The parametric curves are suitable for computer applications. They present an alternative sufficient method to represent and make a curve. Parametric curves grant additional parameters to get better approximation for the stated problem. This approach is used to get approximations of order 2*n* instead of the classical order of *n*+1 using the Lagrange interpolation, see [10, 12] and the references therein.

The paper has the following construction. The approximation problem is explained in section two. The Bézier curves of degree seven and the Bézier points are examined in section three. The septic Bézier curve with the least deviation is demonstrated in section four, while its roots and extrema are given in section five. Section six contains the conclusions.

2. THE APPROXIMATION PROBLEM

Given the circular arc $c : t \mapsto (\cos(t), \sin(t)), -\theta \le t \le \theta$ as in Fig. 1. As explained before, this form is not proper for computer applications. We need to find a polynomial curve that represents the circular arc with least deviation from the x-axis. Classical methods of approximation depend on the Lagrange and Hermite approximations and yield order of approximation of $n+1$. In this paper, the geometric properties of the curve are utilized in the choice of the Bézier points to get the Bézier curve of degree seven with approximation rate fourteen rather than the classical rate of eight. The circular arc c will be represented by employment of Bézier curve $p : t \mapsto (x(t), y(t)), 0 \le t \le 1$, where $x(t), y(t)$ are polynomials of degree seven, to approximate c with rate of approximation of fourteen.

The issue of approximating a circular arc has been inspected by many researchers. In the literature, assortments of conditions, norms, and degrees are applied, see [1, 2, 3, 4, 5, 8, 11, 13] and the references therein. Albeit, it is the first time that the degree seven is scrutinized; non of these papers debates this case with high order of approximation. Our results are optimal and can not be improved.

The l_∞-norm is harnessed as a measure for the error function for approximating the circular arc c by the polynomial curve p as follows:

$$E(t) := \sqrt{x^2(t) + y^2(t)} - 1. \tag{1}$$

Instead of the l_∞-norm, the following modified Euclidean error form $E(t)$ is applied:

$$e(t) := x^2(t) + y^2(t) - 1. \tag{2}$$

It is rational to apply this error form because $e(t) = 0$, i. e. the components $x(t)$ and $y(t)$ of the approximating polynomial curve p make up for the implicit equation of the circle.

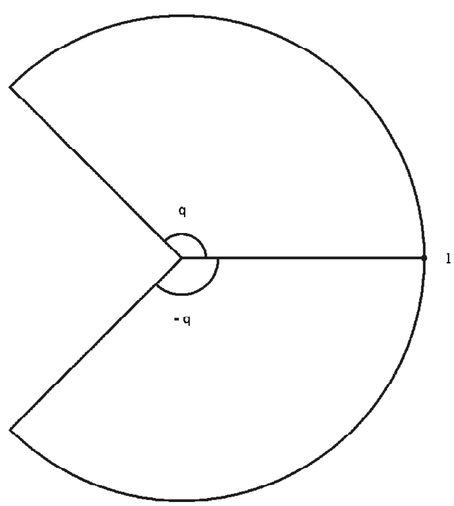

Figure 1: A circular arc

Our aim is to detect the polynomial curve $p: t \mapsto (x(t), y(t)), 0 \le t \le 1$, where $x(t)$ and $y(t)$ are septic polynomials. The following two statuses have to be redeemed:

1. The parametric polynomial p minimizes $\max_{t\in[0,1]} |e(t)|$,

2. The modified Euclidean error term $e(t)$ equioscillates fifteen times over $[0,1]$.

We apply These statuses to get the Bézier points. This is realized by finding the values of the parameters introduced from the geometric properties of the circular arc, see the books [6, 7].

The modified Chebyshev polynomial $e(t)$ will be equalized with the monic Chebyshev polynomial of degree fourteen that has uniform error of 2^{-13}. The angle θ is taken to be as large as possible in order to approximate the largest circular arc with uniform error 2^{-13}. For a future research, one should caliber θ in order to further minimize the uniform error. The process may include semi numerical procedure.

3. SEPTIC BÉZIER CURVE

Write the septic curve $p(t)$ as a septic Bézier curve in the following form:

$$p(t) = \sum_{i=0}^{7} p_i B_i^7(t) =: \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}, \quad 0 \le t \le 1. \tag{3}$$

The points $p_0, p_1, p_2, p_3, p_4, p_5, p_6$ and p_7 are called the Bézier points, and the septic polynomials

$B_0^7(t) = (1-t)^7$, $B_1^7(t) = 7t(1-t)^6$, $B_2^7(t) = 21t^2(1-t)^5$, $B_3^7(t) = 35t^3(1-t)^4$, $B_4^7(t) = 35t^4(1-t)^3$, $B_5^7(t) = 21t^5(1-t)^2$, $B_6^7(t) = 7t^6(1-t)$ and $B_7^7(t) = t^7$ are the Bernstein polynomials.

We are attentive to coerce the error to possess minimum deviation regardless at which part of the curve this error appears in the middle or at the boundary points. To obtain continuity between the approximating curve and the circular arc, the error function can be amended to reign zeros at the boundaries. Though, in this case, the error will be more than the error in the case we explore.

As explained before, we are interested in minimizing the uniform error over the whole segment $[0,1]$. Adopting a harmonic choice for the Bézier points with the circular arc will enable us to get the best uniform approximation with the least deviation. The key point is the proper choice of the Bézier points to be content with the approximation problem. The circle takes possession of many symmetries that will be used to detect the adequate positioning for the control points. We begin with the first control point p_0; it has to be as follows: $p_0 = (\alpha_0 \cos(\theta), \beta_0 \sin(\theta))$. The values of α_0 and β_0 will be found later to satisfy the approximation problem. The other end control point p_7 should reflect the existing symmetry in the circle, and therefore has the form $p_7 = (\alpha_0 \cos(\theta), -\beta_0 \sin(\theta))$. Initiate $p_1 = (a_1, b_1)$, then the point p_6 should reflect the existing symmetry in the circle, and therefore $p_6 = (a_1, -b_1)$. Initiate $p_2 = (a_2, b_2)$, then the point p_5 should reflect the existing symmetry in the circle, and therefore $p_5 = (a_2, -b_2)$. Initiate $p_3 = (a_3, b_3)$, then the point p_4 should reflect the existing symmetry in the circle, and therefore $p_4 = (a_3, -b_3)$. For the sake of simplification, set $a_0 = \alpha_0 \cos(\theta), b_0 = \beta_0 \sin(\theta)$. The Bézier points have the following settings, see Fig. 3,

$$p_0 = \begin{pmatrix} a_0 \\ b_0 \end{pmatrix}, p_1 = \begin{pmatrix} a_1 \\ b_1 \end{pmatrix}, p_2 = \begin{pmatrix} a_2 \\ b_2 \end{pmatrix}, p_3 = \begin{pmatrix} a_3 \\ b_3 \end{pmatrix}, p_4 = \begin{pmatrix} a_3 \\ -b_3 \end{pmatrix}, p_5 = \begin{pmatrix} a_2 \\ -b_2 \end{pmatrix}, p_6 = \begin{pmatrix} a_1 \\ -b_1 \end{pmatrix}, p_7 = \begin{pmatrix} a_0 \\ -b_0 \end{pmatrix}. \quad (4)$$

Substitute the Bézier points in (4) in the Bézier curve $p(t)$ in (3) to get $\forall t \in [0,1]$:

$$p(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} a_0\left(B_0^7(t) + B_7^7(t)\right) + a_1\left(B_1^7(t) + B_6^7(t)\right) + a_2\left(B_2^7(t) + B_5^7(t)\right) + a_3(B_3^7(t) + B_4^7(t)) \\ b_0\left(B_0^7(t) - B_7^7(t)\right) + b_1\left(B_1^7(t) - B_6^7(t)\right) + b_2\left(B_2^7(t) - B_5^7(t)\right) + b_3\left(B_3^7(t) - B_4^7(t)\right) \end{pmatrix}. \quad (5)$$

Later on, we will find out that there are several solutions. The solution that is in agreement with the tuning of the circle has to cycle counter clockwise starting from the second quadrant and terminating in the fourth quadrant. Subsequently, the Bézier curve p is ought to behave alike by cycling counter clockwise starting from the second quadrant and terminating in the fourth quadrant. To achieve this the following terms have to be contented:

$$a_0, a_1, a_2, b_2, b_3 < 0, \; a_3, b_0, b_1 > 0. \quad (6)$$

The parameters in the last equation designate the Bézier curve in (5); they are used to obtain the best approximation with least deviation. The conditions are enjoined on the approximating curve p. The components of p are substituted into the error function $e(t)$. Since the components $x(t)$ and $y(t)$ are septic polynomials, then the error function $e(t)$ is a polynomial of degree fourteen. In the coming section, the conditions are applied on the approximation polynomial to get the free parameters in (6).

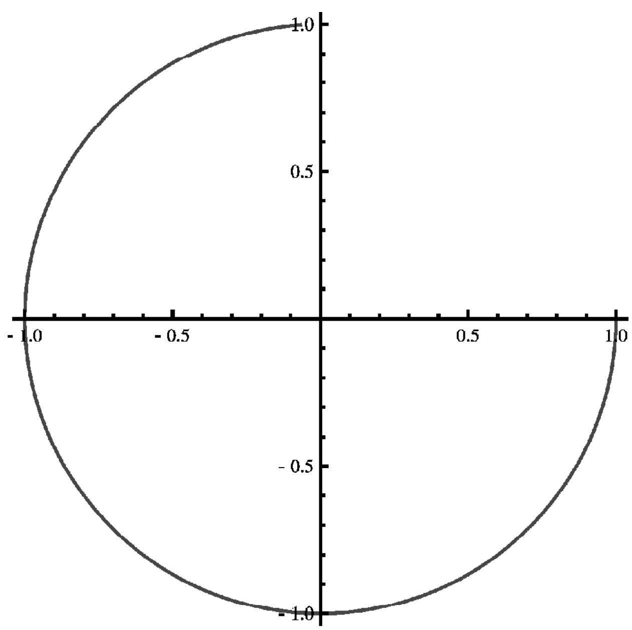

Figure 2: The half of the solution for $0 \le t \le \frac{1}{2}$

4. BEST SEPTIC BÉZIER CURVE

In this section, the parameters $a_0, a_1, a_2, a_3, b_0, b_1, b_2, b_3$ are set so that the statuses of the approximation problem are given as in the following theorem:

Theorem 1: Substituting the following values of the parameters:

$$a_0 = -0.06793067774776883, a_1 = -1.8405489886602024, a_2 = -2.4131651725084513, \quad (7)$$

$$a_3 = 3.646409595766575, b_0 = 0.997751218148063, b_1 = 0.8736389848122498, \quad (8)$$

$$b_2 = -2.7866952694531855, b_3 = -3.6468092095444162 \quad (9)$$

in the Bézier points (4) and thereafter in the Bézier curve (5) realizes the following statuses: p minimizes the Chebyshev norm of the error function $\max_{t\in[0,1]} |e(t)|$, and the error function $e(t)$ equioscillates fifteen times in $[0,1]$. The error functions realize $\forall t \in [0,1]$:

$$-\frac{1}{2^{13}} \le e(t) \le \frac{1}{2^{13}}, -\frac{1}{2^{13}(2-\varepsilon)} \le E(t) \le \frac{1}{2^{13}(2+\varepsilon)}, \; where \; \varepsilon = \max_{0\le t\le 1} |E(t)| \approx 2^{-14}. \quad (10)$$

Proof: These values of the parameters are realized in the septic polynomials $x(t)$ and $y(t)$ in equation (5) and thereafter realized in the Chebyshev error function $e(t)$ in (2). Simplifying the resulting equation leads to the following equation:

$$e(t) = 4(b_0 - 7(b_1 - 3b_2 + 5b_3))^2 t^{14} - 28(b_0 - 7(b_1 - 3b_2 + 5b_3))^2 t^{13} + 7\big(7a_0^2 + 175a_1^2 + 567a_2^2$$
$$-70a_1(9a_2 - 5a_3) - 630a_2a_3 + 175a_3^2 - 14a_0(5a_1 - 9a_2 + 5a_3) + 19b_0^2 - 242b_0b_1 + 763b_1^2$$
$$+678b_0b_2 - 4242b_1b_2 + 5859b_2^2 - 1090b_0b_3 + 6790b_1b_3 - 18690b_2b_3 + 14875b_3^2\big)t^{12}$$
$$-14\big(21a_0^2 + 525a_1^2 + 1701a_2^2 - 210a_1(9a_2 - 5a_3) - 1890a_2a_3 + 525a_3^2 - 42a_0(5a_1 - 9a_2 + 5a_3)$$
$$+31b_0^2 - 362b_0b_1 + 1015b_1^2 + 942b_0b_2 - 5082b_1b_2 + 6111b_2^2 - 1450b_0b_3 + 7630b_1b_3 - 17850b_2b_3$$
$$+12775b_3^2\big)t^{11} + 7\big(133a_0^2 + 2975a_1^2 + 8883a_2^2 - 70a_1(147a_2 - 80a_3) - 9660a_2a_3 + 2625a_3^2$$
$$-14a_0(90a_1 - 156a_2 + 85a_3) + 153b_0^2 - 1600b_0b_1 + 3955b_1^2 + 3756b_0b_2 - 17430b_1b_2 + 17703b_2^2$$
$$-5410b_0b_3 + 23940b_1b_3 - 45360b_2b_3 + 27125b_3^2\big)t^{10} - 14\big(140a_0^2 + 2625a_1^2 - 8400a_1a_2 + 6615a_2^2$$
$$+4375a_1a_3 - 6825a_2a_3 + 1750a_3^2 - 35a_0(35a_1 - 57a_2 + 30a_3) + 146b_0^2 - 1349b_0b_1 + 2919b_1^2$$
$$+2757b_0b_2 - 11004b_1b_2 + 9261b_2^2 - 3570b_0b_3 + 13265b_1b_3 - 19845b_2b_3 + 9100b_3^2\big)t^9 + 7\big(427a_0^2$$
$$+6370a_1^2 - 18186a_1a_2 + 12348a_2^2 + 8820a_1a_3 - 11550a_2a_3 + 2625a_3^2 - 14a_0(241a_1$$
$$+180(-2a_2 + a_3)) + 431b_0^2 - 3490b_0b_1 + 6566b_1^2 + 6048b_0b_2 - 20622b_1b_2 + 14112b_2^2 - 6720b_0b_3$$

$$+20580b_1b_3-23730b_2b_3+7525b_3^2\Big)t^8-2\Big(1715a_0^2+19355a_1^2+26019a_2^2-2058a_1(23a_2-10a_3)$$
$$-20580a_2a_3+3675a_3^2-98a_0(122a_1-162a_2+75a_3)+1717b_0^2-12068b_0b_1+19453b_1^2$$
$$+17388b_0b_2-49686b_1b_2+26901b_2^2-15750b_0b_3+38220b_1b_3-32340b_2b_3+6125b_3^2\Big)t^7$$
$$+7\Big(429a_0^2+3465a_1^2-6888a_1a_2+2835a_2^2+2520a_1a_3-1680a_2a_3+175a_3^2-4a_0(643a_1-729a_2$$
$$+300a_3)+429b_0^2-2576b_0b_1+3465b_1^2+3024b_0b_2-6972b_1b_2+2835b_2^2-2100b_0b_3+3780b_1b_3$$
$$-2100b_2b_3+175b_3^2\Big)t^6-14\Big(143a_0^2+770a_1^2-1155a_1a_2+315a_2^2+a_0(-715a_1+657a_2$$
$$-225a_3)+315a_1a_3-105a_2a_3+143b_0^2-715b_0b_1+770b_1^2+663b_0b_2-1155b_1b_2+315b_2^2$$
$$-325b_0b_3+385b_1b_3-105b_2b_3\Big)t^5+7\Big(143a_0^2+462a_1^2-462a_1a_2+63a_2^2+70a_1a_3$$
$$-4a_0(143a_1-99a_2+25a_3)+143b_0^2-572b_0b_1+462b_1^2+396b_0b_2-462b_1b_2+63b_2^2-120b_0b_3$$
$$+70b_1b_3\Big)t^4-14\Big(26a_0^2+42a_1^2-21a_1a_2+a_0(-78a_1+36a_2-5a_3)+26b_0^2-78b_0b_1$$
$$+42b_1^2+36b_0b_2-21b_1b_2-5b_0b_3\Big)t^3+7\Big(13a_0^2-26a_0a_1+7a_1^2+6a_0a_2$$
$$+13b_0^2-26b_0b_1+7b_1^2+6b_0b_2\Big)t^2-14\Big(a_0^2-a_0a_1+b_0(b_0-b_1)\Big)t-1+a_0^2+b_0^2.$$

It is known in approximation theory that the Chebyshev polynomial of first kind of degree fourteen, $\widetilde{T}_{14}(2t-1)/8192$ is the unique polynomial among all polynomials of degree fourteen that has the least deviation from the x-axis, where $\widetilde{T}_{14}(u)=\cos(14\arccos(u)),\ u\in[-1,1]$. Since the resulting error function is a polynomial of degree fourteen, thus, it attains the uniform error when it equals the monic Chebyshev polynomial of degree fourteen. This feature is utilized to realize the approximation statuses. Making the error function equal to the Chebyshev polynomial of first kind of degree fourteen, $\widetilde{T}_{14}(2t-1)/8192$ by equating the coefficients in both polynomials gives eight equalities that are used to find the values of the parameters $a_0,a_1,a_2,a_3,b_0,b_1,b_2,b_3$. With the help of the computer algebra system (Mathematica) we get the values of the parameters; these values are sophisticated, and it is unwieldy to present them, thus, they are documented in the decimal forms as specified in equations (7) - (9). Consequently, the septic polynomial p realizes the statuses of the approximation problem. The two error functions $e(t)$ and $E(t)$ are related by the following formula:

$$e(t)=x^2(t)+y^2(t)-1=(\sqrt{x^2(t)+y^2(t)}+1)\,(\sqrt{x^2(t)+y^2(t)}-1)=\big(2+E(t)\big)E(t).$$

We get $E(t)$ in the form:

$$E(t)=\frac{e(t)}{2+E(t)}.$$

Since the error $e(t)$ is confined uniformly by $\frac{1}{2^{13}}$, thus we get

$$-\frac{1}{2^{13}(2-\varepsilon)} \le E(t) \le \frac{1}{2^{13}(2+\varepsilon)}, \quad where \quad \varepsilon = \max_{0\le t\le 1} |E(t)| \approx 2^{-14}, \ t\in[0,1].$$

Thus the error formula is established and Theorem 1 is proved.

Fig. 2 shows the figures of part of the circular arc and its septic Bézier curve $\forall t \in [0,\frac{1}{2}]$, while the complete circular arc and its septic Bézier curve are shown in Fig. 3 with the error in Fig. 4. Fig. 4 reveals that the human eyes are not capable to distinguish the resulted error.

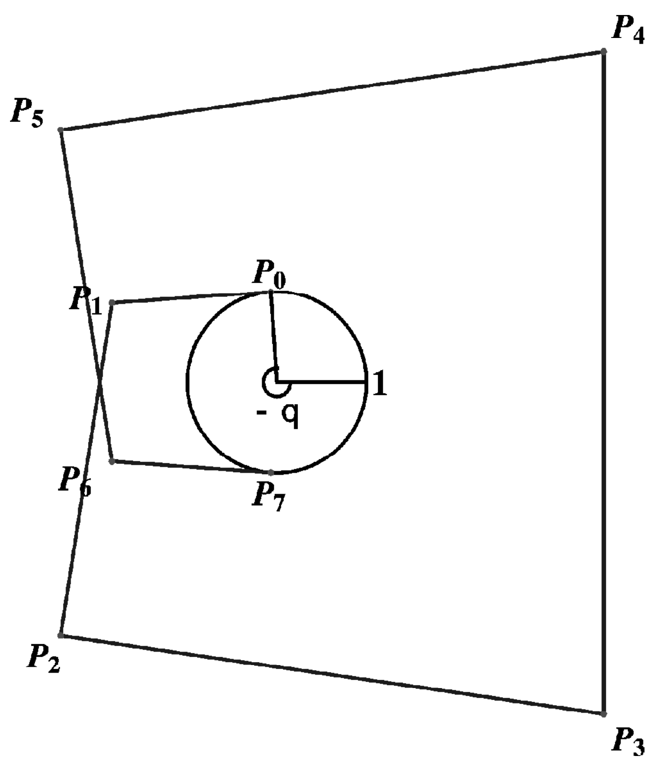

Figure 3: Circular arc and it's septic Bézier curve in Theorem 1.

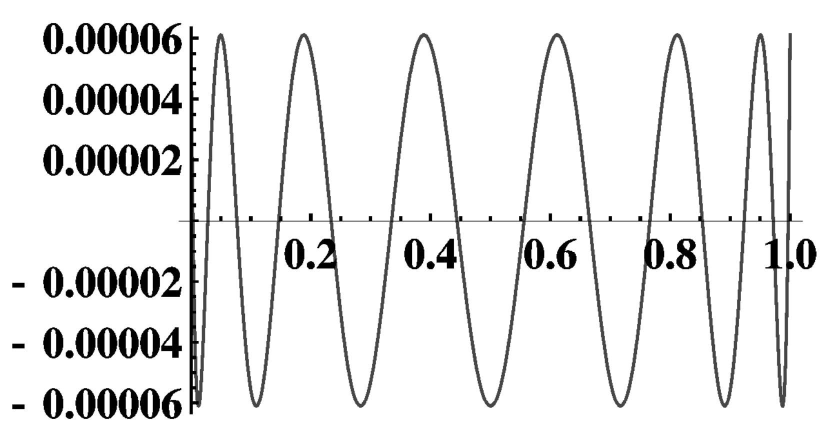

Figure 4: Euclidean Error of the septic Bézier curve in Theorem 1.

The approach in this paper offers a skillful placement of the Bézier points that can not be speculated by a designer to inclose one and half of the circle with the best uniform error.

5. ROOTS AND EXTREMA

The roots and the extrema of the Chebyshev error functions are given in this section. We first specify the roots of the error functions $e(t)$ and $E(t)$:

Proposition I: The roots of the Chebyshev error functions $e(t)$ and $E(t)$ are given by:

$$t_1 = \frac{1}{2}(1+\cos(\frac{\pi}{28})) = 0.996856, t_2 = \frac{1}{2}(1+\cos(\frac{3\pi}{28})) = 0.971942, t_3 = \frac{1}{2}(1+\cos(\frac{5\pi}{28})) = 0.923362,$$

$$t_4 = \frac{1}{2}(1+\cos(\frac{7\pi}{28})) = 0.853553, t_5 = \frac{1}{2}(1+\sin(\frac{5\pi}{28})) = 0.766016, t_6 = \frac{1}{2}(1+\sin(\frac{3\pi}{28})) = 0.66514,$$

$$t_7 = \frac{1}{2}(1+\sin(\frac{\pi}{28})) = 0.555982, t_8 = \frac{1}{2}(1-\sin(\frac{\pi}{28})) = 0.444018, t_9 = \frac{1}{2}(1-\sin(\frac{3\pi}{28})) = 0.33486$$

$$t_{10} = \frac{1}{2}(1-\sin(\frac{5\pi}{28})) = 0.233984, t_{11} = \frac{1}{2}(1-\cos(\frac{7\pi}{28})) = 0.146447, t_{12} = \frac{1}{2}(1-\cos(\frac{5\pi}{28})) = 0.0766379,$$

$$t_{13} = \frac{1}{2}(1-\cos(\frac{3\pi}{28})) = 0.0280583, t_{14} = \frac{1}{2}(1-\cos(\frac{\pi}{28})) = 0.0031439$$

Due to symmetry, they have the property:

$$t_i + t_j = 1, \quad for \quad i + j = 15.$$

Proof: Substituting the values of t_i in $e(t)$ yields $e(t_i) = 0, \forall i = 1,2,\ldots,14$. These are all of the roots of $e(t)$ because it is a polynomial of degree fourteen and has exactly fourteen roots. The other error function $E(t)$ has the same roots because $e(t) = 0$ iff $x^2(t) + y^2(t) = 1$ iff $\sqrt{x^2(t) + y^2(t)} = 1$ iff $E(t) = 0$.

Proposition II: The extreme values of the error functions $e(t)$ and $E(t)$ occur at the parameters:

$$\tilde{t}_0 = 1, \quad \tilde{t}_1 = \frac{1}{2}(1+\cos(\frac{\pi}{14})) = 0.987464, \quad \tilde{t}_2 = \frac{1}{2}(1+\cos(\frac{\pi}{7})) = 0.950484,$$

$$\tilde{t}_3 = \frac{1}{2}(1+\cos(\frac{3\pi}{14})) = 0.890916, \quad \tilde{t}_4 = \frac{1}{2}(1+\cos(\frac{2\pi}{7})) = 0.811745,$$

$$\tilde{t}_5 = \frac{1}{2}(1+\cos(\frac{5\pi}{14})) = 0.716942, \quad \tilde{t}_6 = \frac{1}{2}(1+\cos(\frac{3\pi}{7})) = 0.61126, \quad \tilde{t}_7 = \frac{1}{2}(1+\cos(\frac{\pi}{2})) = 0.5,$$

$$\tilde{t}_8 = \frac{1}{2}(1-\cos(\frac{3\pi}{7})) = 0.38874, \tilde{t}_9 = \frac{1}{2}(1-\cos(\frac{5\pi}{14})) = 0.283058,$$

$$\tilde{t}_{10} = \frac{1}{2}(1-\cos(\frac{2\pi}{7})) = 0.188255, \tilde{t}_{11} = \frac{1}{2}(1-\cos(\frac{3\pi}{14})) = 0.109084, \tilde{t}_{12} = \frac{1}{2}(1-\cos(\frac{\pi}{7})) = 0.0495156,$$

$$\tilde{t}_{13} = \frac{1}{2}(1-\cos(\frac{\pi}{14})) = 0.012536,\ \tilde{t}_{14} = 0.$$

Due to symmetry, they have the property:

$$\tilde{t}_i + \tilde{t}_j = 1, \quad for \quad i + j = 14.$$

Proof: Since $e(t)$ is of degree fourteen, thus its derivative is of degree thirteen and has thirteen roots. Substitute these thirteen values of $\tilde{t}_1,\ldots,\tilde{t}_{13}$ to get $e'(\tilde{t}_i) = 0,\ \forall i = 1,\ldots,13$. We check the end points for critical points and get $\tilde{t}_0 = 1,\ \tilde{t}_{14} = 0$.

Since $1-\frac{1}{8192} \le x^2(t) + y^2(t) \le 1+\frac{1}{8192}, \forall t \in [0,1]$, subsequently $\sqrt{x^2(t)+y^2(t)} \neq 0,\ \forall t \in [0,1]$.

Differentiating $E(t)$ and equate to 0 to obtain $\frac{e'(t)}{\sqrt{x^2(t)+y^2(t)}} = 0$. Since the denominator is not equal 0, then this happens only iff $e'(t) = 0$. Thus both error functions $e(t)$ and $E(t)$ possess the extrema at the same parameters. This completes the proof.

Proposition III: The error functions $e(t)$ and $E(t)$ attain their extrema at $\tilde{t}_i$'s with the following values:

$$e(\tilde{t}_{2i}) = \frac{1}{8192}, i = 0,\ldots,6, \qquad e(\tilde{t}_{2i+1}) = \frac{-1}{8192}, i = 0,\ldots,5.$$

$$E(\tilde{t}_{2i}) = \frac{1}{16384}, i = 0,\ldots,6, \qquad E(\tilde{t}_{2i+1}) = \frac{-1}{16384}, i = 0,\ldots,5.$$

Thus

$$\frac{-1}{8192} \le e(t) \le \frac{1}{8192}, \quad \frac{-1}{16384} \le E(t) \le \frac{1}{16384}, t \in [0,1].$$

Proof: By substituting these parameters $\tilde{t}_i$ into the error functions the equalities are realized.

Proposition IV: At any value $t \in [0,1]$, the errors using the septic Bézier curve in Theorem 1 to approximate the circular arc behave according to the following polynomial of degree fourteen:

$$e(t) = \frac{1}{8192} - \frac{49t}{1024} + \frac{3185t^2}{1024} - \frac{637t^3}{8} + \frac{17017t^4}{16} - \frac{17017t^5}{2} + \frac{88179t^6}{2} - 155040t^7 + 379848t^8 -$$

$$-655424t^9 + 793408t^{10} - 659456t^{11} + 358400t^{12} - 114688t^{13} + 16384t^{14}, \quad \forall t \in [0,1].$$

Proof: The proposition is a consequence of Theorem 1.

Using the link between the errors $E(t)$ and $e(t)$ gives:

$$E(t) \cong \frac{1}{16384} - \frac{49t}{2048} + \frac{3185t^2}{2048} - \frac{637t^3}{16} + \frac{17017t^4}{32} - \frac{17017t^5}{4} + \frac{88179t^6}{4} - 77520t^7 + 189924t^8 -$$

$$- 327712t^9 + 396704t^{10} - 329728t^{11} + 179200t^{12} - 57344t^{13} + 8192t^{14}, \quad \forall t \in [0,1].$$

6. CONCLUSIONS

Classical septic approximation of the circle yields an eighth order of approximation. We showed in this paper that by a proper choice of the Bézier points that the septic approximation of the circle yields fourteenth order of approximation. Moreover, the approximation attains statuses of Chebyshev quality; it equioscillates fifteen times. The numerical demonstrations expose the effectiveness of the proposed method. The approximation intersects the curve fourteen times with uniform error 2^{-13} and thus outperforms any other approximation.

ACKNOWLEDGEMENT

The author owes thanks to the reviewers for helpful and invaluable comments and suggestions for improving an earlier version of this paper.

REFERENCES

[1] Y. J. Ahn and C. Hoffmann, Circle approximation using LN Bézier curves of even degree and its application, J. Math. Anal. Appl. (2014), 257-266.

[2] P. Bézier, The mathematical basis of the UNISURF CAD system, Butterworth-Heinemann Newton, MA, USA, ISBN 0-408-22175-5, (1986).

[3] J. Blinn, How many ways can you draw a circle?, Computer Graphics and Applications, IEEE 7(8) (1987), 39-44.

[4] C. de Boor, K. Höllig and M. Sabin, High accuracy geometric Hermite interpolation, Comput. Aided Geom. Design 4 (1988), 269-278.

[5] L. Fang, Circular arc approximation by quintic polynomial curves, Computer Aided Geometric Design, Volume 15, Issue 8, (1998), 843-861.

[6] G. Farin, Curves and Surfaces for Computer Aided Geometric Design, Academic Press, Boston (1988).

[7] K. Höllig, J. Hörner: Approximation and Modeling with B-Splines, SIAM, Titles in Applied Mathematics 132, (2013).

[8] S. W. Kim and Y. J. Ahn, Circle approximation by quartic G^2 spline using alternation of error function, J. KSIAM , V 17(5) (2013), 171-179.

[9] I. P. Natanson, Constructive Function Theory, Vol. 1, Ungar, (1964).

[10] A. Rababah, Taylor theorem for planar curves, Proc. Amer. Math. Soc. Vol 119 No. 3 (1993), 803-810.

[11] A. Rababah, Approximation von Kurven mit Polynomen und Splines, Ph. Dissertation, Stuttgart Universität, 1992.

[12] A. Rababah, Best sextic approximation of circular arcs with thirteen equioscillations, to appear in Proceedings of the Jangjeon Mathematical Society (2017).

[13] A. Rababah, The best uniform cubic approximation of circular arcs with high accuracy, Communications in Mathematics and Applications 7(1), (2016), 37-46.